OPENING EYES TO MATHEMATICS

LESSONS / VOLUME 3

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Published by The Math Learning Center
Salem, Oregon

Opening Eyes to Mathematics
Lessons / Volume 3
Includes a set of blackline masters, packaged separately

Opening Eyes to Mathematics consists of
Teaching Reference Manual (Grades 3 and 4)
Contact & Insight Lessons / Volume 1 and blackline masters (Grade 3)
Contact & Insight Lessons / Volume 2 and blackline masters (Grade 3)
Lessons / Volume 3 and blackline masters (Grade 4)

Library of Congress Catalog Card Number: 91-62683

Prepared for publication on Macintosh Desktop Publishing system.

Printed in the United States of America

ISBN 1-886131-31-7

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Introduction

Anyone who writes down to children is simply wasting his time. You have to write up, not down. Children are demanding. They are the most attentive, curious, eager, sensitive, quick and generally congenial readers on earth. They accept, almost without question, anything you present them with, as long as it is presented honestly, fearlessly and clearly. Some writers for children deliberately avoid using words they think a child doesn't know.

This emasculates the prose and, I suspect, bores the reader. Children are game for anything. I throw them hard words, and they backhand them over the net. They love words that give them a hard time, provided they are in a context that absorbs their attention.

E. B. White, *The Writer's Chapbook*

Many of us who are teachers have come to accept the idea that reading can't really be "taught". We can read to children and share big books. As experienced users of print, we can model such strategies as moving our eyes from left to right, pointing to individual words as we recite familiar text, ... We can surround our students with print and share our own love of literature. Many of us believe that in the end, children must sort reading out for themselves, in much the same way they learn language.

Why then is it so difficult to trust that learning mathematics might work in the same way?

Donna Burk and Allyn Snider,
Posing and Solving Problems with Story Boxes

It may seem strange to introduce a collection of mathematics lessons with two quotes about reading. Yet perhaps our experience with helping children learn to read can provide guidance for helping them learn mathematics. After all, if children enjoy wrestling with hard words, wouldn't they also find a suitably challenging mathematical problem satisfying? Don't children need to "sort mathematics out for themselves", just as they do with reading? We believe the answer to each question is "Yes".

These thoughts about mathematics and reading apply to other areas of learning as well. A child learns to talk by responding to the challenge of communicating. People engage in pastimes such as piano playing, racquetball, bridge or knitting even if they don't have the prerequisite skills to "master" those activities. In fact, if people waited until they mastered the fundamentals needed to be skillful at a game like golf before actually playing a round, most would never play at all.

The lessons in this volume of *Opening Eyes*, like those of previous volumes, reflect our belief that children can learn mathematics in the same way as they learn other things. Just as they learn to read by reading, and possibly golf by golfing, they can successfully learn mathematics by doing mathematics. This is best done by exploring suitably challenging mathematical problems and situations.

A problem-oriented context for learning mathematics promotes mathematical thinking, making and testing conjectures, deciding if an answer or procedure makes sense, and justifying solutions. As children investigate problems, there is opportunity for them to hone their critical thinking skills and become confident, independent thinkers. A natural setting is established within which children can construct mathematical understandings, develop number and operation sense, and become more proficient with basic mathematical skills.

Problem Solving and Problem Posing

Problem solving requires problems to be solved. For that reason, the lessons in this volume typically start with a problem for children to investigate. These problems have been chosen to promote a balance among the different branches of mathematics such as number, geometry, probability, measurement and statistics.

We have found that children are interested in many different kinds of problems. As long as they are appropriately challenging, problems can emerge from situations in school or home life (e.g., Lesson 10, School Events), from games (e.g., Lesson 6, Probability: Who Would You Rather Be?), or from purely mathematical settings (e.g., Lesson 7, Exploring Perimeters).

By its very nature, problem solving involves problems for which solutions are not immediately available. It necessarily involves something the problem-solver does not already know how to do. Thus, a problem to one person may be merely an exercise for someone else. A good problem is one that challenges children, lends itself to variety of strategies, and may very well have more than one solution.

As you will see, we are asking you to offer children problems that are generally a little too hard and to refrain from telling them solutions outright. As we did in Volumes 1 and 2, we propose giving children time to wrestle with the problems and develop their own solutions. Of course, this doesn't mean that you stand aside and leave the children to "sink or swim". Your role is critical as you facilitate the learning process through your encouragement, questioning and acceptance.

We encourage you to accept the times when children experience periods of anxiety or frustration. These periods are natural parts of problem solving and can become opportunities for new learning to occur. It's also likely that some children will not be able to solve every problem. Nonetheless, a surprising amount of growth will likely occur for everyone. In addition, with our help and trust, all children can develop constructive responses to being stuck and strengthen their ability to analyze problems.

A companion to problem solving is problem posing, and we urge your class to become a community of problem-posers. This can be fun and energizing, as children generally enjoy responding to one another's challenges. Be sure to join in the fun yourself! There will be times when you will be the problem-poser. There will also be times when you will have to make some decisions about a child's problem: Is the problem mathematically worth exploring? Is it challenging? Should it be presented to the whole class? If it is of interest only to the problem-poser, would it be okay for that person (or group) to explore on their own while the class does other things?

While there are no general rules for generating good problems, there are some helpful guidelines:

- Consider problems which involve mathematics from the next one or two grade levels. For example, division situations can generate good problems for children who don't already know division algorithms.
- Varying the parameters of a question often yields good "What if?" questions. Here is an example from Lesson 7, Exploring Perimeter: On a square *geoboard*, how many different *rectangles* can be formed which have a *perimeter* of *12*? One can change the italicized parts of this question and produce added problems for investigation. On (*a square geoboard, graph paper, isometric dot paper*), how many different (*rectangles, triangles, other shapes*) can be formed which have a (*perimeter, area*) of (*12, 17, 33½, 44.5*)?
- Reversing the question is often an effective way to generate good problems. As an example, instead of asking "What is 4 + 3?", try asking, "The answer is 7. What is the question?" See Lessons 17 and 47, Operations on Numbers, for additional examples of this strategy.
- Don't forget your own curiosity about mathematics and that of your children!

These and other problem-posing suggestions are included in a number of lessons. Remember, it's okay to ask a question that seems hard (or for which you may not know the answer). We encourage you to join your children in the search for solutions!

General Use of This Book

Opening Eyes was originally conceived as a complete mathematics curriculum for third grade. Most of the activities in Volumes 1 and 2 grew from Debby and Libby's experiences with their third graders. It quickly became apparent, however, that the program satisfied the needs of teachers in other settings. Many teachers in fourth grade, multi-aged, and even some second grade classrooms were able to successfully implement all or part of the program.

My children have already had a year of Opening Eyes. What can I do for them this year? This became a very important question for us as we thought about this volume. These new lessons, therefore, extend children's experiences with the mathematical models and concepts developed in Volumes 1 and 2, and contain frequent cross references to key lessons in those volumes.

We foresee teachers using this volume in two different classroom environments:

- The most natural use will likely be with children in fourth grade, or in multi-age classrooms, who have had previous *Opening Eyes'* experiences.
- However, you can use this book even if this is your children's first encounter with *Opening Eyes*. In large part, the lessons can stand on their own. Of course, the experiences are enriched if you are able to spend some introductory time on referenced activities from Volumes 1 and 2.

In either case, we feel you will find it helpful to have access to Volumes 1 and 2 and to become familiar with the philosophy and background information contained in the Teaching Reference Manual.

Planning for the Year

This book is organized somewhat differently than Volumes 1 and 2. Perhaps the most noticeable difference is that there are "only" 51 lessons! Also, the new lessons are reflective of previous Contact and Insight lessons, but haven't been identified as such.

The lessons themselves promote a broad experience with mathematics. Activities from important branches of mathematics, such as number, geometry, probability, measurement and statistics, are included. The lessons are organized in a cyclical manner so that concepts and problems from these branches are revisited periodically.

Perhaps you will feel most comfortable doing the lessons as they are sequenced, especially during your first year with the book. If so, don't be concerned if there isn't time for everything—there is nothing rigid about the sequence or number of lessons. We have, however, tried to maintain a balance among the different areas of mathematics and have indicated connections with the models developed in Volumes 1 and 2. We identify the few times when particular lessons are best done in a specific order. So we encourage you to make these lessons "your own" and to implement them in a way that best meets your classroom situation.

We don't suggest the number of days for lessons in this volume, as in Volumes 1 and 2. Because of the focus on problem solving, these lessons may take several days. Trust your professional instincts in determining how much time to spend on a particular lesson. Allow enough time for investigation and discussion to occur, but also seek an appropriate balance between exploring a given problem and moving ahead. If your children seem tired of a problem, go ahead and leave it. There is always the option of returning to it later. However, if the children are eager to continue an investigation and fruitful mathematics seems to be forthcoming, then follow the children's lead and provide time for further work.

Keep in mind that mathematical problems, like those in life, aren't always solved completely. It isn't necessary for your children to master or solve everything that occurs in a lesson. Don't feel that all problems have to be resolved by the end of class. There will always exist varying degrees of understanding and success. In fact, even for the most able children, it is mathematically healthy if they leave your class discussing some unresolved issue. We often send the children off with question to "think about tonight with your friends or family." Please remember, too, that specific concepts will come up for discussion several times in the book.

Preparing to Use These Lessons

Veteran teachers of *Opening Eyes* already know that preparation for these lessons is not only important but significantly different from "traditional" preparation. Perhaps the two most important things one can do to get ready are: (1) become thoroughly familiar with the Teaching Reference Manual and (2) investigate the problems within each lesson in advance. These two steps can provide guidance for making decisions about the lesson.

Here are some other questions to consider as you think about an upcoming lesson:

- What are the "big ideas" of the lesson? Think broadly here to include such things as problem solving, decision making, communication, generalizing, visual thinking, etc., as well as understanding of specific mathematical concepts and proficiency with particular mathematical skills.
- How can the initial problem of the lesson be introduced?
- Would it be helpful to revisit an activity from Volumes 1 or 2?
- How might the investigative parts of the lesson be organized? What questions might be needed to facilitate the investigation?
- How might the children share their thinking and results?
- What instruments would be useful for assessing the lesson and the children's work?

Devote some classroom time to help your children develop good listening and communication skills, particularly when the children are using *Opening Eyes* for the first time. It's important that children learn to cooperate and value one another's ideas, and that classroom procedures facilitate this. It may help, for example, to invite your children to discuss questions such as, "How can we tell if someone is listening (speaking) well?"

A lot to think about! But we're confident you'll see the rewards of advance planning in the enthusiasm and mathematical thinking your children display!

Implementing the Lessons

The spirit and philosophy of *Opening Eyes*, described in Chapter 1 of the Teaching Reference Manual, underlies this volume of lessons. The caring, loving atmosphere we all strive for in our classrooms is critical to implementing these lessons. By creating a supportive environment that is nonthreatening, a place where it is safe to risk, you will find your children more willingly and confidently participating in the activities.

Problems generally require time to solve and the problems within these lessons are no exception. Provide ample time for children to investigate, make decisions, hypothesize, communicate with one another, and explain their thinking. These kinds of activities are an integral part of problem solving and can help children recognize (and respect) different ways of thinking about mathematics. The problems and accompanying investigations also provide a context within which children can work together, construct mathematical understandings and develop mathematical skills.

Many lessons in this book are quite open in structure. This is a natural characteristic of any program which emphasizes problem solving. There may well be times when you or your children may feel uncomfortable. We, too, have experienced that feeling on many occasions. When those times occur, take a deep breath, be open to learning along with your children, and know that it's okay if you don't have all the answers at your fingertips. Be assured that your children will be gaining from the experience and that you will be helping them become better problem-solvers.

Communication and mathematical discussion are important in this program. Children need plenty of opportunities to discuss work, exchange ideas, explain thinking and present results. There are several ways to organize this part of the lessons, some of which are listed here.

- For large group discussions, you might invite volunteers to make presentations at the overhead or have small groups present chart paper displays of their work.
- At other times, it may be more natural to let the discussion occur entirely within small groups or between neighboring groups. We also like to have the children take "field trips" to nearby tables to view and discuss each other's work.
- For larger projects, you may wish to have the children generate "published" pieces, such as Big Books, poster displays, letters, etc.

Your role as facilitator of the activities is important. Several ideas for fulfilling this role are listed in the Teaching Reference Manual (pages 3–5). Additional ideas, particularly ones related to questioning strategies, have been included in many lessons.

Conduct the lessons in an informal, investigative manner. Many of the activities and concepts will be explored in greater depth later in *Visual Mathematics* (Linda Foreman and Al Bennett, MLC Publications), a program for grades 5–9. We have referenced those activities accordingly. The philosophy and instructional approach of *Visual Mathematics* are consistent with those of *Opening Eyes.*

Daily Computation and Number Study Chart

If you are concerned that computation will be lost in the *Opening Eyes* world of problem solving, you will be especially interested in Lessons B, Daily Computation, and Lesson C, Number Study Chart. These lessons provide ongoing opportunities for children to develop number sense and strengthen their use of various calculating options. These options include mental arithmetic, estimation, models, sketches, and calculators (see pages 45–46 in the Teaching Reference Manual).

The calculating experiences children receive in Lessons B and C complement those which occur naturally within the problem-solving activities of other lessons (e.g., Lesson 22, Primes and Composites, and Lesson 28, Football Scores).

Assessment

The learning that takes place in *Opening Eyes* is reflected daily within an atmosphere of mathematical exploration and communication. Assessing this learning, therefore, is an ongoing process using the techniques described in Chapter 12 of the Teaching Reference Manual.

We also recommend the National Council of Teachers of Mathematics booklet, *Mathematics Assessment: Myths, Models, Good Questions, and Practical Suggestions* (NCTM, 1991). The assessment instruments and practices discussed in that booklet are compatible with the goals of *Opening Eyes.*

NOTES

A bby Book

You Will Need

- unlined paper and scissors for each child

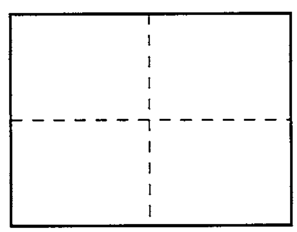
Your Lesson

The lessons of this program reflect our belief that communication is an integral part of learning mathematics. Children share their thinking and explain their procedures in small and large groups. They make displays of their work, capture their feelings in journals and create several published pieces. To facilitate the written communication that occurs, we often ask our children to record their thoughts and problem solutions in a "bby book". ("bby" book—named by and for De*bby* Head and Li*bby* Pollett.) The following activity can be used to create this book in a way that promotes spatial problem solving.

Have the children follow these six steps.

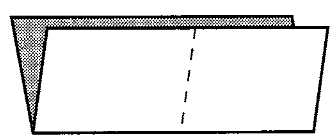
Step 1. Fold an 8½" by 11" rectangular sheet of paper (A) on both lines of symmetry.

A



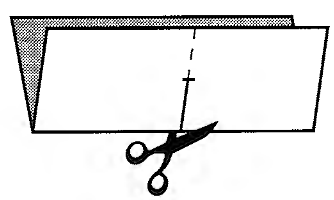
Step 2. Fold and hold.

A



Step 3. Estimate ½ of the way directly up from the fold on the crease and cut.

A

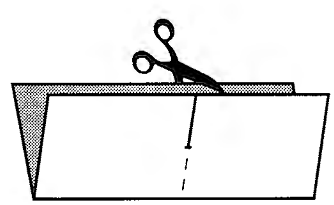


Step 4. Fold a second sheet of paper (B) as in Step 1.

Step 5. Fold and hold the second sheet as in Step 2.

Step 6. Estimate ½ of the way directly down from the open edges and cut along the crease.

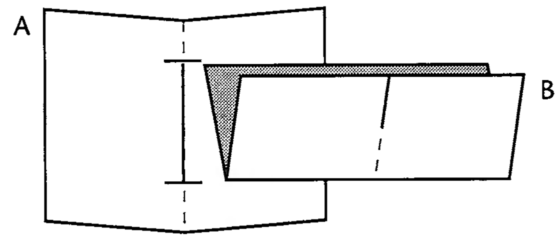
B



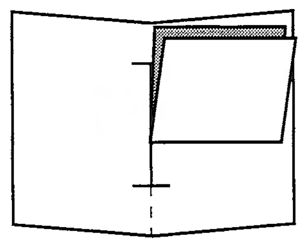
A bby Book (continued)

The children now have two folded papers with cuts in them (Steps 3 and 6). How can these papers be combined to make a book? Have the children address this problem together and then discuss their attempts. Here is one method.

Step 7. Insert B into the slit in A.



Step 8. Slide B up.



Step 9. Open B by folding down.

Explore these questions: How can more pages be added to the book? How many different strategies can you identify?

Teacher Tips

As the children proceed through Steps 1–6, occasionally ask them to pause and think about the folds that have been made. What would be seen if the papers were unfolded? Also, take advantage of any opportunity to review related geometric vocabulary.

Journal Entry

The act of writing can serve as a means of "cementing" one's understanding of a concept or a process. For that reason, we often ask our children to review an activity by writing in their journals.

A journal-writing assignment for this lesson could be: Write a letter to a friend who is absent, or who has moved away, that explains how you made your book. Feel free to include illustrations in your letter.

B Daily Computation

You Will Need

- materials for modeling numbers and performing calculations: base ten pieces, grid paper, calculators, etc.

Your Lesson

A major goal of *Opening Eyes to Mathematics* is to help children develop various approaches to understanding mathematical concepts and solving problems. This is particularly true when it comes to working with numbers and developing number sense. The numeration lessons of this program provide many opportunities for children to evaluate and use various calculating options. These options include mental arithmetic, estimation, use of manipulatives or sketches, paper-and-pencil procedures, calculators and computers.

Many lessons in this volume revisit the numeration concepts, models and calculating options explored in Volumes 1 and 2. These lessons are generally set in a problem-solving mode in which children work with numbers as part of larger mathematical investigations.

In addition to these lessons, we encourage you to offer short, daily experiences with numbers whereby children can apply and practice their skills. Conduct these experiences in an informal open way, much in the spirit of Calendar activities, perhaps devoting about 5 to 10 minutes at the beginning of the day to them. Occasionally, you may wish to expand them into a full lesson (see, for example, Lessons 17 and 47, Operations on Numbers, Parts 1 and 2, in this volume).

Organize these experiences in a variety of ways. For example, you might pose a particular computation problem suitable to your class, and invite the children to solve the problem in different ways. Methods of solution can be shared in a show-and-tell manner. Follow this by asking the children to share story problems that have solutions involving this computation.

Alternatively, you might present several problems of varying difficulty and ask the children to select ones that are challenging. Or else, pose a story, estimation or mental arithmetic problem to be solved (see Teacher Tips). Below are some examples of problems you might use. We have included other suggestions at the end of many lessons.

- June's family traveled 280 miles on the first day of their vacation. On the second day, they only traveled 135 miles. How much farther did they travel on the first day compared to the second day?
- OK, now—no pens or pencils! Try doing this one in your head: 145 + 88. Do the same for 14 × 7, for 1000 – 271, etc.

B Daily Computation (continued)

- I am thinking of a rectangle whose area is 200 square inches. One of the dimensions is less than 25 inches. What can you say about the other dimension?
- If you were going to paint your room, how would you estimate the amount of paint needed? Many people think it's better to overestimate the amount of paint. Do you agree? Why or why not?
- How much time do you allow for getting ready in the morning? How do you feel about your estimate?
- Ask others in your family or neighborhood when they have used estimates? Have they ever deliberately overestimated? or under estimated?

While you may wish to encourage children to explore different methods for solving a problem, this is a time to leave the initial choice to them. In fact, this is an opportunity for you to do some informal assessment of your children's preferences and areas of strength. Note the choices they make. Who prefers to do the problems "in their head"? Who likes to use manipulatives or sketches? Did any opt to estimate or use calculators? Who is struggling and may need encouragement to try a particular method?

Teacher Tips

The primary intent of these experiences is to help children "cement" the numerical processes they have explored previously and to practice their calculating skills. It generally is not a time to investigate new concepts.

Continue this lesson in an ongoing manner. It is a good one to work with daily, twice a week, every Wednesday or whatever meets your needs.

Vary the problems to "cover" all operations. Don't forget problems that have several addends, or ones that involve missing addends. You might also use ones with fractions, decimals, negative numbers or n as an unknown. Mix in some equivalent fraction questions or some averaging.

The National Council of Teachers of Mathematics 1986 Year book, *Estimation and Mental Computation*, contains additional information related to teaching estimation and mental arithmetic in the classroom.

Journal Entry

Consider using a bby book for journal responses. How would you complete the following sentences? Today's problem was _________. This is how I solved the problem: ________________. Then, after a sharing time: I also liked _______'s method, which was _____________.

C Number Study Chart

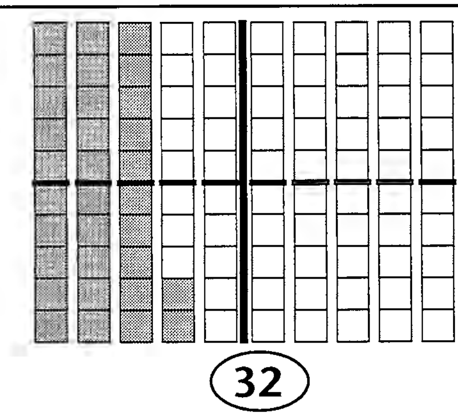
You Will Need

- a transparency of Blackline 14 (Ten Strip Board)
- base ten pieces for the overhead
- a copy of Blackline 14 attached to a piece of chart paper
- markers and tape
- (optional for each child) a copy of Blackline 114 (Small Ten Strip Board) shaded as described in the lesson

Your Lesson

As teachers, we recognize the importance of helping children develop a strong sense of numbers. People with good number sense feel confident in their understanding of numerical situations. They understand number relationships, make helpful estimates and judge the reasonableness of their answers. Good number sense can, in turn, help children acquire the computational facility and problem-solving skills needed to analyze numerical problems.

In this lesson, number study charts offer a visual context for promoting number sense. Charts are created by modeling a number on a ten strip board and then listing the mathematical relationships evident in that model.



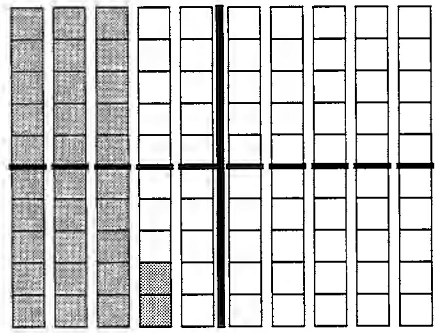
What mathematical relationships do you see in this picture of 32?

We recommend several variations of the lesson.

Version 1: Shade in a collection of units on a ten strip board, as illustrated above, and post the board on chart paper. If you wish, have a smaller version made for each child/table to examine during the lesson.

Ask the children to describe the mathematics they "see" when examining the shaded board. This can include different ways of totaling the number of shaded units, story problems, comparisons, etc. Be sure to give the children adequate time for verbalizing what they see and to demonstrate their observations in a show-and-tell manner. Record the children's observations on the chart paper, being sure to demonstrate a variety of recording options.

$$\begin{array}{r}10\\10\\10\\\underline{+\,2}\\32\end{array}\qquad\begin{array}{r}10\\10\\10\\1\\\underline{+\,1}\\32\end{array}\qquad\begin{array}{r}10\\\underline{\times\,3}\\30\\\underline{+\,2}\\32\end{array}$$



32

$$\begin{array}{r}100\\\underline{-\,32}\\68\end{array}\quad\begin{array}{r}90\\\underline{-\,32}\\58\end{array}\quad\begin{array}{r}80\\\underline{-\,32}\\48\end{array}\quad\begin{array}{r}70\\\underline{-\,32}\\38\end{array}\quad\begin{array}{r}60\\\underline{-\,32}\\28\end{array}\quad\begin{array}{r}50\\\underline{-\,32}\\18\end{array}\quad\begin{array}{r}40\\\underline{-\,32}\\8\end{array}\quad\begin{array}{r}35\\\underline{-\,32}\\3\end{array}$$

$32 = 30 + 2$
$32 = \boxed{2} + 30$
$32 < 40$
$32 < 50$
$32 < 100$
$40 > 32$
$50 > 32$
$100 > 32$

$10\overline{)32}$ = 3 r 2

$2\overline{)32}$ = 16

What mathematical relationships do you see in this picture of 32?

32 rounds off to 30
32% shaded
68% not shaded
32% < 68%
68% > 32%
32% + 68% = 100%

$32 > \frac{1}{4}$ of 100
$32 < \frac{1}{2}$ of 100
base 5: 1 mat, 1 strip, 2 units
or: 6 strips and 2 units
or: 32 units

$5 \cdot 7 \neq 32$

even #

$\frac{32}{100}$ shaded
$\frac{68}{100}$ not shaded
$\frac{32}{100} + \frac{68}{100} = \frac{100}{100} = 1$
$1 - \frac{32}{100} = \frac{68}{100}$

3 dimes and 2 pennies
6 nickels and 2 pennies

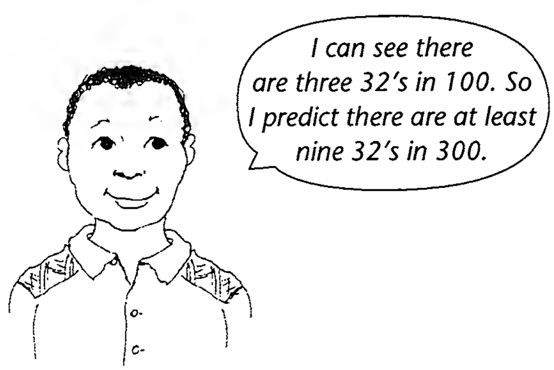
$(3 \times 5) + (3 \times 5) + 2 = 32$

$(2 \cdot 4) + (8 \cdot 3) = 32$
$32 = (2 \times 10) + (2 \times 5) + (2 \times 1) = 20 + 12$
$40 - 8 = 32 \quad 50 - 18 = 32 \quad 100 - 68 = 32$
$2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 = 32 \quad 16(2) = 32$

1, 2, 4, 8, 16 and 32 are factors of 32
$4(8) = 32 = 8 \times 4 = 8 \cdot 4 = 4 \cdot 8 = 8(4)$

Darrell wanted a package of gum. The price was 32¢ per package. He had \$1.00 to spend. Does he have enough money? What would his change be from \$1.00? Could he buy 2 packages? 3 packages? 4 packages? Why or why not?

The completed chart above reflects the spirit we attach to this activity. We encourage you to conduct the lesson in a very open way. Ask your children to look for different ways of describing relationships and to use visual thinking to predict beyond the chart.

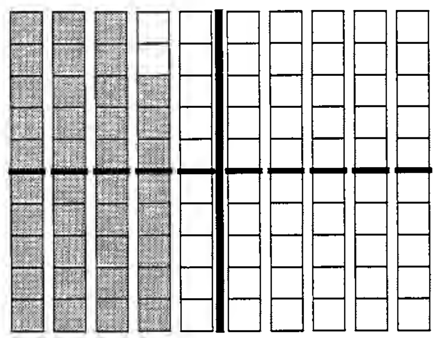


C Number Study Chart (continued)

This activity could comprise a 30-minute lesson for one or more days. Alternatively, you might choose to have the same number posted for several days and spend 10 minutes or so each day discussing added observations.

Version 2: Shade in a collection of units on a ten strip board and post the board on chart paper, as in Version 1. This time, however, record the first observation a child gives you and then invite suggestions for recording the same observation in other ways.

For example, suppose a child notices that 38 is "3 tens and 8 ones." Ask the class to brainstorm different ways of recording that observation.



38

38 = 10 + 10 + 10 + 8

(10)(3) + 8 = 38

3 dimes and 8 pennies

$38 \div 10 = 3$ r 8

3 strips and 8 units (base 10)

3 x 10 + 8 = 38

8 + 30 = 38

38 = 8 + (10 x 3)

$38 \div 10 = 3\frac{8}{10}$

38 = 4(10) – 2

$$\begin{array}{r} 10 \\ \times\ 3 \\ \hline 30 \\ +\ 8 \\ \hline 38 \end{array} \qquad \begin{array}{r} 3 \\ \times\ 10 \\ \hline 30 \\ +\ 8 \\ \hline 38 \end{array}$$

38 – 8 = 30

8 = 38 – 30

Version 3: Conduct Version 1 as a small group activity. Have each group of children shade a number of units (of their choice) on a ten strip board and record as many mathematical observations as they can about that number.

Version 4: (Homework) After your children have had several opportunities to examine, discuss and create number study charts in class, the children can generate some charts of their own with the involvement of family members or friends.

Teacher Tips

Be sure that each number study chart includes at least one story problem related to the shaded number.

Some teachers like to have their children make their number study charts into a bby book. The shaded ten strip board can be part of a cover page and mathematical observations can be recorded on inside pages.

1 Pattern Possibilities

You Will Need

- Chapter 3, Patterns
- square tile
- Blacklines 117–125 (Pattern Possibilities), note materials indicated

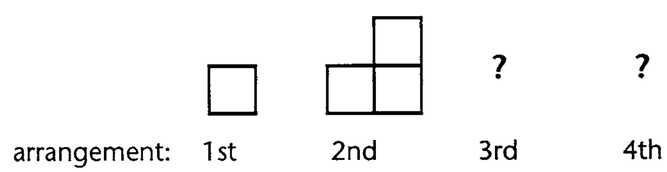
Your Lesson

Patterns are found throughout mathematics and can be used to help children understand concepts and solve problems. It is important, therefore, to provide opportunities for your children to recognize, create and extend patterns. This was done in Contact Lesson 16, Apples—Patterns, Volume 1, and Contact Lesson 95, Toothpick Patterns, Volume 2. It also happened whenever children experienced repeating (e.g., ABCABC...) or growing (e.g., 5, 10, 15...) patterns.

This lesson continues these types of activities, inviting children to generate different patterns from a given situation. In this lesson, your children will draw upon their abilities to think visually and to examine a mathematical setting in different ways.

Activity 1

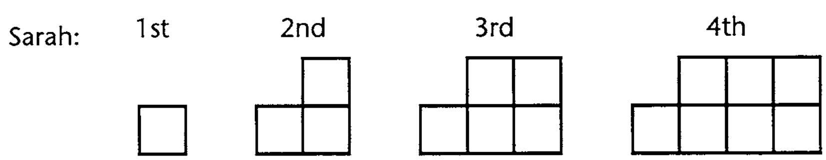
Build the first two tile arrangements shown here at the overhead and have each child do the same at their seats.



Ask the children to build the 3rd and 4th arrangements so all the arrangements form a pattern that can be extended.

We encourage you, in fact, to perform this task yourself before reading on. Did you notice anything in the first two arrangements that prompted your response? How could you describe your pattern so as to predict what the 5th arrangement (or the 20th or 100th ones) would look like?

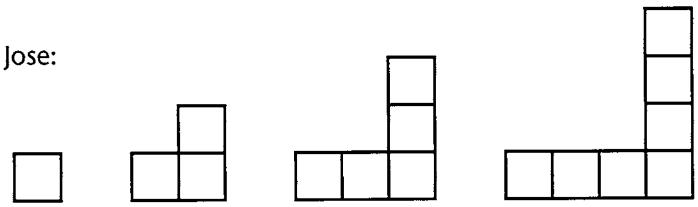
Invite volunteers to show their arrangements and describe their patterns. As you can see from the following sample of classroom responses, there are several possibilities for the 3rd and 4th arrangements. Perhaps you have added a new one to the list!



"The 2nd had 2 more tile than the 1st. So I made the 3rd have 2 more than the 2nd and the 4th have 2 more than the 3rd."

1st 2nd 3rd 4th

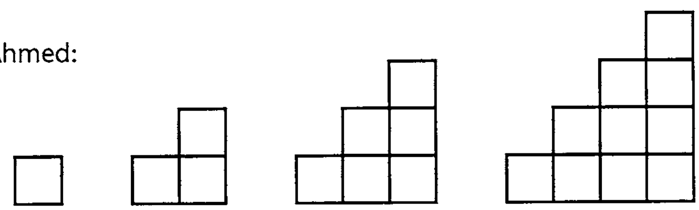
Jose:



"I made the 2nd by putting a tile on top of the 1st and 1 to the left. I did the same for the others—put a tile on top and 1 to the left each time. They sort of look like backward L's."

1st 2nd 3rd 4th

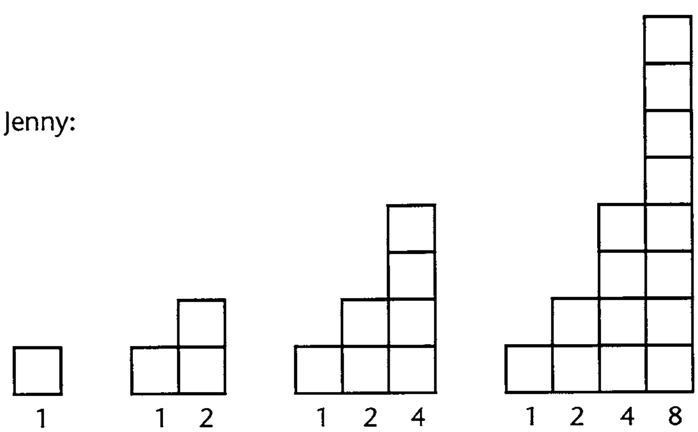
Ahmed:



"The 1st has 1 stack with 1 tile. The 2nd has a stack of 1 and a stack of 2. So I made the 3rd have stacks of 1, 2 and 3 tile. The 4th arrangement has stacks of 1, 2, 3 and 4 tile."

1st 2nd 3rd 4th

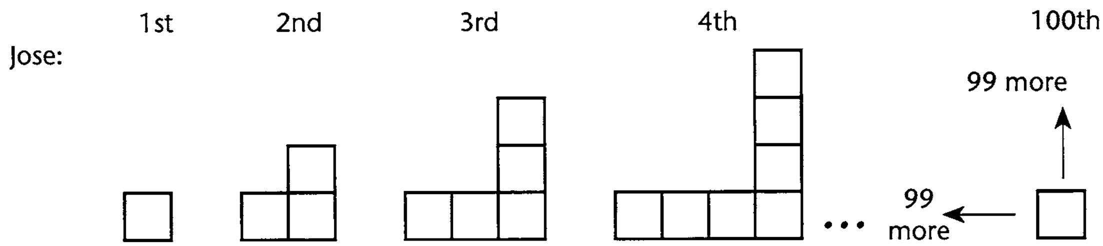
Jenny:



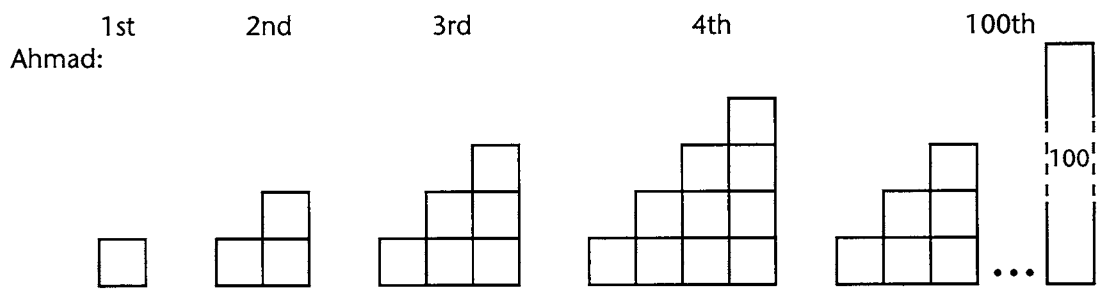
"I'd say each arrangement has one more stack than the one before it. The number of tile in the stacks keeps doubling."

Notice how "clues" from the first two arrangements can be used to establish a pattern. We often invite our children to list the clues they see and, in turn, use these to generate several candidates for the next arrangements. They seem to enjoy the challenge of creating these candidates and explaining how each illustrates a pattern that can be extended.

Each of the patterns suggested by the children can be examined further and perhaps generalized. While this is the main objective of Lesson 26, Volume 3, Pattern Generalizations, you might do this now with a few of the responses. What would further arrangements look like if a particular pattern is continued?



"To make the 100th arrangement, I'd start with the 1st arrangement and add 99 tile to the top and 99 to the left. I'd get a big backward L."



"My 100th arrangement would have 100 stacks. There would be 1 tile in the 1st stack, 2 tile in the 2nd stack, 3 tile in the 3rd, and so on. The last stack would have 100 tile."

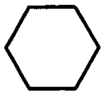
Activity 2

Repeat Activity 1 by examining the start of another sequence in the same way. Additional starters are shown on Blacklines 117–125, along with room for children to record their answers. At this time, we generally choose one of these; such as the pattern block starter on Blackline 119 (shown on next page). We then examine other starters in the days to come (reminders for doing this appear at the end of selected lessons). We also encourage you and your children to generate starters of your own.

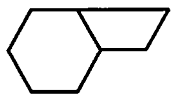
The general procedure is the same for whatever starters you select: Present the children with the first few arrangements in a sequence and invite them to suggest what arrangements can come next if the sequence follows a pattern. Ask the children to find several possible extensions for each sequence and to justify their responses in terms of the patterns they have in mind. Results may then be shared in the usual ways (at the overhead, within small groups, published pieces, etc.).

Starter | **Possible Responses**

arrangement: 1st

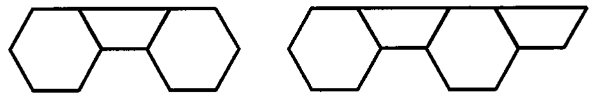


2nd



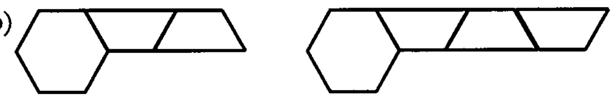
3rd | 4th

a)



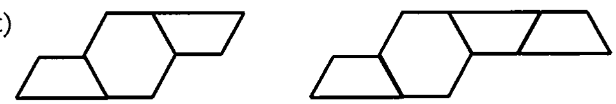
Continue by alternately adding hexagons and trapezoids.

b)



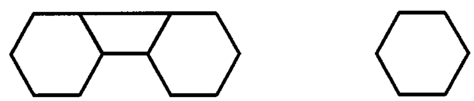
Add one more trapezoid each time.

c)



Add a trapezoid each time, but alternate where it is placed, as shown.

d)



Continue with an ABC pattern using the first 3 arrangements.

Teacher Tips

We often begin this lesson by forming "possible people patterns". With the children in a line, have the first person stand and the second one sit. Now, what standing (S) and sitting (s) patterns can be illustrated by the other children? Here are some possibilities:

S-s-S-s (children alternate between standing and sitting)

S-s-S-S-s (then 3 stand, 1 sits, 4 stand, 1 sits, etc.)

S-s-S-S (repeating in like manner to form an ABAA pattern)

S-s-Jump-S-s-Jump (and so on, to form an ABC pattern)

Choose one of the suggestions and ask each person in the line to determine what they will be doing in the pattern. Each child indicates when they know what to do (perhaps by touching their nose). When all are ready, they count to three and form the pattern.

Was the pattern extended successfully? Suppose the class had 100 children. What would the 50th person in line be doing? How about the 73rd person? or the 100th one?

An interesting variation of this activity is to have the children initially form a circle. What happens then?

Published Pieces

Here are two possibilities:

Have each child or group choose a starter for a sequence and create a chart of clues that may be used to generate patterns. Have them also show possibilities for extending the sequence and justify their responses in terms of their patterns.

Ask each child to create a book using a starter sequence and a pattern extension of their choice. Once again, their reasoning should be noted. How would their friends extend this starter? Perhaps each child can collect of sample of responses from others and include these responses in their book.

Homework

Brainstorm with your children and create a class chart of possible starter sequences. Ask each child to select a few of these starters and conduct a survey of how people at home might extend them.

Journal Entry

Describe an idea that you especially liked about today's pattern. What pleased you about that idea?

2 Birthmonth Buddies

You Will Need

- materials for solving problems and displaying results
- copies of Blackline 126 (Birthmonth Buddies) and Blackline 127 (Birthday Money) for each child

Your Lesson

Problem solving is a very important part of *Opening Eyes to Mathematics*. The investigations which begin most lessons are opportunities for children to construct (or review) particular mathematical concepts and, at the same time, hone their problem-solving skills (see, for example, Volume 3, Lesson 7, Exploring Perimeter). Sometimes, however, we pose situational problems specifically for the purpose of offering children additional problem-solving experiences.

The problems of this lesson, found on Blacklines 126 and 127, fall into the second category mentioned above. Problems of this sort generally require little introduction. Simply pose them and provide time for your children to explore them and to share their thinking.

Opening Eyes to Mathematics **Birthmonth Buddies** **Blackline–126**

Part 1
If all our names are placed in a jar and randomly drawn out one at a time, how many names must be drawn to guarantee that 2 names will be of people with the same birthmonth?

Part 2
1. If all our names are placed in a jar and randomly drawn out one at a time, how many names must be drawn to guarantee (3, 4, 5, etc.) people will have the same birthmonth?

2. How many children would have to be in a class to guarantee that 2 children will have the same birthweek?

3. Jim has socks of 5 different colors mixed up in his drawer. If he starts randomly pulling socks from the drawer, one at a time, how many draws must he make in order to guarantee getting a color match?

2 Birthmonth Buddies (continued)

Ask the children to work with a partner on the problem of Part 1, Blackline 126. Partners will need time to arrive at an understanding of the problem, principally because of the word *guarantee* in the question. They may be tempted to report an answer of "two", since the first 2 names selected may have the same birthmonth just by luck of the draw. However, that event is not guaranteed. It will be interesting to see how your children account for the randomness of the draws.

Continue the lesson by asking the children to work on Part 2 of Blackline 126. Notice how the setting or the parameters of the original problem have been changed to generate new questions. *Varying the problem* is often a very good strategy for problem posing.

The other "birthday" problem for today (Blackline 127) lends itself to several strategies and calls for a decision involving money.

Opening Eyes to Mathematics	Birthday Money	Blackline–127

Amy's birthday is March 15, and Aunt Betsy offered her a choice of presents. Amy could receive $10.00 on her birthday or get a quarter a day every day, starting on her birthday and continuing through July 4. Which choice should Amy make and why?

Can you support your answer with a model or picture? Also, write a few sentences to Amy that justifies your advice.

Teacher Tips

Today's activities are especially good to use at the beginning of the year when your children are just getting to know each other.

Here is some background information for you. In the NCTM 1991 Yearbook, *Discrete Mathematics Across The Curriculum: K-12* (pages 55–58) these problems are discussed in terms of a useful counting technique entitled "The Pigeonhole Principle". This principle derives its name from thinking about placing pigeons in a coop. If there are more pigeons than there are pigeonholes, then for sure there will be at least 1 hole with 2 pigeons. In the birthmonth problem, the months of the year can be modeled by an empty egg carton (representing 12 "pigeonholes"). The slips of papers with names are the "pigeons". How many draws will have to be made to guarantee at least 1 hole will have 2 pigeons?

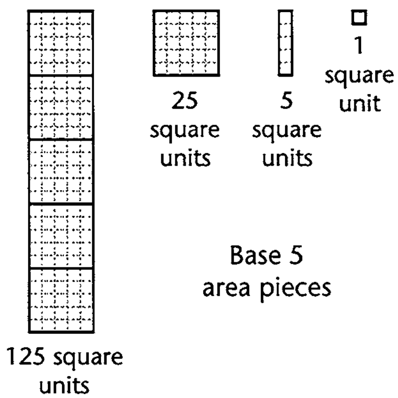
3 *Exploring Base Four, Part I*

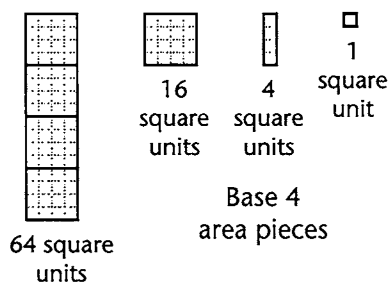
You Will Need

- Chapter 4, Numeration: Place Value

See further materials listed with each activity.

Your Lesson





"My children have already experienced the base five and base ten place value lessons in Volume 1. Do you have any suggestions for additional experiences with place value that I might provide this year?"

This is a question often asked of us, particularly as multi-age classrooms become more and more prevalent in school districts. Teachers have found it helpful to revisit some of the Volume 1 lessons, conducting those activities again at perhaps a different pace or with different examples. Others have found a need to change the setting, perhaps using a base other than five. For those, the activities of Volume 1 can be used again, only this time with, say, base four pieces, or the following activities below may be used.

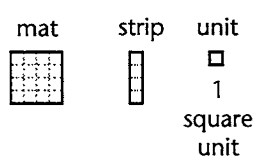
These activities require the use of base four pieces. These pieces are like base five pieces except the area of each piece is four times that of the next smaller one. Also, beginning with the unit, the different pieces in both sets form an alternating "square, rectangle (non-square)" pattern.

Activity 1

- base four area pieces for the overhead

for each child

- square tile
- scissors and tape
- copies of Blackline 73 (2-cm Grid Paper)
- an envelope for storing base four pieces



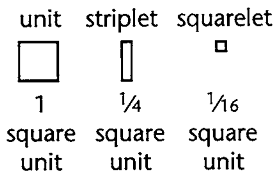
Review the patterns and structure of the base five (or base ten) area pieces that the children have used before. In particular, discuss the growing and visual patterns described above. Then, at the overhead, display the base four pieces shown to the left in order of decreasing size.

Have the children use square tile to build the same collection of pieces at their desks, and invite the class to make observations about these pieces. Record these observations on chart paper and discuss the following questions:

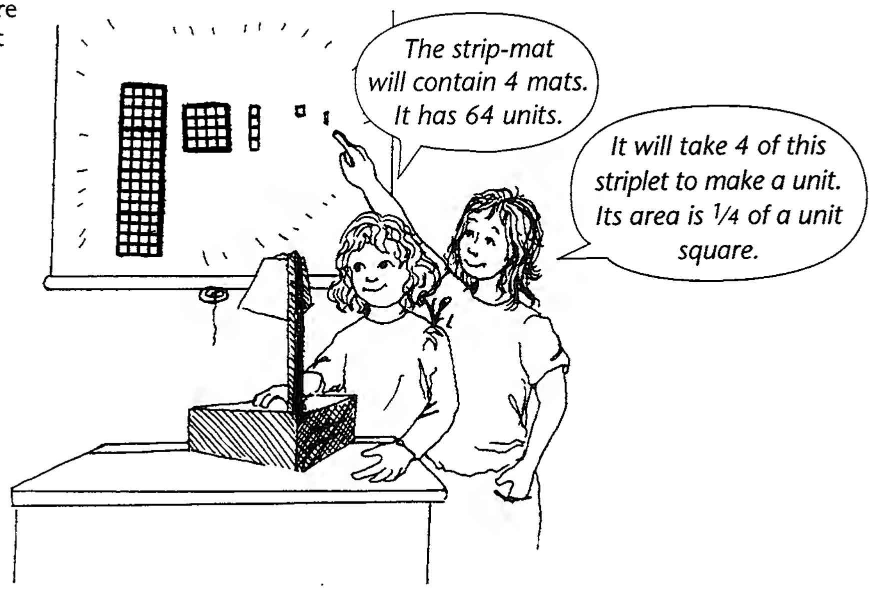
What patterns are present in the base four area pieces?

How are the base five and base four area pieces similar? How are they different?

Imagine bigger area pieces. How could they be described?



Imagine the next smaller area pieces to the right of the unit. How could they be described? (The next two pieces, referred to as striplets and squarelets, are pictured to the left.)



Ask each child to create a set of base four area pieces that includes 2 strip-mats, 8–10 mats, 10–12 strips, 12–15 units, 8 striplets and 8 squarelets. These pieces can be cut (and pasted) from Blackline 73 (2-cm graph paper) and stored in envelopes. They are used in Activities 2 and 3 below and need to be saved for the activities of Lessons 4 (Exploring Base Four, Part 2) and 30 (Shifting the Unit, Part 2) of this volume.

Activity 2

- yarn necklaces
- large 0-1-2-3-1-2 number cube
- base four area pieces (made in Activity 1) for each child
- (optional) 0-1-2-3-1-2 wooden number cubes for individual play

Play the following version of Up and Back Again (originally described in Volume 1, Insight Lesson 11).

Use yarn necklaces to divide the class into two teams. Alternately roll a 0-1-2-3-1-2 number cube for each team to determine the number of units to add to the team's collection. (Each team begins with 0 units.)

As this is done, the team creates the minimal collection of base four area pieces by trading.

The object of the game is for each team to reach an exact total of 64 units (1 strip-mat) and reverse to reach an exact total of 0 units. The first team to do so wins.

Note: As the game proceeds, periodically stop play and have the teams describe their current collection and discuss the number of units left to roll.

The children can also play this game individually against one another, in which case each pair of children will need number cubes.

Activity 3

- Number Card packets (beige cards only)
- transparency of Blackline 96 (More and Less Spinners) for each team of two children (Use paper clip or bobby pin for the pointer.)
- base four area pieces (made in Activity 1) for each child

Play this version of Mine or Yours (originally described in Volume 1, Insight Lesson 7).

Divide the class into teams of two and distribute number cards among the teams. At the start of each round, each player on a team draws a number card and forms the minimal collection for that number. Players then count the number of base four area pieces required to make their minimal collections. Spin the More and Less spinner to determine whether the player with more (less) pieces wins the round.

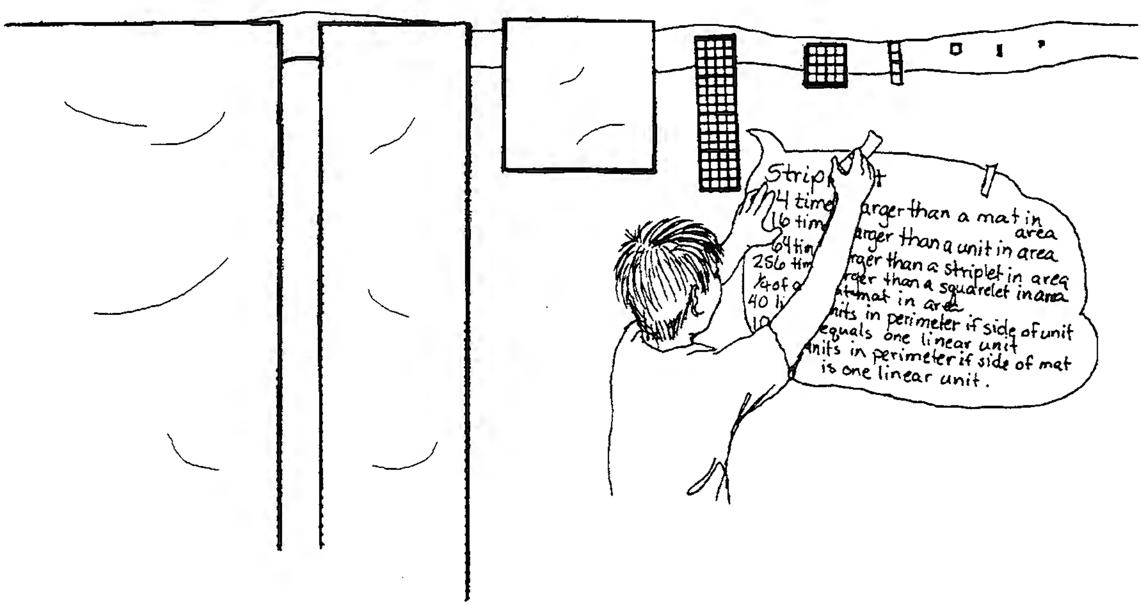
Teacher Tip: The cards with higher numbers may be removed if you feel the difficulty level needs to be lowered.

Activity 4

- scissors, glue, tape available
- large paper for display
- unlined newsprint
- calculators
- copies of Blackline 25 (1-cm Grid Paper) for each child

Have each child set before themself a mat, strip and unit (cut from Blackline 25) in order of decreasing size from left to right. Ask the children to imagine their collections being extended to the right and to the left. Have them cut out the next four pieces to the left of the mat and the next two smaller pieces to the right of the unit. The pieces larger than a strip-mat can be cut from unlined newsprint. In so doing, your children will need to be mindful of how the dimensions and areas of these pieces grow. In fact, we often ask the children to calculate the dimensions and areas of the larger pieces.

When ready, have the children work together to create group displays of the different pieces in their collections. Invite them to make observations about each piece and to visualize further extensions of the collections. (We like to have the children attach their pieces to adding machine tape and describe them as illustrated below.) What would they predict for the next larger/smaller pieces?



Journal Entry

You have made nine different base four pieces. Compose a letter describing how you made these pieces and why you feel they are appropriate. Feel free to illustrate your letter.

Keep a copy of the letter for your working portfolio. You may also wish to mail a copy to an author at one of these addresses:

Debby Head, 919 Circle Drive, Shelbyville, KY 40065

Libby Pollett, 1410 St. Andrews Drive, Shelbyville, KY 40065.

4 Exploring Base Four, Part II

You Will Need

- Chapter 4, Numeration: Place Value
- calculators
- base four area pieces for the overhead
- Number Cards (optional)

  for each child
- base four area pieces made during Activity 1, Lesson 3, Volume 3
- copies of Blacklines 128–130 (Exploring Base Four)

Your Lesson

The activities of this lesson use base four area pieces to revisit place value concepts such as positional notation, minimal collections, larger and smaller places and regrouping. We are assuming that all the different base four pieces made in Activity 1, Lesson 3, are available for each activity. By choosing from all these pieces, your children can gain broader experiences with place value. As you conduct the activities, look particularly for the spots where the pieces smaller than a unit are used in a manner that parallels decimals in base ten.

Activity 1

How can a total of 24 units of area be shown with the base four pieces? Divide the class into small groups and ask the groups to find different ways of answering this question. Have the groups prepare a report of their results that includes comments about the following questions:

How many different ways are there for showing 24 units of area? How can you be sure you have them all? Explain.

What is the minimal collection for 24 units of area?

Note: The first chart below shows all the possible representations of 24 units of area if one doesn't use any piece smaller than a unit. This may be the most common set of answers reported by your children. The minimal collection is indicated with an asterisk (*).

Mats (16)	Strips (4)	Units (1)
0	0	24
0	1	20
0	2	16
0	3	12
0	4	8
0	5	4
0	6	0
1	0	8
1	1	4
1	2	0*

Some groups may report answers that use pieces smaller than a unit (or you may wish to raise the possibility). In that case, several other collections are possible, some of which are reported here.

Mats (16)	Strips (4)	Units (1)	Striplets ($\frac{1}{4}$)	Squarelets ($\frac{1}{16}$)
1	1	3	4	
1	0	7	4	
1	1	2	8	
1	1	2	4	16
0	0	0	96	

Continue this activity by forming collections for other numbers of units. (Number Cards can be a good way to generate examples.) You might also consider reversing the question. That is, begin with a collection of pieces (it needn't be minimal) and invite your children to determine the total number of units of area represented by that collection.

TEACHER *Set out this collection of base four area pieces: 2 mats, 3 units, and 2 striplets. How many total units of area are represented in this collection?*

SUSAN *The 2 mats are 32 units. Three more units make 35. I know each striplet is $\frac{1}{4}$ of a unit. So the total is $35\frac{2}{4}$ units.*

JAMAAL *I agree with Susan. But I also see the 2 striplets make up $\frac{1}{2}$ of the unit square. You could write $35\frac{1}{2}$ units.*

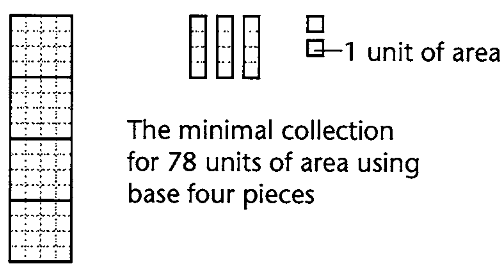
Activity 2

Ask your children to:

Set out the minimal collection for 78 units of area using base four pieces.

Show-and-tell their methods for doing this.

Devise a way of reporting the pieces in this collection to a friend.



The minimal collection for 78 units of area using base four pieces

JERIMIAH *I'd send my friend a picture of a strip-mat, 3 strips and 2 units. A strip-mat is 64 units; the 3 strips and 2 units make 14 more units. That's 78 altogether.*

MARIA *I'd tell my friend all about the base four pieces and their names. Then I'd say, "Put out a strip-mat, 3 strips and 2 units. That's the minimal collection for 78."*

LI *Could I just write "1032" to my friend and tell him to use base four area pieces? Perhaps it works like base ten.*

As suggested by Li's comment, this may be an appropriate time to use positional notation when describing minimal collections of base four area pieces. If so, one can use it just as one does in base ten. That is, the digit in each position reports the number of times a particular piece is used in a minimal collection. Using Li's idea, it is customary to report the base four minimal collection for 78 with notation such as 1032 (base four) or 1032_{four}. Your children may suggest other ways. In any case, it is important to indicate the base as 1032 alone typically refers to the base ten number one thousand thirty-two. You can decide what role exists for this notation in your class.

Provide further independent practice by having the children repeat this activity for other areas. Examples are given below, each with their corresponding minimal collection written as base four numerals.

Area	Minimal Collection
136	2 strip-mats, 0 mats, 2 strips, 0 units; or 2020_{four}
88	1 strip-mat, 1 mat, 2 strips, 0 units; or 1120_{four}
103	1213_{four}
$6\frac{1}{2}$	$12{\odot}2_{four}$ (1 strip, 2 units, 2 striplets)
$17\frac{3}{8}$	$101{\odot}12_{four}$

Notice the last two examples in the above chart require the use of pieces smaller than a unit. The corresponding base four notation, therefore, requires the analog of a "decimal point"— a punctuation mark to identify the units place. In this case, we've elected to use a $\odot$ (reserving the period for base ten decimals). Or, let the children suggest a symbol of their own.

For further independent practice have your children work the problems on Blacklines 128 and 129 (shown on following page). Conduct a show-and-tell discussion about these problems at a subsequent class session.

Set out a collection for 2 mats, 3 strips and 1 unit with base four area pieces.	What is the total area in your collection? Describe the total area in different ways. Blackline 128	Describe your collection ... to the closest mat. ... to the closest strip. ... to the closest strip-mat.

Opening Eyes to Mathematics   Exploring Base Four

Using your base four area pieces, build a collection of your choice. Sketch a picture of it below.	Your collection has now tripled in area. Sketch the minimal collection for your new total. Blackline 129	The area of your original collection has been cut in half. Sketch your new collection below.

Opening Eyes to Mathematics   Exploring Base Four

Teacher Tips

The goals of these excursions into base four are to revisit place value concepts in another visual context and to provide problem-solving experiences that can contribute to the development of number sense. Conduct the activities in an informal manner; your children need not become base four "experts". This will, therefore, call for some decisions on your part. Should base four notation be used? Should Activity 2 focus only on finding minimal collections of whole numbers? How much time should be spent on these activities? Only you can answer these questions, knowing your children as you do. We are always surprised by what children can do, so we encourage you to try the activities.

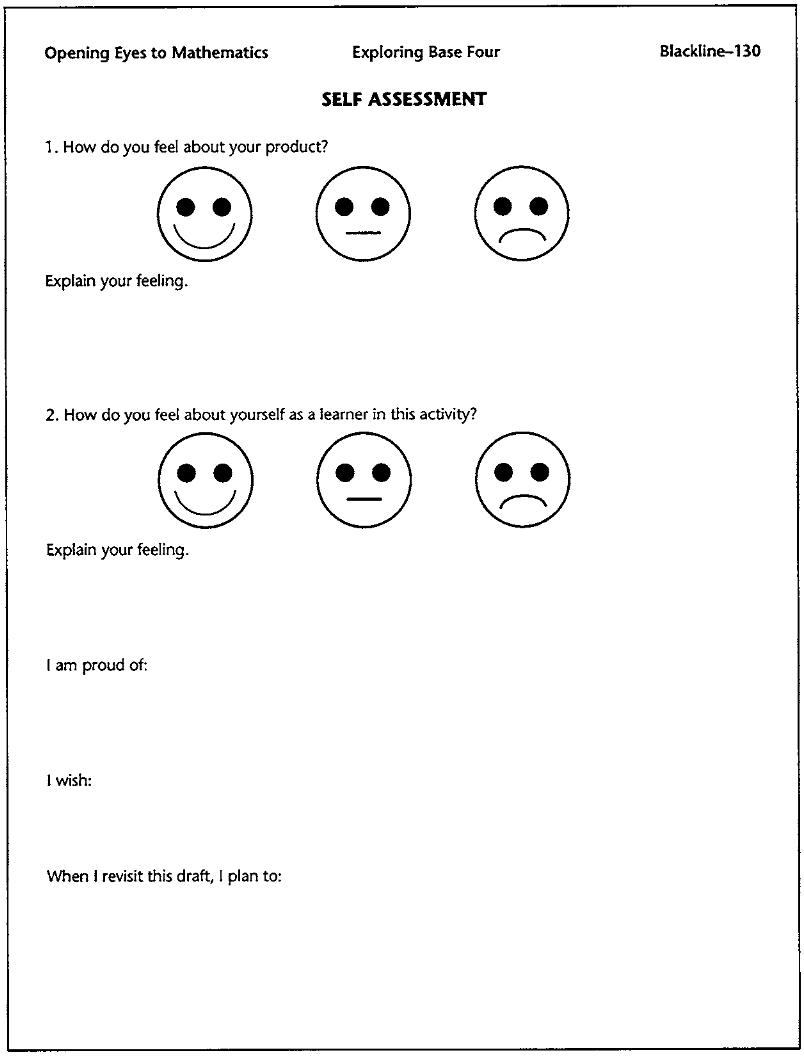
4 Exploring Base Four, Part II (continued)

Journal Entry

Suppose you have this collection of base area four pieces: 2 mats and 1 unit (i.e., 201_{four}). What is the difference in units between this collection and a strip-mat (1000_{four})? Explain your thinking, using sketches as needed.

Assessment

Ask your children to assess their own work and feelings related to these base four activities. Blackline 130 may be used for this purpose.


Opening Eyes to Mathematics — Exploring Base Four — Blackline–130

SELF ASSESSMENT

1. How do you feel about your product?

Explain your feeling.

2. How do you feel about yourself as a learner in this activity?

Explain your feeling.

I am proud of:

I wish:

When I revisit this draft, I plan to:

Pattern Reminder

In order to keep the theme of Lesson 1, Pattern Possibilities, alive, you might explore a sequence starter suggested by your children or one of the remaining starters on Blacklines 117–125.

5 Fractions—Make A Whole

You Will Need

- Chapter 8, Fractions
- people fraction bars for Activity 1 (Making instructions in Teaching Reference Manual, page 133)
- a set of Fraction Bars for every two children
- transparencies of Blackline 61 (Circle) and Blackline 138 (Make A Whole Recording Sheet)
- overhead circular fractional parts (cut from transparencies of Blacklines 131–137, Circular Fractional Parts)

for each child

- Blacklines 131–137 (Circular Fractional Parts)
- Blackline 61 (Circle)
- Blackline 138 (Make a Whole Recording Sheet)
- Blackline 139 (Make A Whole Activity Sheet)
- chart paper, glue, scissors and markers
- group fraction bars (optional) (Making instructions in Teaching Reference Manual, pages 131–132)

Your Lesson

In working with fractions, it is important for children to establish a sense for how fractions are related to one another. Also important is being aware of the whole to which the fractions in a discussion refer. This lesson seeks to reinforce both of these goals. The title, "Make A Whole", refers to the main activity of the lesson, in which children explore different ways to combine fractions so as to make a whole unit.

Activity 1

Begin by revisiting Fraction Bar War (Volume 2, Contact Lessons 148–150). Play this game as a means of reminding children how fraction bars model important concepts such as equivalence between fractions.

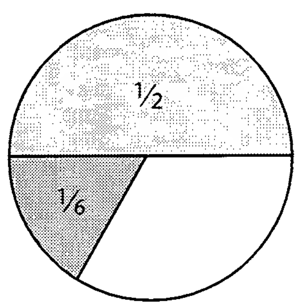
Now ask the class to demonstrate ways of putting various people fraction bars together to make a whole bar. Allow enough time for several solutions to be shared. (See Volume 1, Contact Lesson 31, People Fraction Bars and Contact Lesson 32, Diagramming People Fraction Bars.)

(Alternatively, the above activity can take place in small groups using group fraction bars. See Volume 1, Contact Lessons 33–35, for information on using group fraction bars.)

Activity 2

Distribute copies of Blacklines 61 and 131–137. Ask the children to examine these blacklines and describe some of the fractions that are modeled. Once the children are familiar with the blacklines, display a transparency of Blackline 61 on the overhead. Using transparent circular parts, begin to fill in one of the circles as shown here:

Diagram 1



Invite suggestions for adding pieces to complete the whole.

ALYSSA *I think two more 1/6 pieces will fit.*

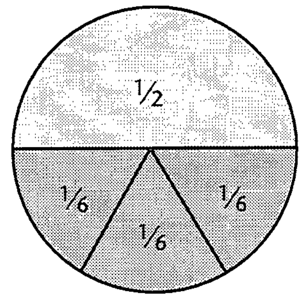
TEACHER *What makes you think that?*

ALYSSA *Well, 1/2 and 1/6 are already there. I remember that 3/6 also shows 1/2. So two more 1/6's ought to do it.*

(Alyssa now tests her hypothesis at the overhead.)

TEACHER *Very nice. Let's record Alyssa's solution on chart paper and describe it with fractions.*

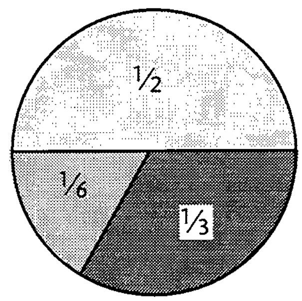
$\frac{1}{2} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$



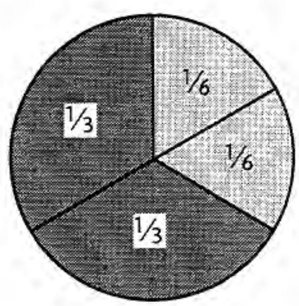
TEACHER *(showing Diagram 1 again) Can anyone suggest another possibility?*

CARLO *Look! Alyssa's $\frac{2}{6}$ can be replaced by a $\frac{1}{3}$ piece. That should work.*

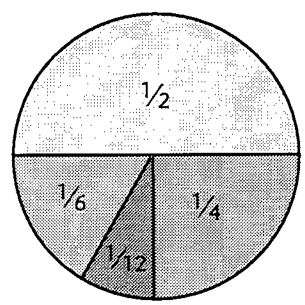
$\frac{1}{2} + \frac{1}{6} + \frac{1}{3} = 1$



Continue the discussion, inviting children to predict possibilities and then test their predictions. Keep a record of everything that is explored, including suggestions that may not make a whole exactly. This is an opportunity to also visualize sums of fractions that may not total 1 exactly. As indicated in the above dialogue, encourage children to explain their thinking and demonstrate any relationships among fractions they observe.

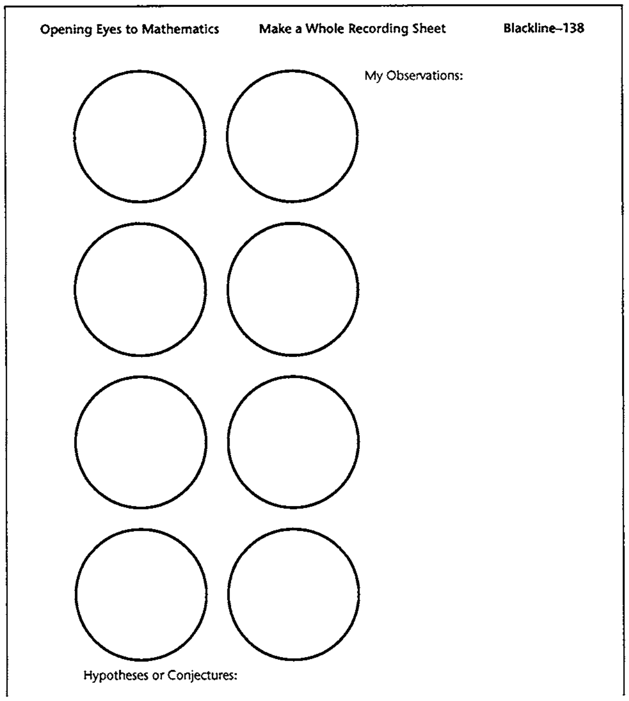


$\frac{1}{3} + \frac{1}{6} + \frac{1}{6} + \frac{1}{3} = 1$



$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{4} = 1$

Now turn the children loose in groups to find other ways to make a whole! They may use the circles on Blackline 61 as the "wholes" and search for combinations of circular fractional parts that will work. Ask the groups to follow the "predict, test, record" procedure described above. Solutions may be recorded on Blackline 138 (see following page.


Opening Eyes to Mathematics Make a Whole Recording Sheet Blackline–138

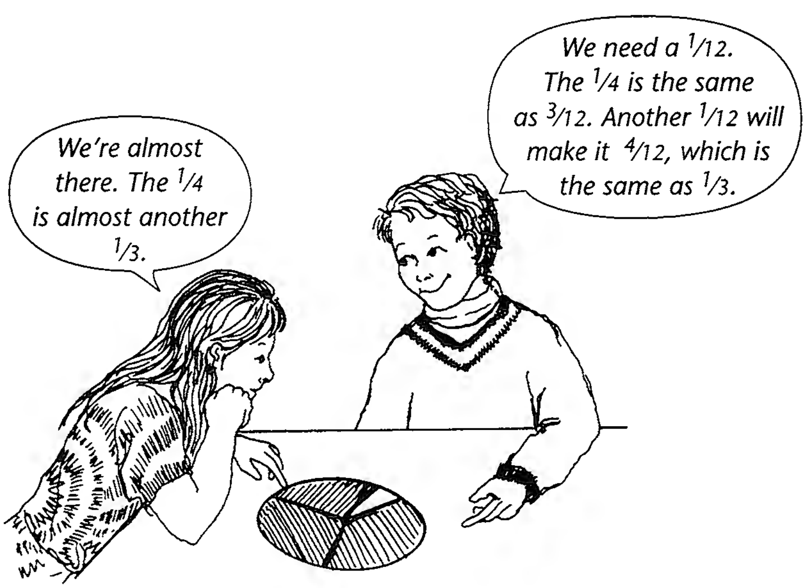
My Observations:

Hypotheses or Conjectures:

Encourage any movements to organize work or to make generalizations. For example, some groups may find it helpful to sort their results into categories: combinations that total one whole exactly vs. those that don't. Others may feel that some fractional parts seem to "fit" more often than others and attempt to explain why.

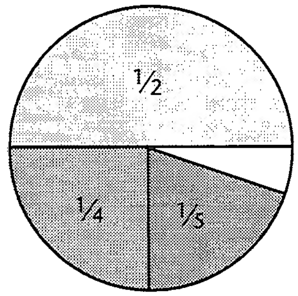
It is important that groups spend time discussing their solutions, whether or not they have made a whole exactly. Have them share their thinking. What led them to a particular combination? How do they know for sure that it makes (doesn't make) a whole?

Note: Since children will sketch their examples, they may not be precise, which is okay—this provides good experiences with estimating sizes and relationships.



5 Fractions—Make a Whole (continued)

This discussion is especially helpful when considering combinations that appear to make a whole exactly, but in fact might be off just a bit. Sometimes, it will be hard to tell if the pieces fit exactly together in the circle.



"This sure comes close to a whole. There must be more because the $\frac{1}{4}$ and $\frac{1}{5}$ don't make another half. I know $\frac{1}{4}$ and $\frac{1}{4}$ make $\frac{1}{2}$."

In some cases, the children may not be reach a consensus. It's okay if they remain at a point where they're just not sure if a particular combination "works".

The lesson may be continued in several ways, some of which are listed here:

Repeat the lesson trying to "make" other totals, such as $\frac{1}{2}$, $1\frac{1}{3}$, etc.

Attempt some of the problems on Blackline 139.

Opening Eyes to Mathematics **Make a Whole Activity Sheet** **Blackline–139**

1. Karen had a dollar. She spent one-fourth of it on candy. She spent half of her dollar on pencils. How much money does she have yet to spend? What fraction of her dollar remains? Sketch a picture of your work and write some equations to match your picture.

2. One-half of the children in Mr. Thomas' class had only a brother. Four children had only sisters. One-sixth of his class of 24 pupils had no brothers or sisters at all. How many people had both brothers and sisters? brothers only? were an only child? Sketch a picture and write some related equations to match.

3. It's the end of the school year and few crayons survived. Ray and Henry decided to combine their remaining crayons and share them. Ray had one-half of his original box of 16. Henry had one-third of his original box of 24. How many crayons did they have in their combined collection? Which boy contributed more crayons? Sketch a picture. Write some related equations to match your picture.

4. The mail carrier delivered mail to the Head's home. Randy got 8 letters. He was thrilled because half of the delivery was addressed to him! Amy and Debby each got 2 letters. Curtis got the remainder of the mail. What fraction of the day's mail was addressed to Curtis? How many letters did he get that day? Sketch a picture of your work. Write some related equations to match your picture.

5 Fractions—Make a Whole (continued)

Repeat the lesson using pattern blocks, where a particular block represents a whole. (See Volume 2, Contact Lessons 151–153, Modeling Fractions With Pattern Blocks.)

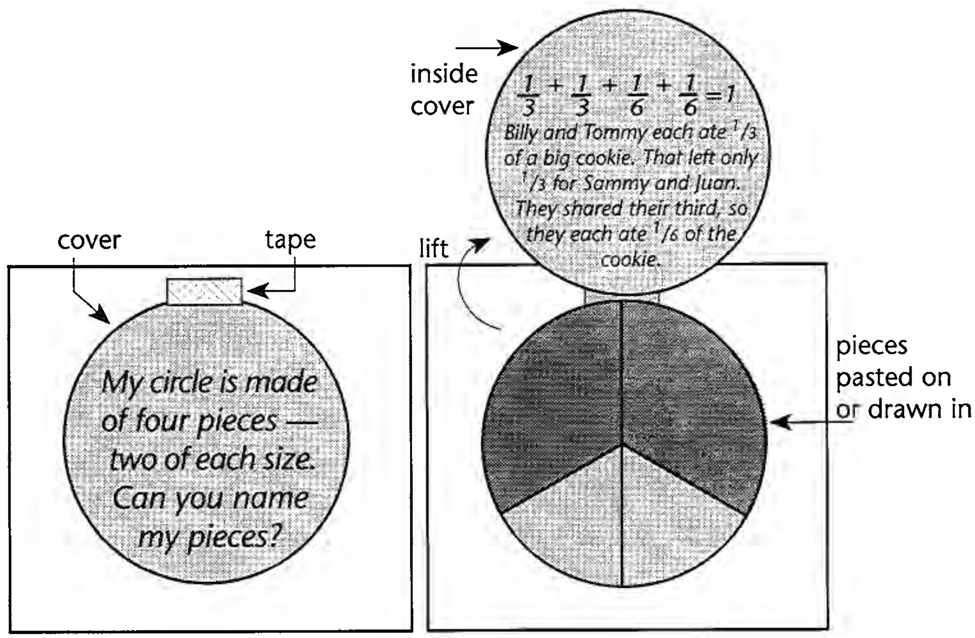
Read *The Doorbell Rang* by Pat Hutchins (Greenwillow: NY, 1980) even if it's an old familiar story!

Journal Entry

What did you discover about combining fractions during this activity? What was easy for you? What was hard for you?

Published Piece

Create a book that has a circle on each page. Divide each circle into sections and estimate the fractional part of the circle in each section. Write any descriptions, observations, story problems or equations about each picture on the back of the previous page.



Teacher Tips

Shade overhead pieces with permanent markers.

6 Who Would You Rather Be?

You Will Need

- Chapters 9, Probability, and 10, Data Analysis and Graphing
- 2 large number cubes marked 1–6, 1 colored red and 1 colored blue
- chart paper

for each partnership of two children

- yarn necklaces
- 2 number cubes marked 1–6, 1 colored red and 1 colored blue
- materials as needed for graphing
- Blackline 140 (Who Would You Rather Be?)

Your Lesson

(This lesson assumes your children have had previous experience with collecting data and preparing graphs. Please see Chapter 10 of the Teaching Reference Manual for general information about these activities. You might also revisit the graphing activities in Volume 2, Contact Lessons 180–183.) Children seem to enjoy participating in activities that have an element of chance. Perhaps it's the randomness and the opportunity for prediction that are fascinating. Through these activities children examine possible outcomes of a chance situation, gather and analyze data, and make decisions.

In this lesson, your children make decisions related to different games. At various times, these decisions may be made on the basis of personal feelings, experimental evidence, or theoretical analysis, depending on the experience and interests of the class. The title of the lesson refers to the following general situation:

Two people (or teams), A and B, repeatedly play a game, with the winner of each round being awarded a certain amount of points. The children play and think about this game, ultimately trying to decide which player (in the long run) is more likely to be ahead in points.

Games may be structured in different ways and some examples are presented below.

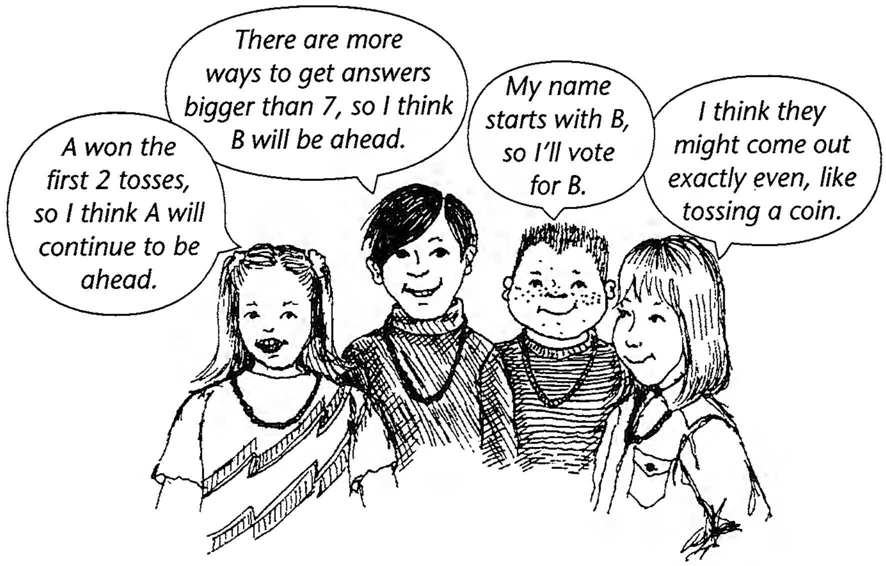
Game 1: Two 1–6 number cubes are tossed repeatedly. On each toss, if the product of the numbers is less than 7, player A scores 2 points. If the product is 7 or bigger, player B scores 2 points.

Post the rules of the game on a chart and tell the class its task is to answer these questions: If this game is played over and over (a large number of times), which player, if any, is favored to have more points? Why?

Here is one way to proceed: Divide the class into two teams, identifying the teams with yarn necklaces. Now form partnerships of two composed of one member from each team. Invite a child to toss one of the number cubes. If the toss yields an even number, that child's team will represent Player A; if the number is odd, that team represents player B.

6 Who Would You Rather Be? (continued)

Demonstrate play of the game by tossing the number cubes a few times and explaining the rules. Invite the class to imagine the game being played a large number of times and to make initial predictions about which team is likely to be ahead. Have volunteers share their predictions and the reasons for them. At this point, predictions may be based purely on personal feelings or guesses.



Distribute two number cubes (one colored red and one colored blue) to each partnership of two children. Have the partnerships complete the questions on Blackline 140.

Opening Eyes to Mathematics — Who Would You Rather Be? — Blackline–140

1. Describe this activity.

2. Before beginning, if the game is played over and over, which player, A or B, do you think is likely to score more points? Why?

3. Test your prediction by playing the game. Your partnership can decide how many times you wish to play. Prepare a graph of the data you collect.

4. Now form a group with another partnership and add your results to theirs. Prepare a graph of the combined results. Glue this graph and the one you made in Problem 3 to a piece of chart paper.

When they have finished, ask the groups to discuss results and to answer the following questions on chart paper:

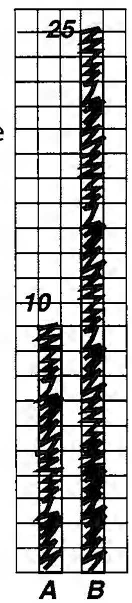
- Are the partnerships' graphs alike in any way? Are they different? Explain.

- Based on the results of your experiment, if you were to play the game again, which player would you rather be? How confident are you that this player would be ahead?
- If you feel the game is unfair to a particular player, how would you change the rules so the game is fair to both players?

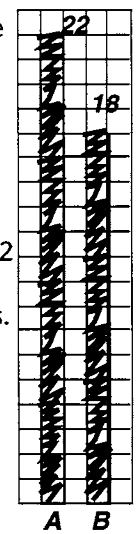
Invite the groups to present their charts and to share their decisions regarding which player is favored in this game. Ask them to also describe the bases for these decisions and the level of confidence they have in their predictions.

If it hasn't occurred already, invite volunteers to share what their decision would be if they based it on personal feelings. Similarly, would it be any different if the results of the experiment were considered? An important underlying point here is that decisions involving chance (or probability) are often revised as additional related information becomes known.

"We played 35 times and B won 25 times. We thought we'd like to be B. Then we saw the other graph. Still, it seems there are more ways for B to win."



"A won 22 times out of 40. The game seemed even to us. But when we combine the results of our whole group, A wins only 32 times and B wins 43 times. Seems better to be B, but we're not sure."



You might note, too, any comments that seem to be theoretical in nature. This may be an appropriate time to discuss the theory related to this game. How can one decide which player is theoretically favored to win? If you wish to pursue this question, invite the children to examine the possible combinations that can arise when the number cubes are tossed.

TEACHER *(holding up the two large red and blue number cubes) Let's think about what can happen when the number cubes are tossed. One possibility is to throw a 4 on the red cube and a 3 on the blue one. Which player would win if that happens?*

SARAH *4 times 3 is 12. B wins since 12 is larger than 7.*

JIM *You could also get 3 on the red and 4 on the blue. That gives 12, too, so B would win. But is that different?*

(Expect some discussion here; theoretically, the answer to Jim's question is "Yes".)

TEACHER *What are some other combinations that could occur when the cubes are tossed?*

MOSES *Well—you could get a 1 on the red and a 1 on the blue. Hey, you could also get a 1 on the red with a 2 on the blue, and with a 3 on the blue, all the way up to a 6 on the blue. Looks like A wins whenever a 1 comes up on the red cube.*

TEACHER *Yes. In your groups, I'd like you to record all the possibilities that can occur when the cubes are tossed. Find a way to organize your results to be sure you have them all. Then use this information to decide which player is favored to win on each toss.*

Theoretically, if the number cubes are randomly tossed, there are 36 possible ways they can land, each of which is equally likely to occur. Of these 36 possibilities, A wins 14 times and B 22 times, so the game favors B. This can be seen by examining the following chart which reports the products corresponding to each possibility.

Multiply		numbers on red cube					
		1	2	3	4	5	6
numbers on blue cube	1	1	2	3	4	5	6
	2	2	4	6	8	10	12
	3	3	6	9	12	15	18
	4	4	8	12	16	20	24
	5	5	10	15	20	25	30
	6	6	12	18	24	30	36

B wins 22 out of 36.

From this chart, we see that the (theoretical) chances of A winning 2 points on each toss are $^{14}/_{36}$. This can be visually depicted by coloring in the possibilities that win for A as shown above.

The chart also provides other information. For example, it shows that a product of 12 can occur in 4 different ways. So the chances of obtaining a product of 12 on any toss are $^{4}/_{36}$.

Here are examples of other games that may be used in this lesson. These may be conducted in the same way as Game 1.

Game 2: Two 1–6 number cubes, one colored red and one colored blue, are tossed repeatedly. On *each* toss, multiply the number that comes up on the red cube by 2 and the number on the blue cube by 3. Then add the results. If the total is even, player A scores 2 points; if the total is odd, B receives 3 points. Which player, if any, is favored to have more points in the long run?

Note: Theoretically, this game is fair to both players. As the following chart shows there are 18 ways to obtain an even total and 18 ways to get an odd result.

6 Who Would You Rather Be? (continued)

Multiply red by 2 and blue by 3, then add		numbers on red cube 1	2	3	4	5	6
	1	5	7	9	11	13	15
	2	8	10	12	14	16	18
numbers on blue cube	3	11	13	15	17	19	21
	4	14	16	18	20	22	24
	5	17	19	21	23	25	27
	6	20	22	24	26	28	30

It is possible to continue this lesson as long as interest is high, or to revisit it periodically. As Game 2 illustrates, rules can be varied to emphasize certain number combinations or properties. Consider varying the rules, the numbers on the cubes, or even the pieces that are tossed (see Teacher Tips).

Teacher Tips

The results of playing these games will vary. However, they should approximate theoretical answers, provided the number cubes are fair and a large number of random tosses have been made.

As the children plan their responses to the questions on Blackline 140, we often invite those with the same color necklaces to confer about their results and feelings. Teams can meet as a whole or break apart into subteams for this purpose.

We're confident your children will enjoy proposing other games and analyzing them. They may need some help with this, however, especially if the lesson occurs early in the year. If so, consider spending some class time creating a chart of "What if" game situations.

Some of the children's suggested games may be examined as part of a homework activity.

Technology

If you have computer software that generates random data, then your children might enjoy simulating these games. Large numbers of trials can be conducted in a short amount of time. It will be interesting to see how the results of such an experiment compares with the theory related to each game.

Journal Entry

Think of a game that came up in this lesson. Write a letter to a friend describing the rules of this game. Would you advise your friend to be Player A or Player B? Why? What tips would you give to your friend?

7 Exploring Perimeter

You Will Need

- Chapter 7, Measurement
- colored squares for the overhead
- a transparency of Blackline 141 (Square Dot Paper)

for each child

- square tile
- linear pieces (optional)
- Blackline 141 (Square Dot Paper)
- copies of Blacklines 142 and 143 (Exploring Perimeter)

Introduction

By its very nature, problem solving requires problems to be solved and time in which to solve them. When solving problems, people must often make decisions and actually assume the role of problem-poser: "How shall income be budgeted each month?" "What explains why the car is making that funny noise?" "Is a rectangular floor plan as efficient as a different one with the same area?"

Making decisions and problem posing are also integral parts of doing mathematics. This can occur in our classrooms whenever we vary the conditions and parameters related to a particular problem situation. As an example, the following problems were posed in Volume 2, Contact Lessons 133–134, Geoboard Perimeters:

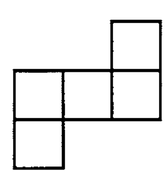
On a *square geoboard,* how many *rectangles* can be formed which have *perimeter 12*? How many with perimeter 16? perimeter 50?

The italicized parts in these questions may be changed to generate other problems. *Square geoboards* may be changed to *pattern blocks, square dot paper, isometric dot paper* or *square tile,* to name a few. Similarly, *rectangle* can become *triangle* or *shape;* and *perimeter* might be replaced by *area.* The decisions are ours, or better yet, the children's.

Your Lesson

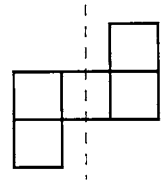
This lesson describes one of the variations mentioned above. You might begin by revisiting some of the ideas from Contact Lessons 133–134 and then proceed with the following activities.

TEACHER *(at the overhead) What observations do you have about this tile arrangement?*

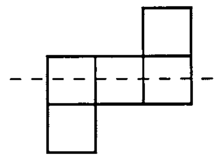


LUCAS *It's made with 5 squares. That means its area is 5—if each tile is*

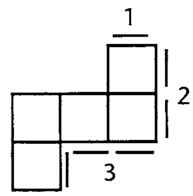
MARIA *I can make two congruent parts with this line.*



ALYSSA *I see another pair of congruent parts.*

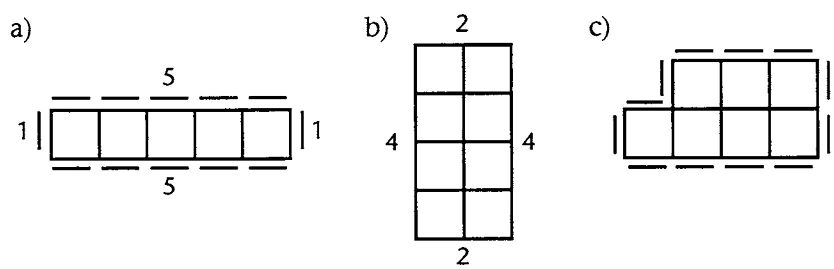


ENOS *I'm looking at its perimeter. I see 3, 2 and 1—that's 6 linear units. This happens twice, so the perimeter is 12 linear units.*



Observations will vary, of course, and so what happens next is up to you. (We encourage you to add your own observations to the list.) Based on the above dialogue, should an area problem be posed? Or one on congruence? Or perhaps one on perimeter. In any case, there is a decision to be made, and what follows is merely one possibility. (Please see the Teacher Tips for additional comments.)

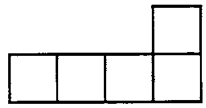
Continue, asking your children to each build a shape which has a perimeter of 12 linear units and to share their results. Here are some possibilities:



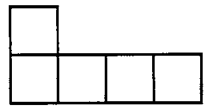
It is now natural to pose this question as one to be explored in groups of four: Assuming the sides of the tile have to match end-to-end, how many different shapes can be made which have perimeters of 12 linear units?

Children can record their shapes on square dot paper (Blackline 141). Provide plenty of time for group discussion and exploration. Encourage the children to make their own decisions about how to proceed.

DEVON *Here's one.*

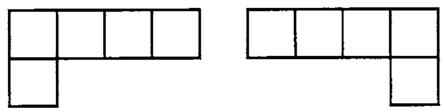


PEGGY *I've got a different one.*



DEVON *Isn't that the same as mine only turned a bit? I don't really think it's different.*

ROBERTO *If they're different then so are these:*

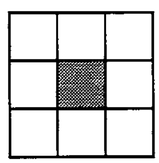


ALICIA *I agree with Devon. They're all the same, only turned. If we say they are different, there will be a ton of shapes!*

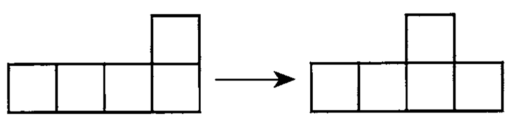
GROUP *Suppose we say they're the same. How many "really different" ones can we find?*

Look for other ways groups make decisions.

GROUP 1 *Let's not count shapes with "holes" such as this.*

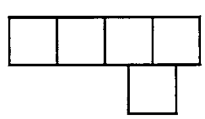


GROUP 2 *We know this one works. Let's first move the tile that sticks out around and see how many others we can find.*

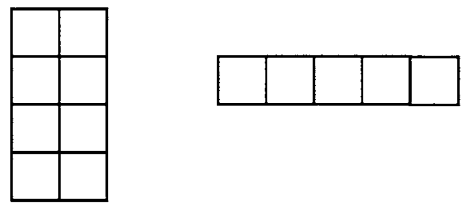


GROUP 3 *Let's start small. Can we find some which use 3 tile? Then we can try 4 tile, 5 tile, and so on.*

GROUP 4 *We also wanted to count ones where the ends don't match entirely. There will be a lot of that kind.*



Responses such as the ones above make it highly likely that groups will proceed in different ways, and you may have to respond accordingly. In general, as you observe your children, encourage any attempts to classify or generalize results. Questions such as these will be helpful here: How did you determine that? Will that always be true? How can you be sure you've obtained all the possibilities? Or, if little observation seems to be occurring, invite the groups to classify their attempts, as Group 3 above suggests, and to look for relationships among their answers, as illustrated by Group 2 above.



"These both work, but the 2 by 4 array has more sides that are shared."

Much of the discussion will likely occur within groups. As this takes place, encourage the children to reflect about their work. How do they feel about their work? Do their results make sense? Can they be explained? How can they be shared with others?

When ready, you might ask the groups to "publish" their results for the rest of the class.

Teacher Tips

The variation on Contact Lessons 133–134 described in this lesson focuses on perimeter. Other lessons in this volume will focus on different concepts such as area or congruence (e.g., Lessons 16 and 49).

Note the suggestions in the Introduction about using pattern blocks or different kinds of dot paper in this exploration. These materials may be used during this lesson by the children, at their request, or at a later time when visiting perimeter again.

Journal Entry

Explain how you would find all the shapes that have a perimeter of 10 linear units.

*** Reminder**

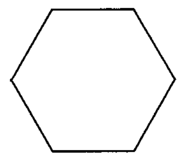
For Lesson 8 your children will need to bring standard tools for measuring length, such as carpenters' rules, rulers, tape measures, yardsticks, metersticks, trundle-wheels, etc.

Homework

Invite your children to work with a friend or a family member on problems such as those found on Blacklines 142 and 143.

Opening Eyes to Mathematics | Exploring Perimeter | Blackline 142

Suppose you are arranging tables for a banquet or birthday party. There are five hexagonal-shaped tables to be arranged. Each table looks like this:



1. If the tables must join along at least one full side, what different table arrangements are possible?

2. If only one person sits along a side, what is the fewest number of people who could be seated? The largest number of people? Explain your thinking.

3. Is it possible to arrange the tables so there is one person seated along each side and the total number of people is odd? Explain.

Opening Eyes to Mathematics | Exploring Perimeter | Blackline 143

1. You are building a rabbit cage and have enough money to buy 24 feet of fencing. What size options do you have for the rabbit cage? Which cage would you build? Why? Explain your thinking with the help of diagrams or sketches.

2. King Arthur had 32 knights. He ordered his servants to slide small square tables together to make one big long table for the banquet. If only one person can sit along a side, what is the fewest number of tables that could be used to seat everyone? Explain your thinking.

Pattern Reminder

Revisit the theme of Lesson 1 (Pattern Possibilities) by exploring a sequence starter of your choice or one found on Blacklines 117–125.

8 *Standard Units of Length*

You Will Need

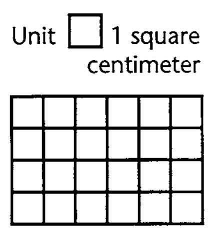
- Chapter 7, Measurement
- standard tools for measuring length: carpenter's rule, rulers, tape measures, yardsticks, metersticks, trundle-wheels, etc.
- copies of Blackline 88 (Guess and Check Record Sheet) for each child

Introduction

In previous measurement lessons, your children gained experiences with using both non-standard and standard units of measure. A major purpose of those experiences was to strengthen children's sense of the measurement process:

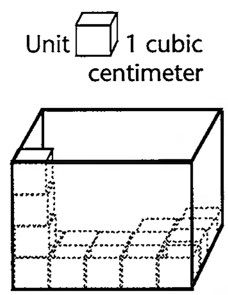
An appropriate unit of measure must first be chosen. The attribute being measured (e.g., length, weight, area, etc.) can then be described in terms of this unit. (Teaching Reference Manual, p. 64.)

Area



The area is 24 square centimeters.

Volume



"I would guess that it will take about 80 units to fill the box. The volume is about 80 cubic centimeters."

It may be helpful to revisit some of those earlier lessons (e.g., Volume 1, Contact Lessons 13, 37, 55, 56 and others). These lessons culminated with a series of activities that focused on measuring lengths with standard tools (Volume 2, Contact Lessons 167–169). These latter activities are extended in this lesson.

Your Lesson

Discuss the measuring tools that your children brought to class. Ask the children to describe instances where they or others have used these tools. Provide a time for the children to identify and compare the various units of length.

Pose these questions for discussion, creating a class chart of children's responses: What lengths in the classroom or school would you like to measure?

What standard unit(s) seem most appropriate for measuring each length?

OBJECT TO BE MEASURED	UNIT OF MEASURE
length of chalk tray	feet
width of chalk tray	centimeters or inches
length of stripe on US flag	centimeters or inches
width of stripe on US flag	centimeters or inches
width of ★ on U.S. flag	centimeters or inches
South wall of classroom	yards or meters

Divide the class into teams of 2 to 4 children. Ask each team to estimate the lengths listed on the chart in terms of different units of length. The teams can then determine the corresponding actual measures. Responses may be recorded on Blackline 88.

Opening Eyes to Mathematics   Guess and Check   Blackline 88

Name____________________ **GUESS AND CHECK**

Item	Unit of Measure	Guess	Check

When this task has been completed, ask the teams to post their actual measurements on chart paper. Discuss the results. How do the measurements for each length compare? What might explain any differences? If a length has been measured with two different units, how do the two measurements compare? Which unit seems more appropriate to use for a given length? How did the children handle fractional parts of a unit?

Reverse the problem: Ask the teams to find something in the room that has a length of approximately 15 cm.

Have them first identify such an object without doing any measurements. Instead, encourage the teams to use their "sense of a centimeter", along with some estimation strategy. Actual measurements may then be made as checks. Repeat, seeking objects of other given lengths.

Teacher Tips

You might keep these activities alive with questions such as these:

What can you find in our classroom that is about half as long as a meterstick?

What object(s) can you think of that almost equals ¼ yard in length?

What is the height of our classroom?

How tall would all of us be if we were stacked on top of each other?

We like to post one question on our calendar each week. A group list of estimates is made about each question. The children then choose an appropriate unit of measure and DO IT TO IT!

Your children might also enjoy the challenge of measuring the lengths of different playing fields.

Homework

Ask the children to work with a friend or family member on this task. What lengths are measured at home? For example, how would you measure the length of your carpet? toothbrush? fishing line? Select 5 objects that have very different lengths. Estimate these lengths and then measure them.

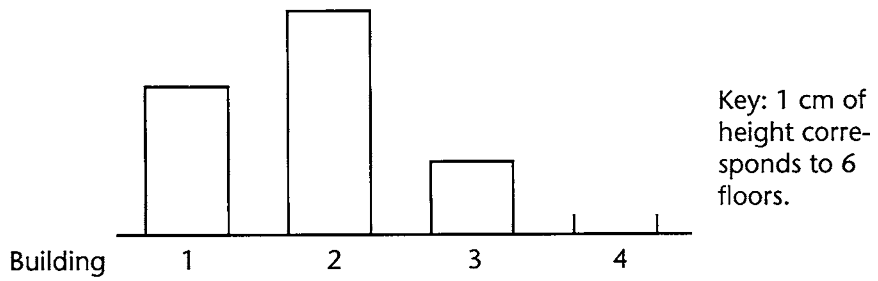
Journal Entry

Briefly describe your feelings during the activities of this lesson. Did you feel comfortable about any of your estimates? Which ones were hard for you?

Suppose the class decides to measure the length of the cafeteria tomorrow and has to choose one of these measuring tools to use: meterstick, ruler, 20-cm strip, tape measure, yardstick or the edge of a 1-inch square. Which tool would you recommend choosing and why?

Daily Computation

Show the following picture to the class.



Questions: How many floors do Building 1, 2 and 3 have? Building 4 has 21 floors. How high should we draw it?

9 Cutting Shapes in Half

You Will Need

- Chapter 11, Geometry
- sheets of paper (from recycle bin)
- scissors
- (optional) chart paper and Blackline 141 (Square Dot Paper)

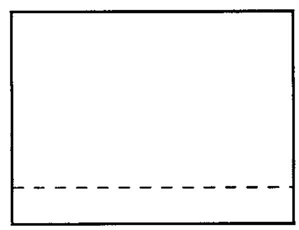
Your Lesson

These activities can help your children strengthen their mental geometry and think about important geometric concepts such as congruence, area, perimeter and symmetry.

Hold up a piece of paper, telling the class you'd like to make one cut that would split the paper in half. Ask the children to imagine the situation and to suggest how to make the cut. Don't cut just yet; instead, encourage the children to elaborate on their thinking. This is a chance for them to mentally reflect about a geometric situation and to refine their vocabulary for describing that situation.

JULES *You ought to be able to cut straight across.*

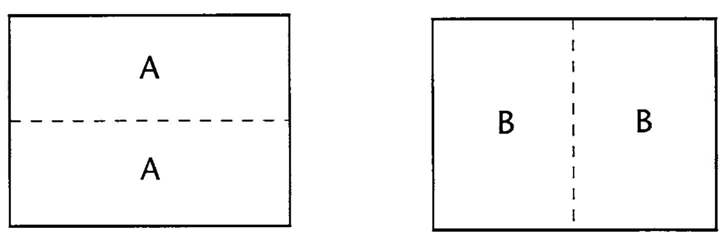
TEACHER *Oh. Like this?*



AMANDA *No, no! Straight across, halfway up.*

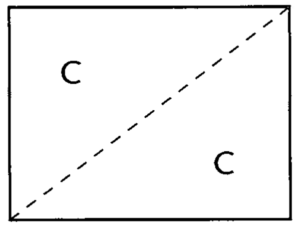
SHIRLEY *Fold it in half across the middle. Here's how—bring the bottom edge up so it fits exactly on the top edge.*

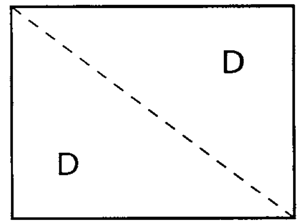
Someone may talk in terms of a line of symmetry; otherwise, you can recall this concept (Teaching Reference Manual, pages 86–88).



"Fold along a line of symmetry."
"If a mirror were placed along the fold, each would be the mirror image of the other."

We've found it helpful to label each half with the same number or letter in order to identify each pair more easily. Often, children will suggest folding and cutting along a diagonal. Ask the class to imagine doing this.





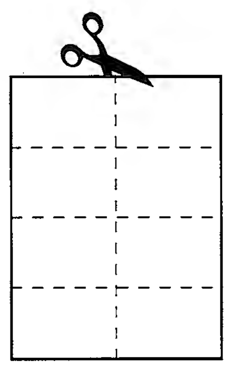
TEACHER *Will the folded parts coincide as they did before?*

LIANA *I don't think the corners will match. Still, it seems that the two parts are the same. If you turn one, it can fit on the other, so they must be halves.*

TEACHER *(demonstrating by folding and subsequently cutting) Yes, this fold is not a line of symmetry. But the two parts can be made to fit. They are congruent to each other, just as before.*

The above dialogue points out why the diagonal of a rectangle is not a line of symmetry (unless the rectangle happens to be a rhombus). The dialogue also alludes to the rotational symmetry of a rectangle. In Lesson 37 of this volume, your children will informally examine the distinction between line and rotational symmetry.

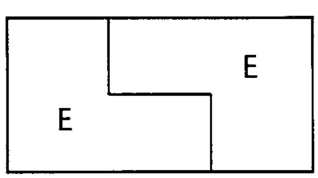
Have the children now fold and cut several rectangles from paper:

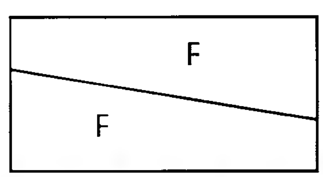


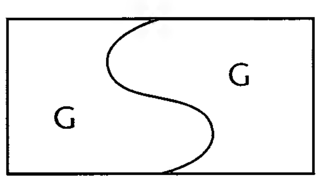
Ask them to cut one of the rectangles in half and label the two parts with an identifying mark. Have them check their results and invite volunteers to describe their cuts and solutions.

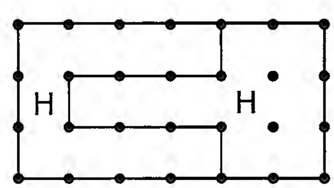
Continue the search for different cuts by posing this problem to the class: How many different ways can you make a single cut that will split a rectangle into two parts that are equal in area?

Here are some solutions we've received from our children:









As you can tell, some children opted to use square dot paper, cutting in such a way that the area could easily be counted!

Discuss the solutions.

- Were any generated by folding across a line of symmetry (as in Solutions A and B)? Are there other solutions of this type?
- Were any tested by turning one piece onto the other (as in solutions C and D. Note this can happen in solutions E, F and G as well.) Other solutions of this type?
- Do any consist of two congruent parts (solutions A–G)? two non-congruent parts (solution H)? Other solutions like these?

Continue the lesson by having the children address these problems: Each time you cut your rectangles, you made a pair of shapes that had the same area. Look at each pair—do the shapes also have the same perimeters?

Teacher Tips

The activities of this lesson lend themselves nicely to writing projects in which your children can describe their observations and results.

Daily Computation

You may wish to make this a mental arithmetic exercise.

There is a "½ Off Sale" at a local store. Find the missing prices in the chart:

Original Price	Sale Price
$84	
$49	
$138	
$64.40	
	$96
$205	
	$33.50

Add some prices of your own along with their "½-Off" Sale Price.

10 *School Events*

You Will Need

- copies of the following blacklines, as needed, for each child: Blackline 144 (Playground Equipment Checklist), Blackline 145 (Buses for a Field Trip), Blackline 146 (Mr. Harper's Bus), Blackline 147 (Operation Care)
- materials for displaying information

Your Lesson

In this lesson, your children are asked to attempt some problems which draw on their familiarity with playgrounds. Introduce the lesson as described below and, according to the needs of your class, assign the problems either to individual children or to small groups.

Begin by describing the following situation to your class:

> A school needed new playground equipment badly. The fourth grade classes decided to raise money by seeking contributions, conducting bake sales and selling fruit. They collected \$90 and agreed to purchase 5 items since 5 classes had helped with the fund-raising.
>
> After conducting a needs assessment, it was decided that the school could use more jump ropes, swing seats, basketballs, soccer balls and soccer nets. A committee checked prices and found that these items could be purchased for the following amounts:

Jump rope	\$10.00
Swing seat	\$10.00
Basketball	\$15.00
Soccer ball	\$25.00
Soccer net	\$30.00

> However, the classes disagreed as to how many of these items to buy. They wanted a total of 5 items, but some wanted to purchase one of each for a total of \$90. Others wanted to buy more than one of certain items and omit others entirely. For example, some people suggested placing this order:

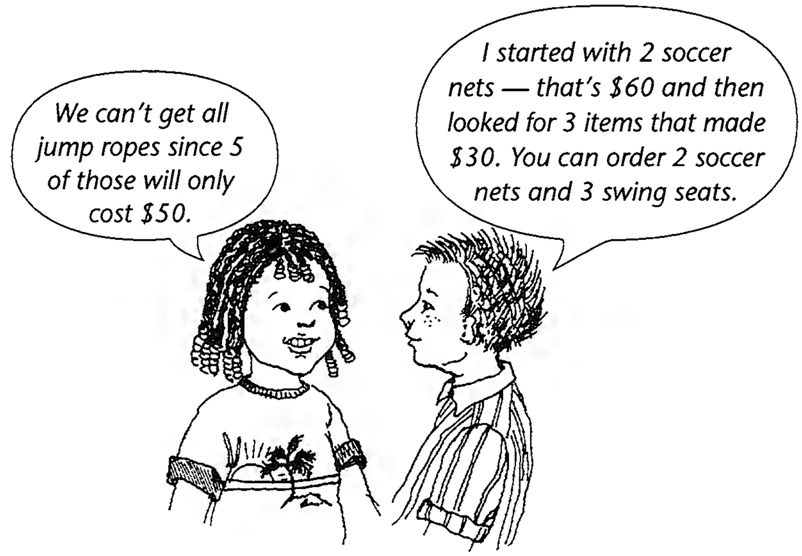
2 soccer nets	\$60
3 jump ropes	\$30
Total	\$90

> What other combinations of 5 items, selected from the above list, are there that total exactly \$90? How can you be sure you've found all the possibilities? Select your preferred combination and explain why you would suggest it be ordered.

Ask your children to address these questions and to prepare a display of their results. This display may contain models, charts, tables, graphs, lists, etc., as determined by the children.

10 School Events (continued)

Note: This activity is another opportunity for your children to develop number sense and problem-solving skills. It's important to provide time for children to explain their procedures for generating appropriate combinations and to discuss strategies for knowing when all possibilities have been found.



The lesson may be continued by examining some related problems that could be suggested by you or your children. You might, for example, remove the restriction that a total of 5 items must be purchased; or perhaps the prices can be changed to generate work with particular number combinations. Other problems related to school events also appear in Blacklines 145–147.

Opening Eyes to Mathematics **Buses for a Field Trip** **Blackline–145**

Suppose a school is chartering buses for a special year-end trip to Chicago. These people plan to make the trip: 142 children, 5 teachers, 18 parents and 1 principal. The law allows a maximum of 2 people per seat. How many buses should be reserved for the trip?

Draw a model or picture to explain your thinking. Describe your pictures with sentences and equations.

Bonus: How can the principal assign people so each bus carries the same number of adults and the same number of children?

Opening Eyes to Mathematics **Mr. Harper's Bus** **Blackline–146**

Mr. Harper picked up 19 boys and 13 girls on his bus last Tuesday. His bus has 30 seats. If each person sits with a partner, how many seats were filled last Tuesday?

Draw a picture to illustrate your answer. Write some sentences and equations to match your picture.

Bonus: How many more people could Mr. Harper have picked up last Tuesday without breaking the law?

Double Bonus: After Mr. Harper filled his bus last Tuesday with children, there were the same number of girls as boys. How many more boys did he pick up? How many more girls?

Opening Eyes to Mathematics **Operation Care** **Blackline–147**

A scout troop has collected $100 to buy food for the local community shelter. How would you advise the troop to spend the money? Use grocery ads to help you plan a recommended shopping list. Use the chart below to keep a record of your purchases. Remember to provide a balanced diet.

food group **budget**

item	quantity	cost for one	total cost

TOTAL SPENT ____________

Assessment

This lesson lends itself well to a checklist similar to that shown on Blackline 144.

Opening Eyes to Mathematics **Playground Equipment Checklist** **Blackline 144**

student	Did the student—	teacher
❐	Retell the problem in their own words?	❐
❐	Include a clear diagram of their solution?	❐
❐	Label diagrams correctly?	❐
❐	Use clear, elegant mathematical language?	❐
❐	Create correct solutions?	❐
❐	Do more than the problem asked?	❐
❐	Work neatly?	❐
❐	Include a variety of solutions?	❐
❐	Communicate their thinking clearly?	❐
❐	List tools used?	❐

To the student:
Did you enjoy this problem? Why or why not?

How do you feel about yourself as a mathematician?

11 *Fractions on a Geoboard*

You Will Need

- Chapter 7, Measurement, and Chapter 8, Fractions
- transparent geoboard and transparencies of Blacklines 148–149 (Fractions on a Geoboard)

for each child

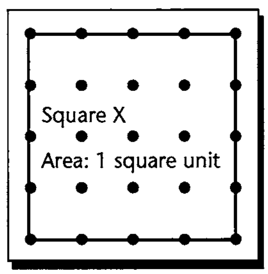
- geoboards and rubber bands
- Blacklines 148–149 (Fractions on a Geoboard)

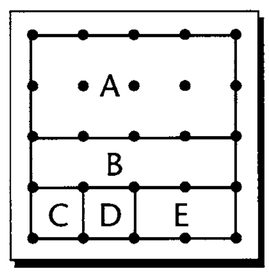
Your Lesson

The activities of this lesson provide experiences with modeling fractional relationships in a visual, problem-oriented setting. They also underscore the role of a unit when working with measurements and numbers.

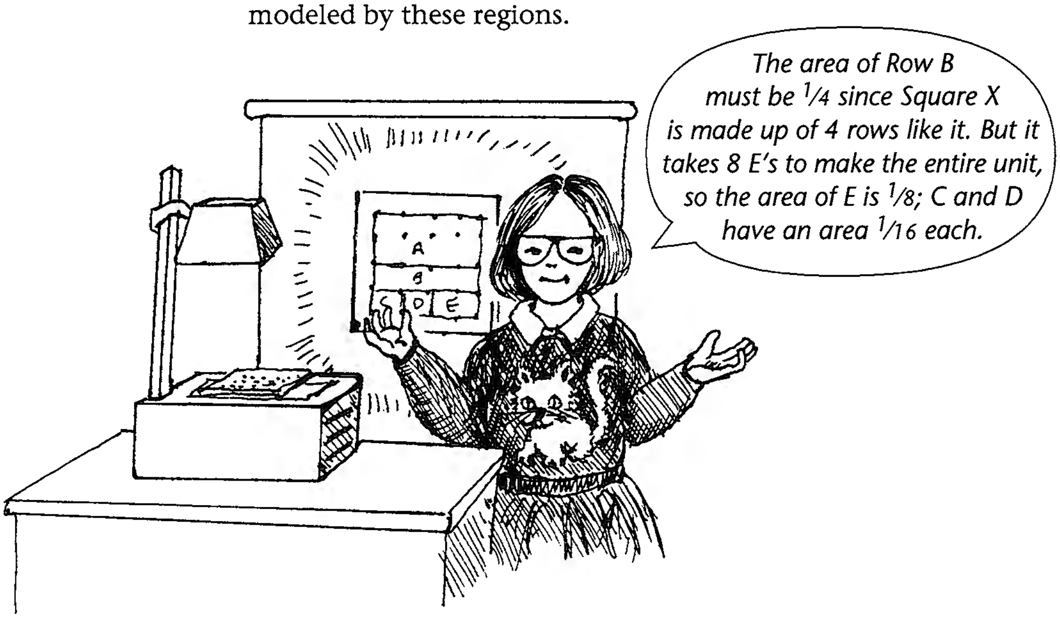
Activity 1

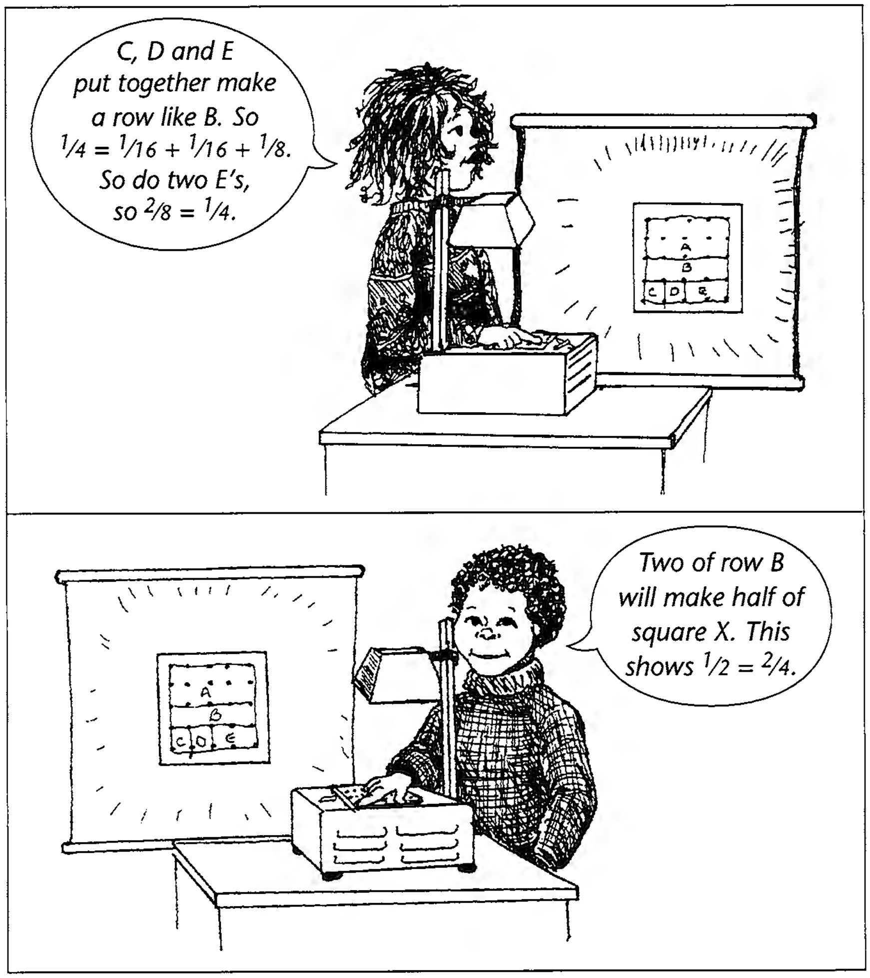
Display Square X shown below and tell the children the area of this square is 1. Then stretch additional rubber bands to form the lettered regions shown in the second illustration.



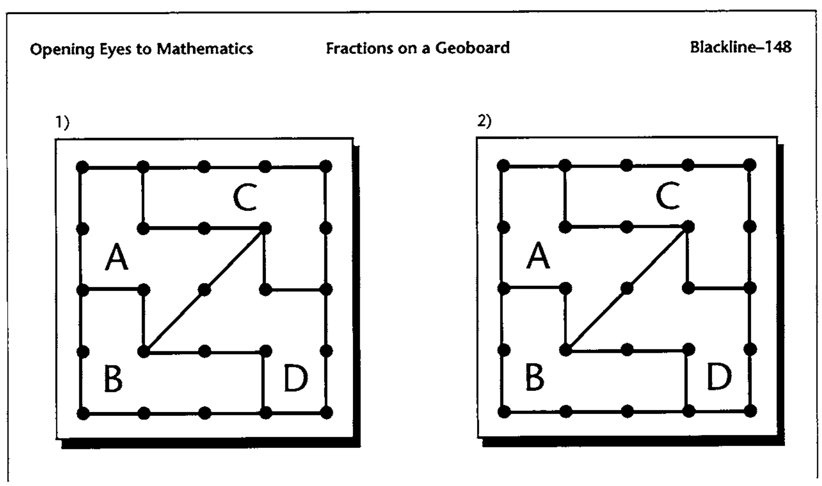


What is the area of each lettered region? Invite volunteers to describe how they would answer this question. Provide time, too, for your children to describe some fractional relationships that are modeled by these regions.



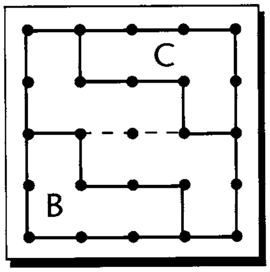


Show Square X once more at the overhead, reminding the children to imagine that its area is still 1. Use other rubber bands to form the regions shown in Problem 1, Blackline 148.



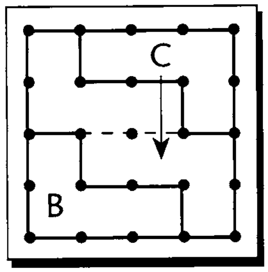
Have the children build these regions on their boards. Then, working in small groups, ask the children to determine the area of each lettered region and to describe on Blackline 148 any fractional relationships modeled by these regions. Some possible responses are shown on the following page.

a)



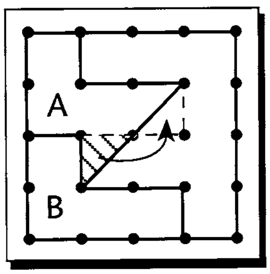
"B and C have the same area. This picture shows the area of B is ¼ since four of the B's make up square X."

b)



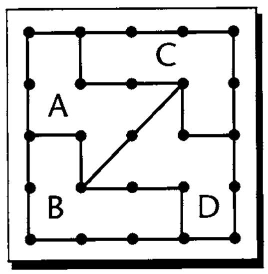
"By sliding C down, I see that B and C make up ½ of square X. So $2/4 = 1/2$."

c)



"By moving the shaded portion of A as shown, I can make A look like B. So A and B have the same area—¼ of square X."

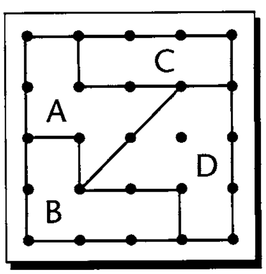
d)



"If D is cut out and turned, it can fit exactly on A. So A and D are congruent shapes and have the same area. The same is true with B and C."

You may wish to repeat this activity using other regions such as those pictured here. Diagrams may be sketched on Blackline 149.

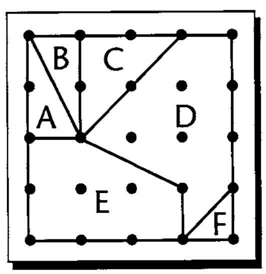
a)



Region	Area
A	$1/4 = 4/16$
B	$1/4 = 4/16$
C	$3/16$
D	$5/16$

Note the total area is
$4/16 + 4/16 + 3/16 + 5/16 = 16/16 = 1$,
the area of square X

b)



Region	Area
A	$1/16$
B	$1/16$
C	$2/16 = 1/8$
D	$6/16 + 1/32 = 13/32$
E	$5/16$
F	$1/32$

c) Regions suggested by your children.

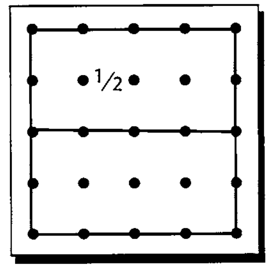
Continue the lesson by inviting the children to explore some of the following activities.

Activity 2

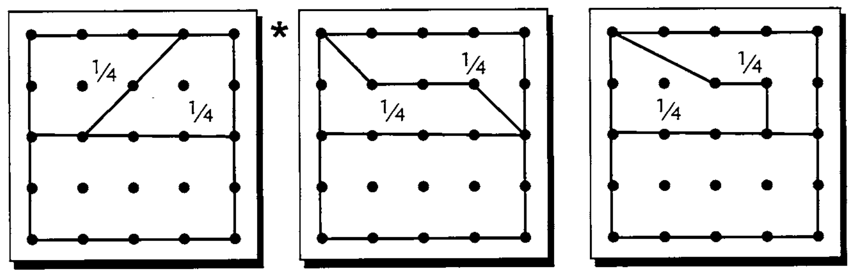
Divide the class into teams of two and have the children build square X on their geoboards, imagining once more that this square has area 1. Ask each team member to use another rubber band to show a region that has area ½.

Have the teams select one of their regions that has area ½ and come up with different ways to split this region so as to model the relationship ½ = 2⁄4. Solutions may be recorded on the blank geoboards on Blackline 149. Here are some examples.

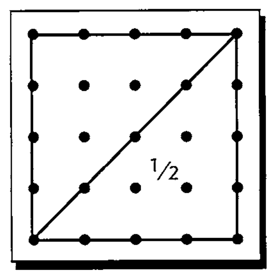
a)



*



b)



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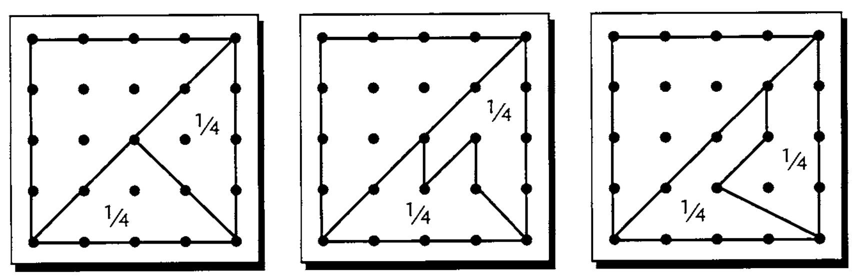
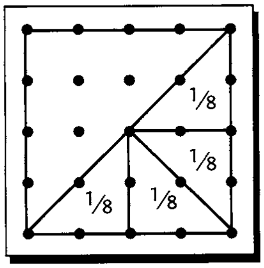
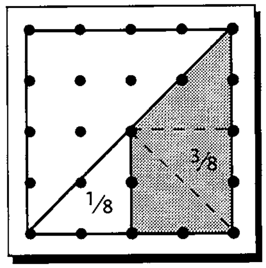


Note: You may wish to discuss which solutions split the ½ into two congruent regions. In the above illustration, congruent regions are shown in the solutions marked with an asterisk.

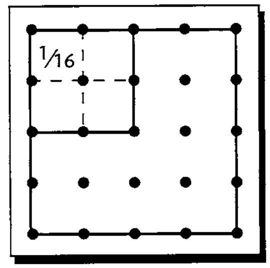
Teams can repeat this activity using a different region with area ½. They can also illustrate ways to model other fractional relationships.



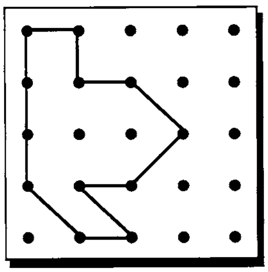
$\frac{1}{2} = \frac{4}{8}$



$\frac{1}{8} + \frac{3}{8} = \frac{4}{8}$



$\frac{1}{4} = \frac{4}{16}$



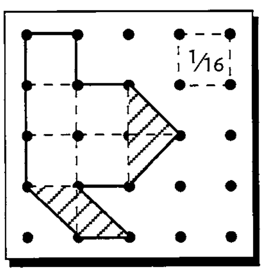
Shape A

When ready, you (or the children) may request that teams take a "field trip" to nearby tables to observe others' work or thinking.

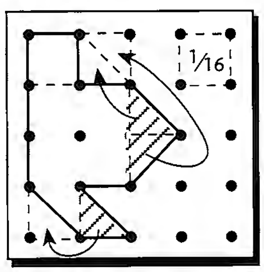
Activity 3

Ask the students to again imagine that Square X has area 1. Have the children work in groups of four to build Shape A (to the left) and determine its area.

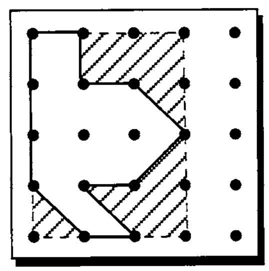
Conduct a show-and-tell discussion of their results.



"Each of these smaller squares has area $\frac{1}{16}$. Shape A has 7 of these smaller squares—5 complete 1's plus 2 more made from the shaded halves. So the area of A is $\frac{7}{16}$."



"If the shaded halves are moved as shown, then we get a shape that is made from 7 complete $\frac{1}{16}$ squares. So the area of shape A is $\frac{7}{16}$."



"Shape A can be surrounded by a rectangle that has area $\frac{12}{16}$. The shaded parts have a total area of $\frac{5}{16}$. So the area of shape A is $\frac{12}{16} - \frac{5}{16} = \frac{7}{16}$."

Continue this activity by dividing each group of four into teams of two children. Each team creates a geoboard shape; the entire group then works together to determine the area of both shapes. (Please see Teacher Tips.)

A nice variation of this activity is to have each team create a shape and determine its area (still using square X as the unit). Shapes are then exchanged and each team attempts to determine the area of the other's shape.

Activity 4

Invite groups to suggest some "What if's" related to this lesson that might be explored.

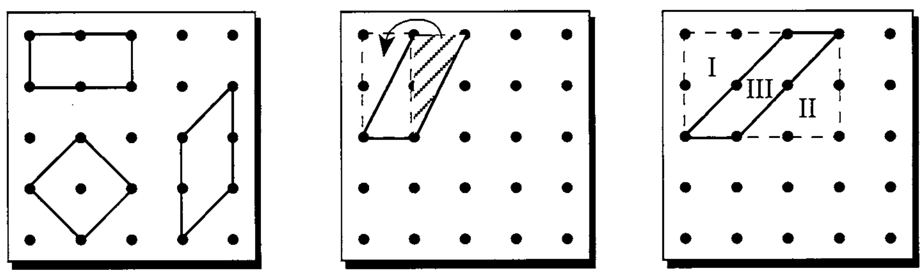
Assessment

Published Piece: Each child can create a "geoboard challenge" of their own. That is, they create a shape and begin to find different ways to determine its area. These ways may emerge from using different units or from different strategies for finding the area. Information may be added periodically to their report throughout the year. Children may wish to communicate their thinking in various ways (e.g., pictures, words, equations, etc.) and use different tools (such as colored pencils or markers) to display their results.

Teacher Tips

By way of introduction, you may wish to revisit Contact Lesson 132, Volume 2; before conducting the above activities. If so, a good problem to use is: How many different shapes having area 2 can be made on a geoboard?

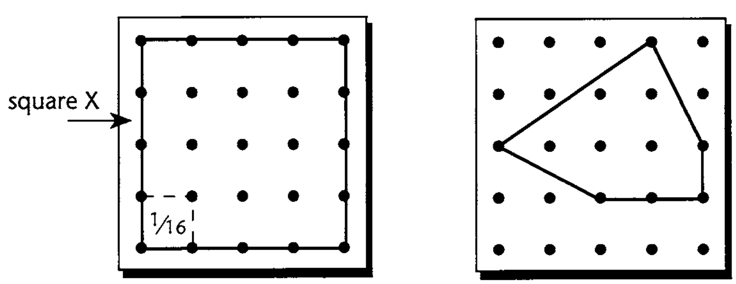
Possible thinking strategies for such a problem are illustrated here (assuming  is the unit of area).



"Regions I and II can be put together to make a square with area 4. So the area of Region III is 6 – 4 = 2."

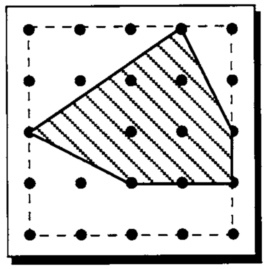
Here are some special notes about the activities of this lesson.

Activity 3: We find that children seem to enjoy creating geoboard shapes and asking each other to determine the corresponding areas. Note that this can be a very challenging task. For example, note the different responses described in the example below.



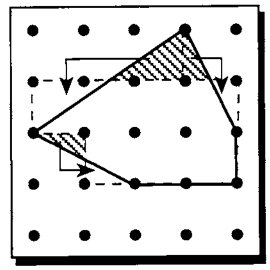
What is the area of this shape, using square X as the unit?

1st response



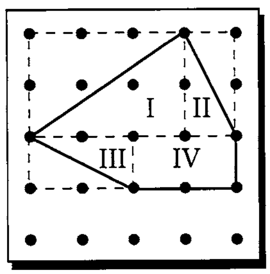
"It looks like the shaded region takes up about $\frac{1}{2}$ of square X. We estimate its area to be $\frac{1}{2}$."

2nd response



"We think the shaded parts can be moved like this. That will make 7 of the $\frac{1}{16}$ squares. We think the area of the shape is $\frac{7}{16}$."

3rd response

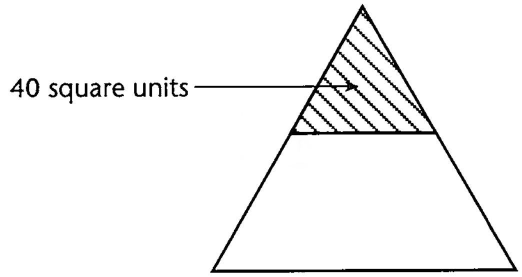


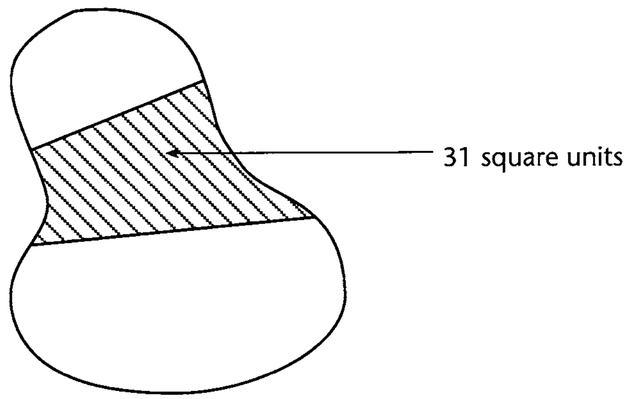
"Regions I, II and III are halves of rectangles.
So: the area of Region I is $\frac{1}{2}$ of $\frac{6}{16} = \frac{3}{16}$.
the area of Region II = $\frac{1}{2}$ of $\frac{2}{16} = \frac{1}{16}$.
the area of Region III = $\frac{1}{2}$ of $\frac{2}{16} = \frac{1}{16}$.
Region IV's area is $\frac{2}{16}$.
So the total area = $\frac{3}{16} + \frac{1}{16} + \frac{1}{16} + \frac{2}{16} = \frac{7}{16}$."

We encourage you to let your children devise strategies for finding these areas. No particular method needs to be "taught". In all likelihood, this will require some judgement on your part. Sometimes it may be most appropriate to estimate the area (as in the first response above); other times, a more accurate strategy (as in the last response) may be offered by a child or by you.

Activity 4: As in other lessons, we help the children recognize that learning continues and that they can be the problem-posers. We are sure your children will have some creative continuations to offer.

Daily Computation Estimate the areas of the unshaded parts. Explain your thinking.





12 Box Covers

You Will Need

- Chapter 7, Measurement
- cereal box (in the shape of a rectangular prism)
- scissors, tape, chart paper

for each child

- at least one snackpack cereal box per child (We recommend boxes about the size of a gelatin box.)
- a copy of Blackline 150 (Box Cover Contest)

Your Lesson

This lesson is a continuation of Contact Lesson 91, Volume 1, and involves a spatial exploration that taps the creativity of all your children. In this exploration, your children are invited to generate different ways of covering a box. In so doing, They can strengthen their spatial awareness, add to their geometric vocabulary and measurement skills, and augment their ability to connect two and three dimensions. The exploration is also related to a more general problem that is of economic concern to many businesses, namely, how to package products in a profitable way.

Ask your children to examine their boxes and, in small groups, make a list of observations about these boxes. Invite volunteers to share their observations in a show-and-tell manner. Your large cereal box can be used for demonstration purposes. Compile a class chart of the observations.

Children may discuss such things as vertices (corners), (right) angles, planes, opposite faces, surface area, parallel (lines or planes), perpendicular (lines or planes), intersecting (lines or planes), congruence or rectangles. This would be a nice time to reacquaint your class with some of this terminology. Don't forget, you can add observations of your own!

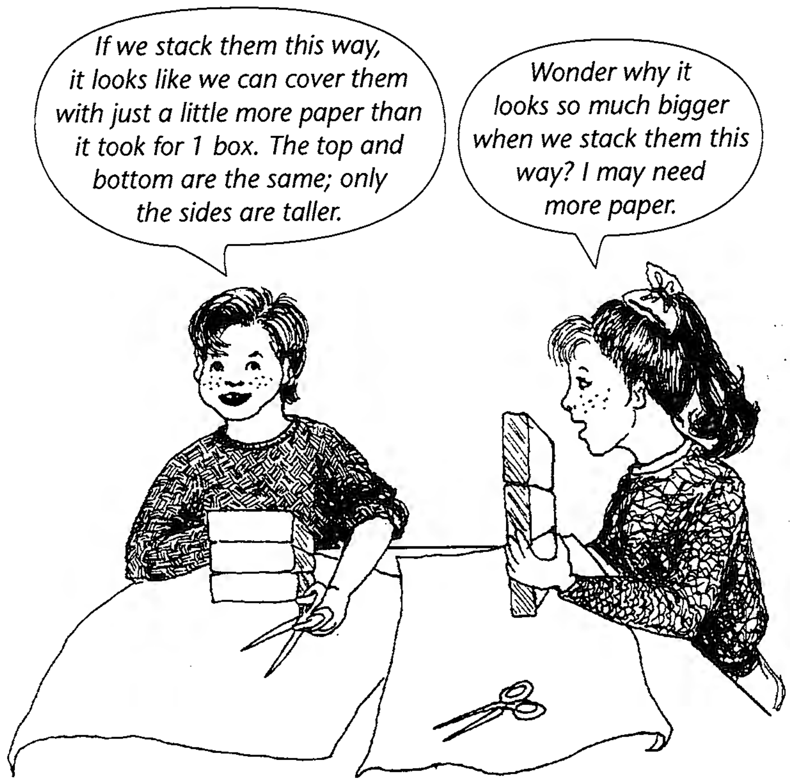
While the children are still working in groups, ask them to imagine their box being unfolded and to make a sketch of what they are picturing. This can be a good mental geometry exercise, so allow some time for discussion of the children's predictions.

Now have the children make some paper jacket covers for their boxes. Challenge them to do this in ways that *cover the box entirely with no overlaps of paper.* Encourage them to search for several solutions. Notice the strategies that are used during this activity. Do some children cut out rectangles for each face and then tape them together? Do some imagine a design first, cut out the proposed design entirely, and then test it? etc. Notice, too, the communication that occurs within the groups. What kind of sharing and cooperation is occurring?

After a while, conduct a sharing of solutions and then allow additional time for the groups to create more designs. At this point, we like to have the children select some of their designs and write about them (see the suggestions for Published Pieces below).

You might also offer additional variations of the activity. Here are two suggestions:

- Have boxes of different sizes and shapes available and ask the children to make covers for them.
- Ask the children to stack at least three congruent boxes together in different ways and make covers for these larger boxes.



What observations do they have about these new covers? Are the new covers related to each other in any way? Are they related to the covers for a single box?

Invite the children to suggest some "What if's" of their own.

Teacher Tips

One *Opening Eyes* teacher asks her children to give oral "sales pitches" about designs of their choice.

Assessment

These presentations could be videotaped as well as used as portfolio entries.

Group Project

Ask each group to create an entry for the Box Cover Contest (Blackline 150). You might wish to make the letter portion of this contest be an individual journal entry.

Opening Eyes to Mathematics **Box Cover Contest** **Blackline–150**

Your team is entering a box cover contest that has the following rules.

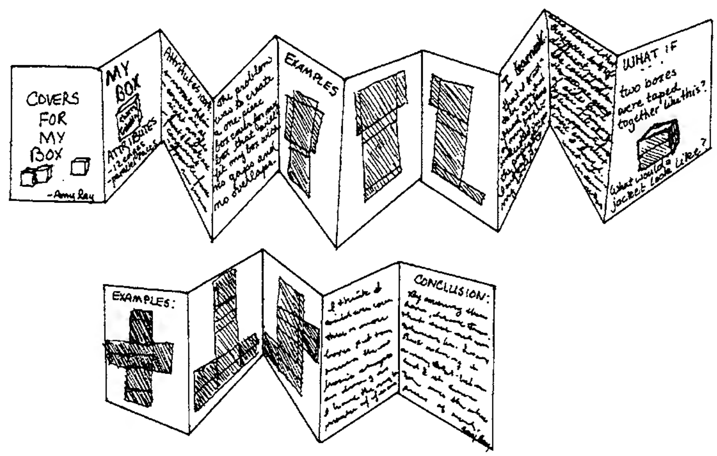
1. Each team must create a paper cover for its box.

2. Each entry must be accompanied by a persuasive letter to the box company. The letter should describe how the entry is superior to all other entries.

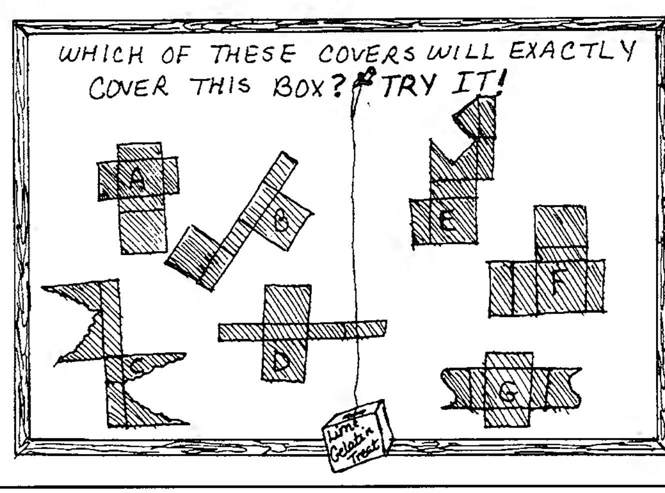
3. Other rules are:
   a) The cover must be in one flat piece.
   b) You are not limited to edge cuts.
   c) You are not limited to straight cuts.
   d) Your cover must fold to cover your box with no gaps and no overlaps.
   e) The use of tape is permissable.
   f) Color or decorative design will not be considered.
   g) You must show evidence of team effort.
   h) You must show evidence of mathematical understanding.

Published Pieces

Have the groups compose an "accordion" book that display covers for their boxes.



Have the groups create bulletin board displays of covers that friends might test.



13 *Base Ten Linear Pieces*

You Will Need

- Chapter 6, Multiplication and Division
- Volume 3, Appendix A, Area Models for Multiplication and Division
- base ten area pieces for the overhead
- transparency of Blackline 151 (Linear Pieces Outlines) for each child

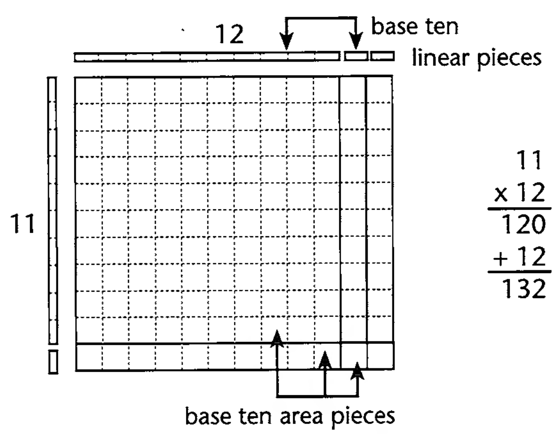
for each team of two children

- base ten area pieces (area pieces were called "counting" pieces in Volumes 1 and 2)
- base ten linear pieces (see Teacher Tips)

Your Lesson

Lessons 13–15 revisit area models for multiplication and division. During these lessons, children can visualize various products and quotients, strengthen their understanding of area and dimension, and gain further experience with "hidden arrays" (i.e., the Distributive Property of Multiplication). Please read the background information discussed in Appendix A before proceeding.

There are two types of base ten pieces used in these lessons. Area pieces are used to form rectangular arrays; linear pieces are used to represent dimensions.

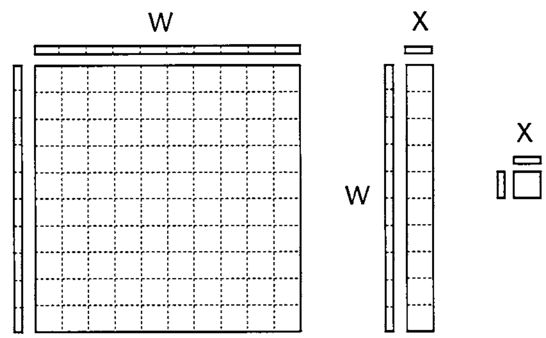


$$\begin{array}{r} 11 \\ \times\,12 \\ \hline 120 \\ +\,12 \\ \hline 132 \end{array}$$

Your children have been using area pieces and linear units throughout *Opening Eyes*. In this lesson, the children are introduced to additional linear pieces.

Activity 1

Distribute base ten area and linear pieces to each team of two children. Ask the teams to examine these pieces and describe any relationships they observe. Be sure to discuss the relative lengths of the different linear pieces. Note, too, that each of these lengths matches a dimension of an area piece.

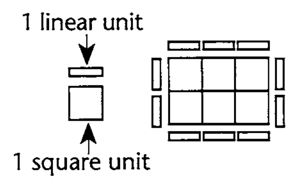


"The length of W is 10 times the length of X."
"W has the same length as the side of a mat."

Tell the class that linear pieces may be used as rulers and have the children use these pieces to measure the lengths of some nearby objects, such as the width of their desks, the length of a book, etc.

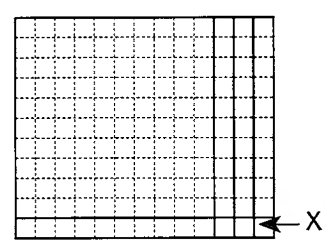
Activity 2

Build the rectangular array shown below at the overhead and discuss its area and dimensions. Review the use of linear units for indicating dimensions, and ask the children to pose a multiplication or division story that could be modeled by the array.

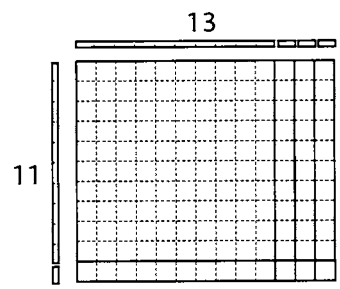


"The area of this 2 by 3 array is 6 square units.
The perimeter is 10 linear units."

At the overhead, build the following rectangular array with base ten area pieces; ask the children to do the same at their workplace.



Ask the children to determine the area, dimensions and perimeter of this array, assuming the square marked X is the unit of area. Conduct a show-and-tell discussion of the children's work. Indicate the dimensions of the array at the overhead with sketches of linear pieces. Have the children also represent these dimensions with their linear pieces. Tell the class that linear pieces (or sketches of them) can be used to represent the dimensions of larger rectangular arrays.



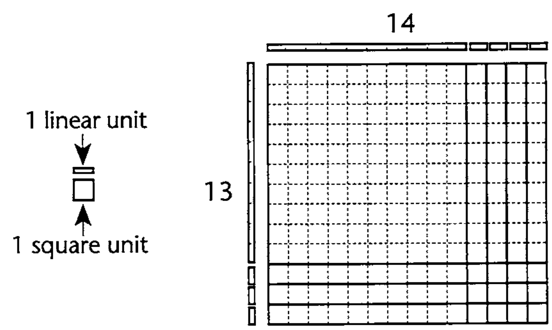
Area = 143 square units
Dimensions: 11 and 13 linear units
Perimeter = 48 linear units

Repeat this activity starting with a different rectangular array.

Activity 3

Continue with exercises such as these:

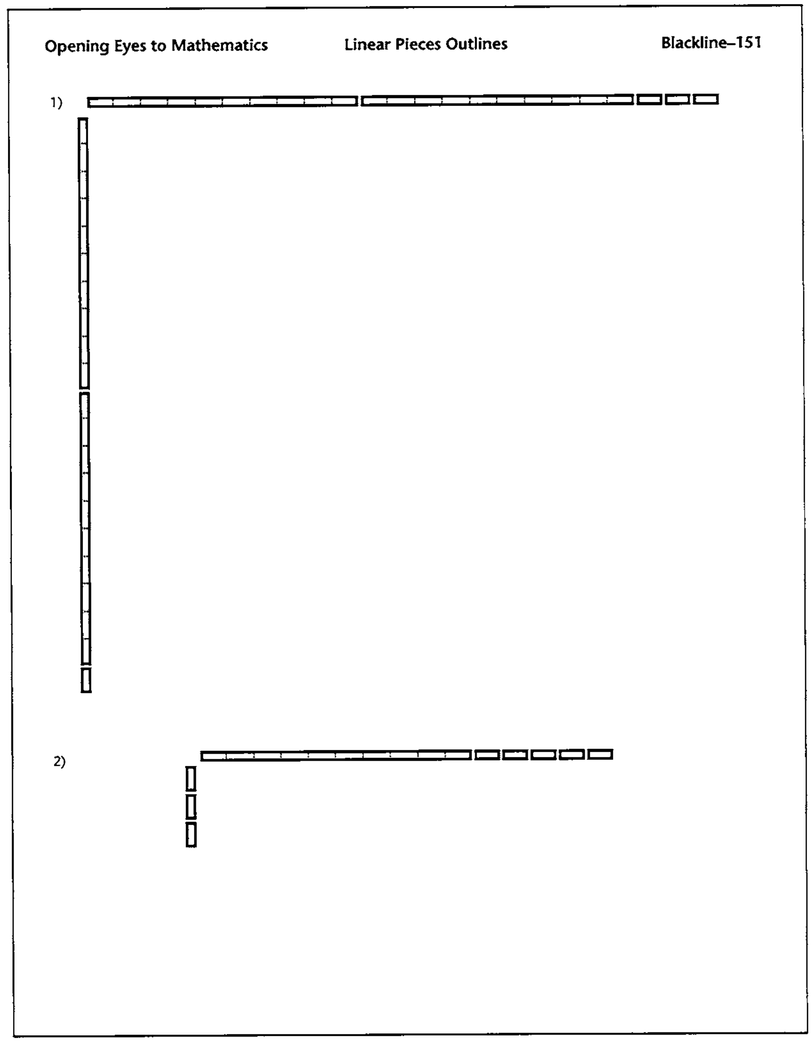
1) Ask the teams to build a 14 by 13 rectangular array with base ten area pieces, and show the dimensions of this array with linear pieces. Have the teams determine the area and perimeter of the array.



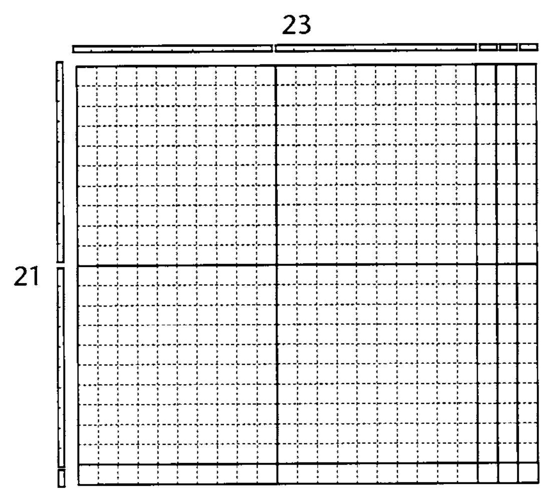
Area = 182 square units
Perimeter = 54 linear units

Repeat, only this time ask each team to build a rectangular array of its choice. Ask neighboring teams to examine each other's arrays, show the dimensions of the array with linear pieces, and determine the area and perimeter of the array.

2) Display the first linear piece outline on Blackline 151 at the overhead. Referring to this, tell the class that linear pieces have been used to form the outline of a rectangular array. What are the dimensions of the array? Discuss.

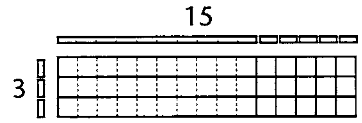


Ask the children to imagine the array itself. Can they picture it and describe it? What is its area and perimeter? Invite a volunteer to complete the array at the overhead.



Area = $21 \times 23 = 483$ square units
Perimeter = 88 linear units

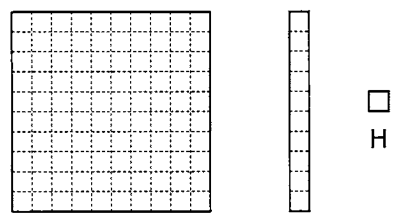
Repeat this activity using the second linear piece outline on Blackline 151.



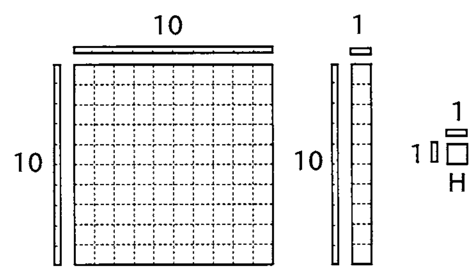
Area = 3 × 15 = 45 square units
Perimeter = 36 linear units

3) Ask each team to build a linear piece outline of a rectangular array of its choice. Have neighboring teams examine each other's outline and determine the perimeter and area of the array.

4) Show the following area pieces at the overhead and have the children set out the same pieces at their seats. Tell the class that piece H (in the diagram) is 1 unit of area.



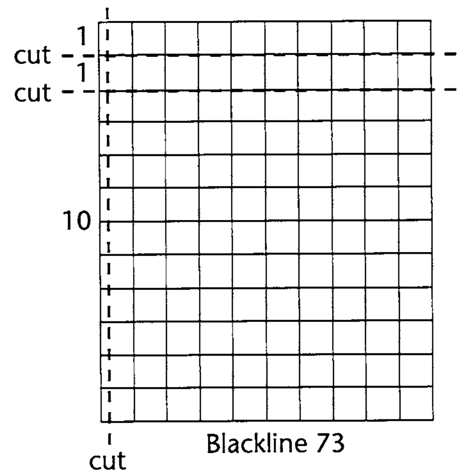
Have the children report the dimensions of each piece and show these dimensions with linear pieces.



Ask the class to imagine larger (and smaller) area pieces. How long would the corresponding linear pieces be? Discuss.

Teacher Tips

You can obtain Large Base Ten Linear Pieces (Catalog #LEP) from the Math Learning Center. These pieces correspond to the Large Base Ten Area Pieces (Catalog #USM) sold by MLC. Alternatively, you can cut linear pieces from Blackline 73, as shown on the following page.



Eventually, of course, children can opt to sketch (or visualize) linear pieces.

Pattern Reminder

Set aside some time in your day to do some Pattern Possibilities. You might examine a remaining sequence starter from Blacklines 117–125.

14 *Area Models for Multiplication & Division, Part I*

You Will Need

- Chapter 6, Multiplication and Division
- Volume 3, Appendix A, Area Models for Multiplication and Division
- overhead base ten area pieces
- a copy of Blackline 7 (Base Ten Area Pieces) for each child (optional)

for each group of four children

- base ten area and linear pieces
- Blackline 8 (Base Ten Grid Paper)
- chart paper and markers

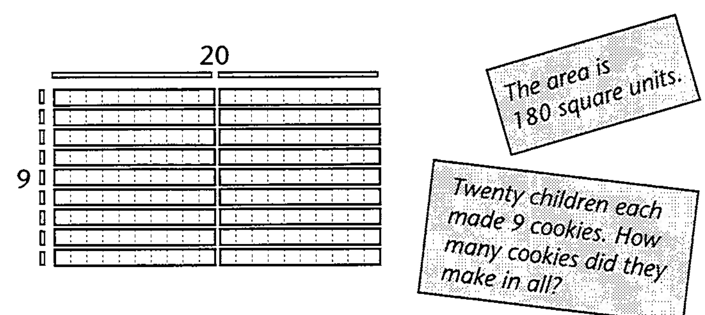
Your Lesson

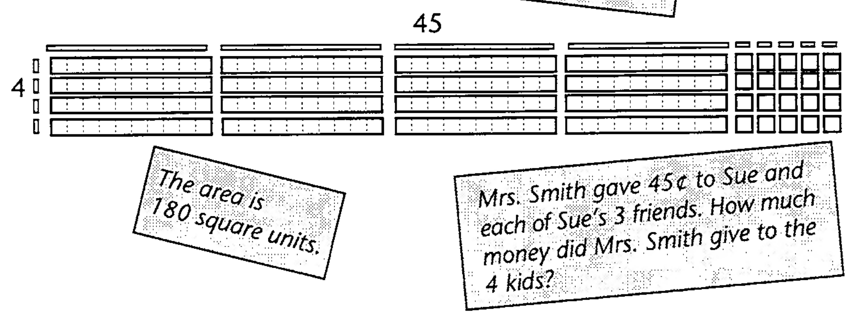
In this lesson, children explore several activities that focus on area models for multiplication and division. These activities extend similar ones found in Volume 2, Insight Lessons 79, 81, 87, 99 and 100; you may wish to revisit some parts of those lessons before proceeding.

Activity 1

Distribute base ten area and linear pieces to each group of four children. Have each group set the following collection of base ten area pieces before themselves: 16 strips and 20 units. Ask each group to do the following exercises: build a rectangular array that uses all of these pieces, determine the area and dimensions of their array, create a multiplication or division story problem that can be modeled by their array.

When ready, invite the children to take a "field trip" to observe each other's work. Several rectangles are possible, two of which are sketched here.





Observe the children carefully as they work. They may need additional time to discuss their arrays and problems before proceeding with the next step of the activity.

Ask the groups to return to their places and have them form other arrays which use their entire collection of 16 strips and 20 units. Have them indicate the dimensions of these arrays with linear pieces and then make a sketch of their work on chart paper. How many arrays are possible? Encourage the teams to seek all the possibilities.

Observe the children as they work and try to gain some insight about their thinking. Some may use trial-and-error completely. Others may apply related knowledge of multiplication or number relationships. Some may also want to use a calculator for the purpose of testing possible divisors.

Ask the groups to also describe any similarities or differences they observe among their rectangles. Post the charts and discuss the results. Invite some of the groups to describe their work and explain their thinking. How can one be sure all the possibilities have been found? (Please see Teacher Tips.)

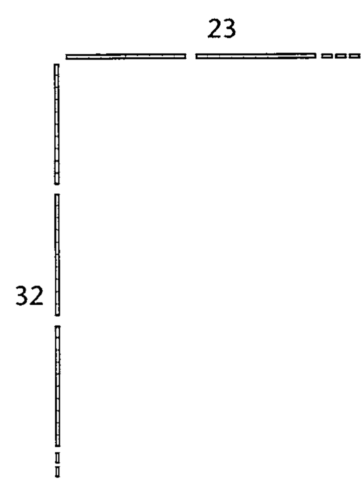
Activity 2

Ask the groups to do this problem: Form a rectangle with dimensions 23 and 32. Make a sketch (or grid paper diagram) of this rectangle and its dimensions. Write a multiplication problem that can be modeled by your rectangle. Describe how you would answer the question posed in your problem.

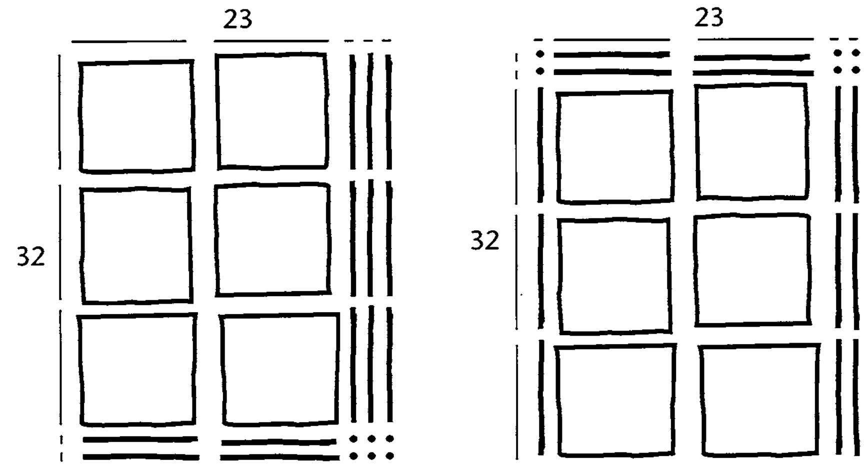
Note how the groups form this rectangle. Do they begin with area pieces and attempt to form the rectangle first? Do any first use linear pieces to show the required dimensions, thereby forming the outline of the rectangle? Or are other ways emerging?

Conduct a sharing time where groups discuss their work with neighboring groups or with the entire class.

This can be an opportunity for the children to begin visualizing the different parts of the array (this is explored further in Lesson 15). To do this, you might have a transparency of the array and its dimensions ready. Display this transparency, exposing only the dimensions. Ask the children if they can picture the area of the array.

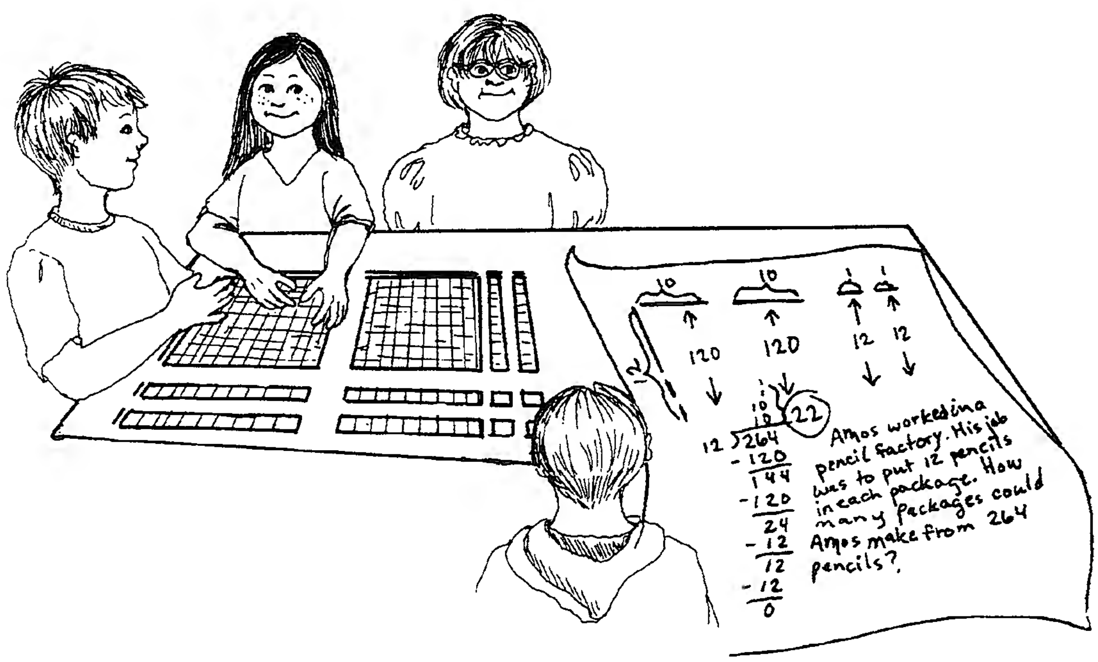


Note that groups may validly form the same array yet have individual pieces in different locations.



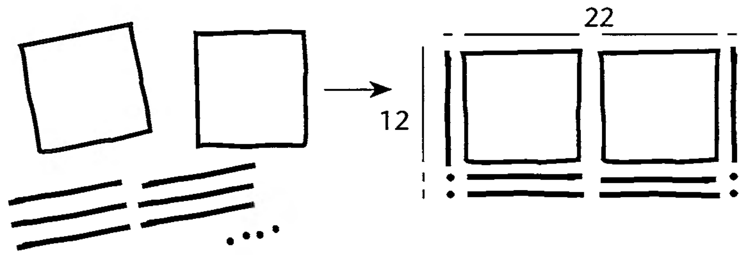
Activity 3

Repeat Activity 2, this time asking the groups to do the following problem. Form a rectangular array that has one dimension 12 and an area of 264 square units. Make a sketch or grid paper diagram of this array and its dimensions. Write a division problem that is modeled by this array. Describe how you see the answer to your problem.

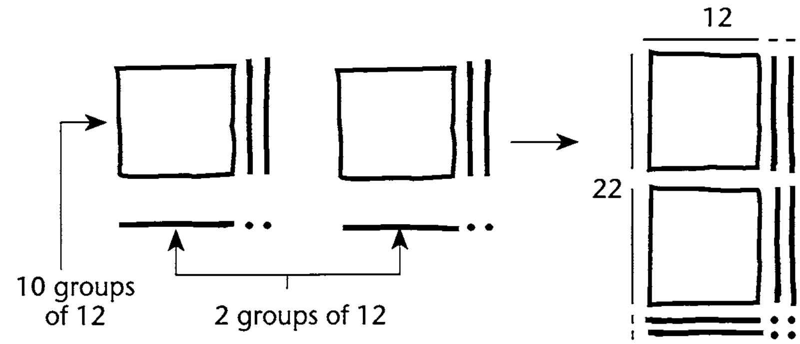


Once again, note the strategies used to form this rectangle.

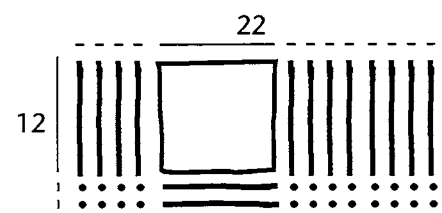
RAUL *We got the minimal collection for 264 and moved the pieces around. The other dimension is 22.*



LI We put the minimal collection for 264 into groups of 12. We made 22 groups and put them together.



RUTH We made a dimension of 12 with linear pieces first. Then we built the rectangle by filling it in with groups of 12 until we had 264 units. After a while we made it go faster by using a mat and 2 strips for 10 groups of 12.



Activity 4

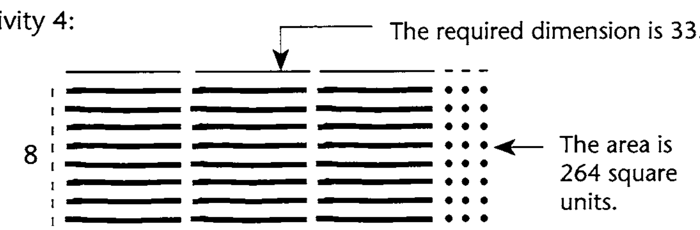
Continue working with an area of 264. Repeat Activity 2, only this time have the groups make a rectangle with area 264 and one dimension 8. What is the other dimension?

Activity 5

Repeat Activity 2 once more, only this time have the groups make a rectangle with area 264 and one dimension 16. What is the other dimension?

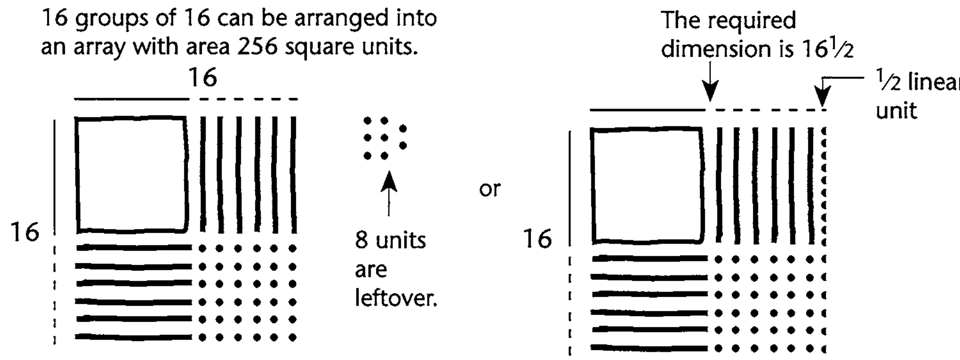
What strategies did groups use in Activities 4 and 5? How did the groups handle the remainder in Activity 5? Here are possible solutions for the problems in these activities.

Activity 4:



This model shows that 264 ÷ 8 = 33

Activity 5:



This shows that 264 ÷ 16 = 16 R 8

Activity 6

As time permits, repeat Activity 2 by exploring other problems that focus on area models for multiplication or division. Be sure to offer problems that provide experiences with both operations.

Teacher Tips

Here are some notes about the activities of this lesson.

Activity 1: Assuming the units have area 1, then a collection of 16 strips and 20 units has a total area of 180 square units. Here are the dimensions of the rectangular arrays that can be formed from this collection:

> 1 by 180, 2 by 90, 3 by 60, 4 by 45, 6 by 30, 9 by 20, 10 by 18 and 12 by 15. (By trading some pieces, a 5 by 36 can also be made.)

Some teams may extend this list by distinguishing an array like 60 by 3 from 3 by 60. This can be their decision.

Activities 2–6: In these activities, children are asked to sketch arrays and their dimensions. Sometimes children with little or no experiences with sketching arrays may be more meticulous than necessary (e.g., drawing the units within a mat or a strip). Some modeling on your part may prevent these children from getting bogged down by unneeded details.

Journal Entry

If you were to ask a friend to build a rectangle, what dimensions would you use? What do you think their rectangle would look like after it was built? Make a sketch of the rectangle showing its dimensions and reporting its area.

(As a suggestion: Some of our children found it helpful to approach this task with the help of Blackline 7, Base Ten Area Pieces. This blackline offers a nice transition from pieces to sketches, since children can cut and paste the pieces on chart paper.)

15 Area Models for Multiplication & Division, Part II

You Will Need

- Chapter 6, Multiplication and Division
- Appendix A: Area Models for Multiplication and Division

for each team of two children

- base ten area pieces, linear pieces and Blackline 8 (Base Ten Grid Paper)
- chart paper and markers
- Blackline 152, 3 pages (Daily Computation Possibilities)

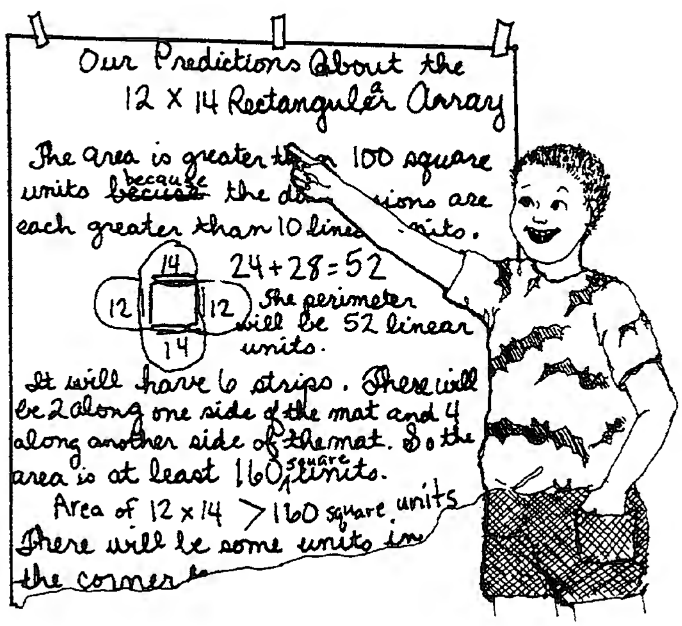
Your Lesson

This lesson provides further experiences with visualizing multiplication and division concepts. Children can use these experiences to construct numerical procedures for determining products and quotients. Proceed with the following activities.

Activity 1

Distribute base ten area and linear pieces to the class, and ask the children to imagine building a 12 by 14 rectangle. Have them refrain from touching the pieces for the moment and instead ask them to offer clues which describe the rectangle. What can they say about the sides of the rectangle? What about the interior of the rectangle?

Chart the clues offered by the class, asking each volunteer to explain the thinking behind their response.



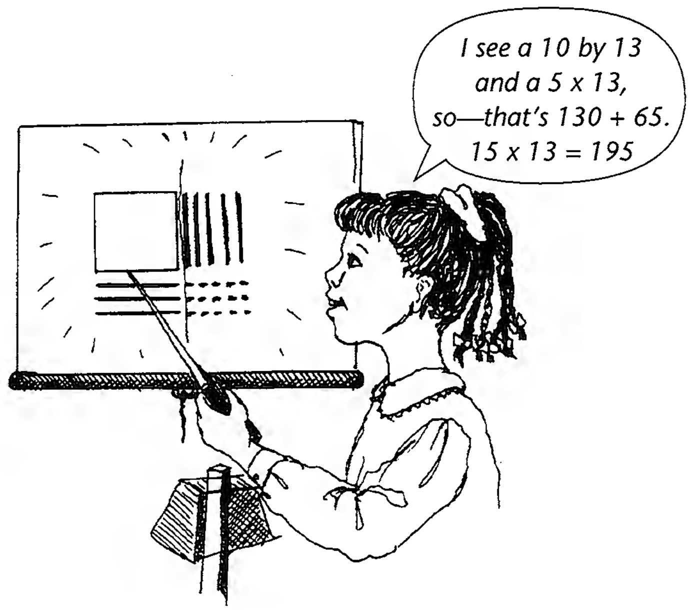
With the children working individually or with a partner, have them build this rectangle and its dimensions. Ask them to compare their results with the list of clues the class compiled. You can circulate among the children, making notes and observations about their work.

Repeat this activity for rectangles of other dimensions, including those suggested by the children.

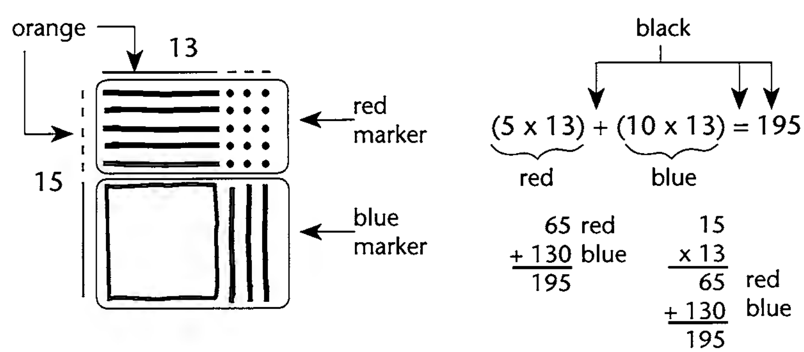
Activity 2

Using the procedure of Activity 1, discuss the construction of a 15 by 13 rectangle. Chart the clues and then have the children build this rectangle at their seats.

Ask a volunteer to build this rectangle at the overhead and then invite a second child to split the rectangle apart into smaller rectangles.



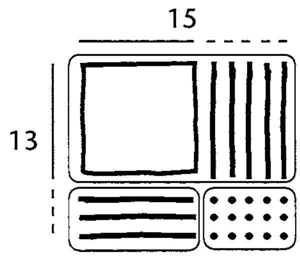
Ask the children to identify each of the smaller rectangles. On chart paper, make a sketch of the original rectangle and its dimensions. Circle the smaller rectangles and note their areas. These areas total the area of the original rectangle; children can suggest different ways of describing this relationship with number statements.



We have found it helpful to color code this chart (see above illustration), so like colors are used to identify particular rectangles and the corresponding part of the number sentences.

Ask the children to examine other ways to partition their 15 by 13 rectangle into smaller rectangles. Have them make a sketch of each partition and sum of the areas of the smaller rectangles.

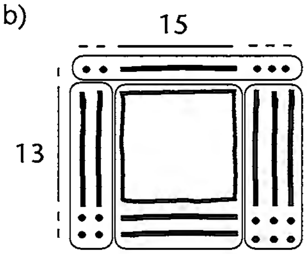
a)



10 x 15 + 3 x 10 + 3 x 5 =
150 + 30 + 15 = 195

$$\begin{array}{r} 13 \\ \underline{\times\ 15} \\ 150 \\ 30 \\ \underline{+\ 15} \\ 195 \end{array}$$

b)



1 x 15 + 12 x 2 + 10 x 12 + 3 x 12 = 195

$$\begin{array}{r} 13 \\ \underline{\times\ 15} \\ 15 \\ 24 \\ 120 \\ \underline{+\ 36} \\ 195 \end{array}$$

The children's partitions provide a context for reviewing the area model of multiplication. Unless it has occurred already, write the statement $15 \times 13 = 195$ beside a sketch of a 15 by 13 rectangular array. Tell the children this statement is one way of reporting the total area of this array. The product (195) is represented by the area and the factors (15 and 13) are modeled by the dimensions of this array.

The total area can also be determined by summing the areas of the smaller rectangles within the children's partitions. This illustrates the Distributive Property of Multiplication. It is this property of multiplication that is the basis for computing products by summing smaller partial products. (Please see pages 54 and 55 of the Teaching Reference Manual for more information and examples.)

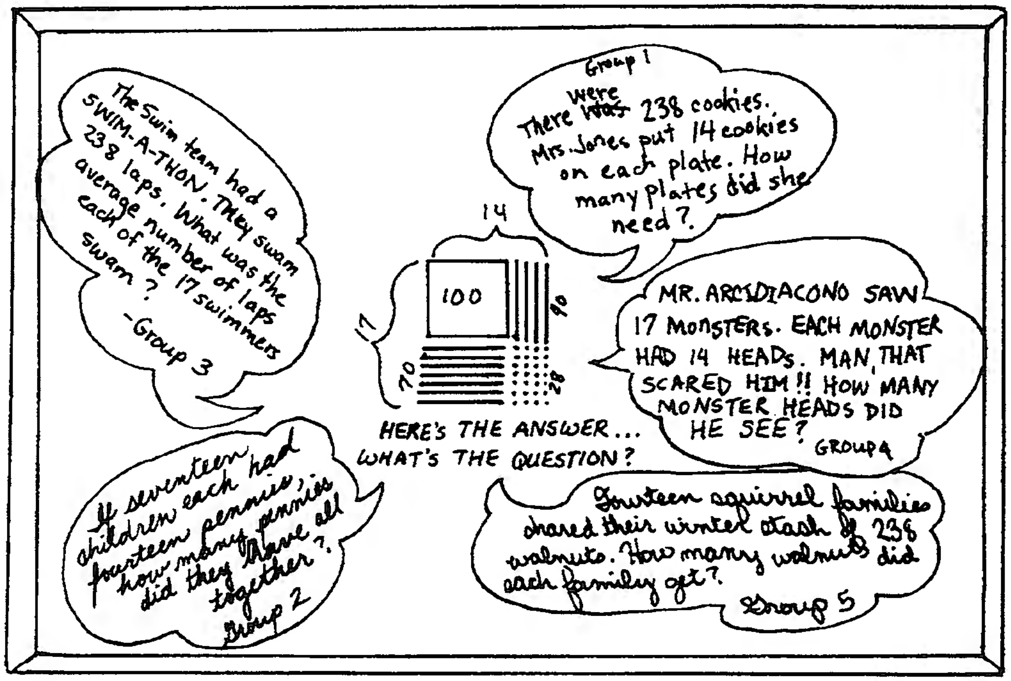
Discuss the Distributive Property with the children, using their partitions as examples. Their work demonstrates that products can be computed in several valid ways. Children can therefore select the way that best reflects their thinking.

Journal Entry

Write a brief paragraph which describes what you know about the Distributive Property of Multiplication. Show how you would determine the answer to 23×17.

Activity 3

Display a sketch of a 14 by 17 rectangular array on a chart. Label the chart "Here's the answer. What's the question?" Working in small groups, invite the children to write multiplication or division story problems that can be modeled by this array.



When ready, bring the class together for a sharing of stories. How is each story modeled by the array? How can each part of the story, including the question and the answer, be visualized? This can also be a time to link mathematics with the language arts portion of your day. Discuss the writing process. Do any of the stories need to be edited?

Repeat this activity beginning with sketches of other rectangular arrays. Of course, you can always create stories of your own, particularly if the children's work seems to be focusing on one operation only.

Journal activity: Sketch a rectangular array of your choice. What story could it tell?

Activity 4

This activity focuses on using rectangular arrays as models for division. Ask the children to work in small groups on this problem:

With base ten pieces, build a rectangular array that has area 198 and one dimension 12. What is the other dimension of this array?.

Take note of the procedures used by the groups as they work. Try to gain some insight into the thinking behind each procedure. At this point, have the groups take a "field trip" to examine each other's work. Ask the appropriate group at each station to explain the process they followed to make their array. Also, invite the class to suggest some division story problems that could be modeled by the array.

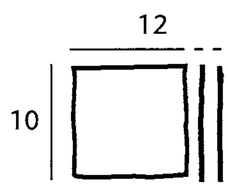
Ask each group to sketch their completed arrays (and its dimensions) on a piece of chart paper. (Please be sure to check that each group's sketch matches their physical model.) Have the groups write a description of their procedure for solving the problem which uses numbers and arithmetical symbols (see Teacher Tips).

As an example, one group might proceed as follows:

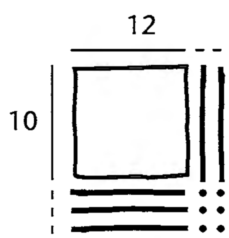
BENJAMIN *Let's get out 1 mat, 9 strips and 8 units. That's an area of 198 for sure. I know for sure it would have 10 twelves in it—I can see it!*

RUTH *It might help to show a dimension of 12 with linear pieces.*

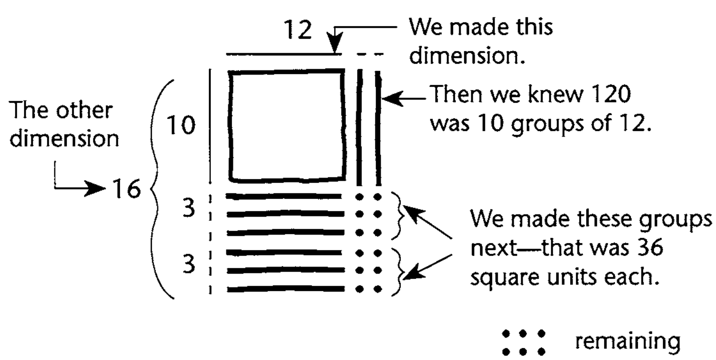
(Benjamin's group begins to build the required array. They also represent part of the missing dimension.)



JOSE *Hmm. Our area is only 120 so far. We still have 76 square units to go. I know we can get 3 more 12's at least.*

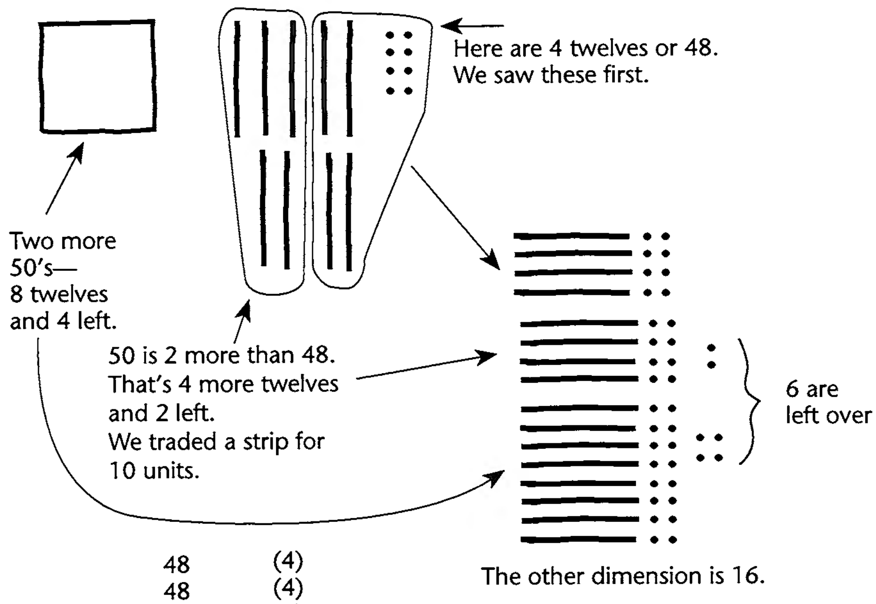


SUZANNE *Look, we've got 42 square units left. That's enough for 3 more groups of 12 or 36. But there will be some left over.*



Twelves	Area
10	10 x 12 = 120
3	3 x 12 = 36
3	3 x 12 = 36
16 groups	192
	plus 6 left over

A second group might start with 198 square units and make groups of 12, perhaps as illustrated here.



48	(4)
48	(4)
96	(8)
192	16 twelves

In a large group, discuss some of the methods that emerged. Have volunteers report how their groups proceeded to build the required array and share their sketches and numerical descriptions.

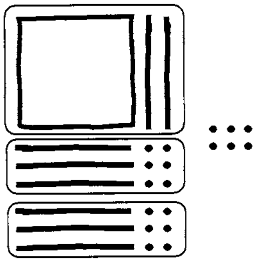
Teacher Tips

Children may prefer to make grid paper diagrams instead of freehand sketches, so have base ten paper available.

The numerical descriptions offered by the groups in Activity 4 represent division algorithms constructed by the children. While they may look different from the standard algorithms we remember, they are still valid. It is important to recognize these algorithms and not feel bound by particular standard procedures. Notice, too, that children were given an opportunity to verbalize their thinking prior to describing it with numbers. This seems to be a critical step in the process of constructing algorithms.

The following illustration shows how the children's procedures in Activity 4 may connect with more familiar looking algorithms.

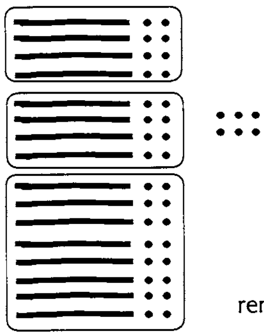
Benjamin's group:



```
              3 \
              3  } 16
             10 /
         12)198
          - 120   (10 x 12)
             78
           - 36   (3 x 12)
           - 42
           - 36   (3 x 12)
   remainder  6
```

Check:
10 x 12 + 3 x 12 + 3 x 12 =
20 + 36 + 36 =192
192 + 6 = 198

Second group:



```
              8 \
              4  } 16
              4 /
         12)198
           - 48   (4 x 12)
            150
           - 48   (4 x 12)
            102
           - 96   (8 x 12)
   remainder  6
```

Check:
4 x 12 + 4 x 12 + 8 x 12 =
48 + 48 + 96 =192
192 + 6 = 198

Daily Computation

This would be a good time to focus your daily computation time on multiplication and division. Ask your children to compute various products or quotients and to solve story problems. Examples appear on Blackline 152, Daily Computation Possibilities (3 pages). Spread the problems over several days, and don't forget to include some mental math, estimation and large numbers! Let the children decide which calculating option they wish to use (see Appendix A).

16 Congruence & Area

You Will Need

- Chapter 11, Geometry
- a transparency of Blackline 153 (Congruence and Area)
- transparencies of Blackline 68 (Geoboard Paper) and Blackline 154 (Isometric Dot Paper)
- overhead geoboard and rubber bands
- blank transparencies

for each child

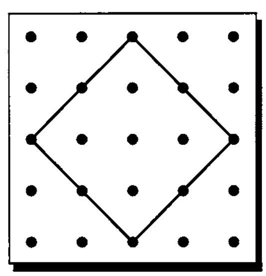
- square geoboards and rubber bands
- 2 copies of Blackline 153 (Congruence and Area)
- copies of Blackline 68 (Geoboard Paper) and Blackline 154 (Isometric Dot Paper) available
- a copy of Blackline 155 (Congruence and Area Homework)
- scissors and tracing paper available

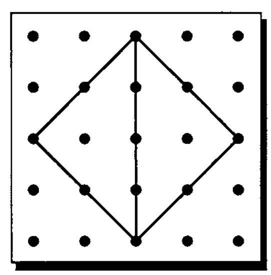
Your Lesson

Some geometric concepts are, at the same time, distinctly different, yet closely related. Congruence and area are two examples of such concepts (later, in Lesson 37, we will include symmetry with these two). Congruent shapes necessarily have the same area; but shapes with equal areas need not be congruent. We seek, in this lesson, to help children distinguish between these concepts. We like to begin with a variation of Contact Lessons 67 and 68, Geometry—Congruence, found in Volume 1, page 89.

Activity 1

Show the class the square (pictured below on the left) and ask the children to make it on their geoboards. Add another rubber band splitting this square into two regions as shown to the right and have the children do the same.

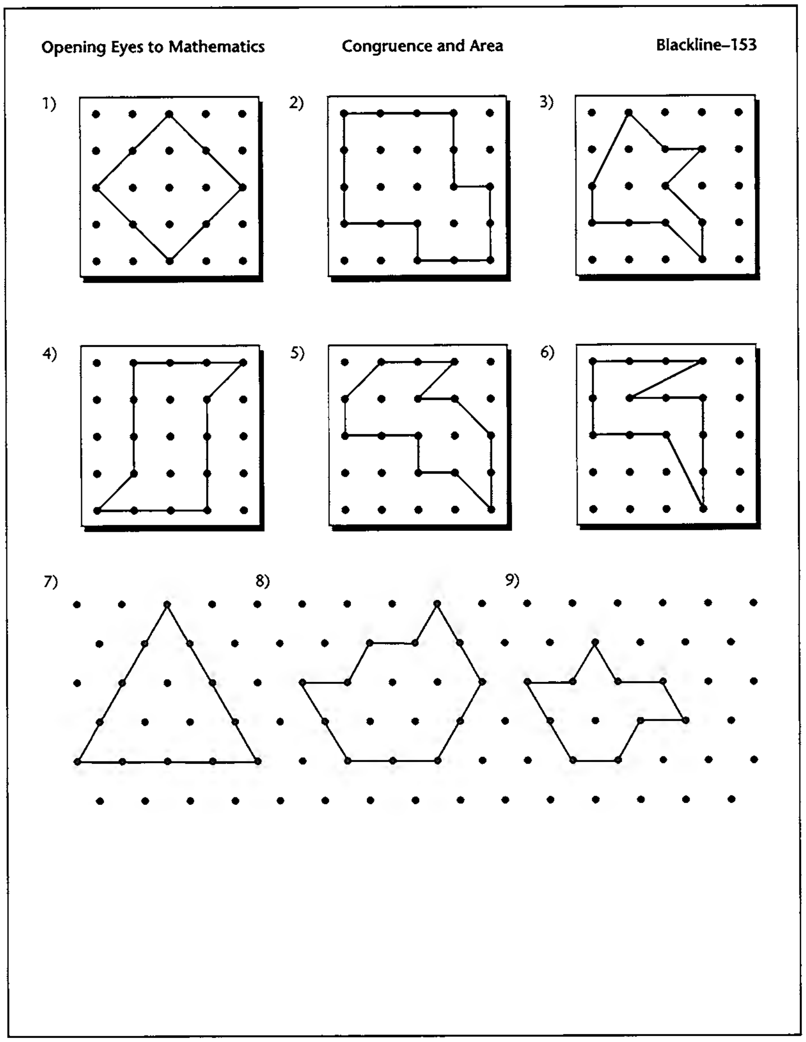




Invite the class to make observations about the resulting figure. Discuss these observations in a show-and-tell manner.

- The square is divided into two triangles.
- The two triangles have the same area.
- Both triangles have a right angle. (See Teacher Tips.)
- The area of each triangle is half of the square's area.

Are the two triangles congruent? Ask the class to devise some methods for answering this question. Be sure to have volunteers demonstrate these methods. Some children might cut out one of the triangles and fit it over the diagonal line of symmetry formed by the second rubber band. Others may describe a rotation that puts one triangle exactly on the other.

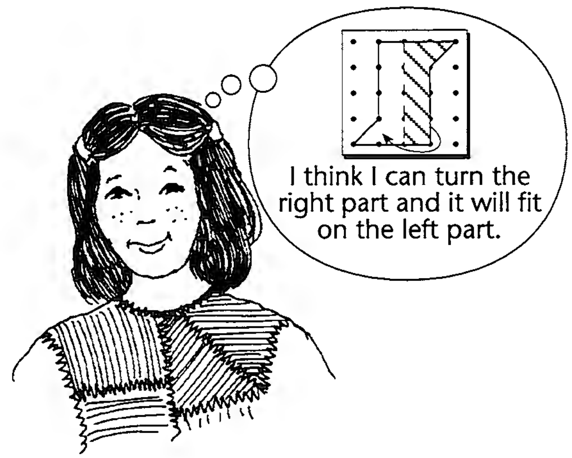


Distribute Blackline 153 and divide the class into teams of two. Ask the teams to work on this problem:

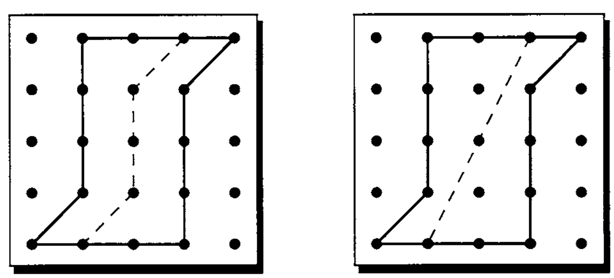
TEACHER *Examine each shape on your blackline. Where possible, see if you can find some ways to split each shape into two regions that have the same area and are congruent. Record your solutions on geoboard paper.*

(Note shapes 7–9 are drawn on isometric dot paper.)

This exploration can strengthen your children's mental geometry. In fact, children often imagine the effect of splitting a shape in a certain way as a first approach to the problem. Encourage them in this effort, for this mental activity is an underlying goal of the lesson. Doubtful solutions can be tested by cutting or tracing one of the two regions created by a split and seeing if that region can be placed exactly on the other.



Some of the shapes can be split into two congruent regions in more than one way. As appropriate, encourage your children to look for multiple solutions. While the exact number of solutions is not necessarily the main concern here, the search for different answers can promote the mental work described above and reinforce your children's awareness of congruence.



Here are 2 ways to split shape 4 into 2 congruent parts. Are there others?

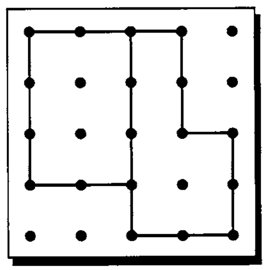
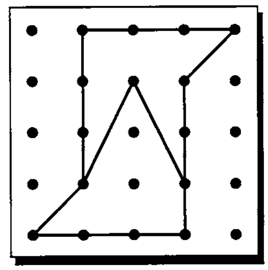
Provide time for the children to share their work in some way (e.g., with a neighboring team, displays on chart paper or as part of other published pieces, demonstrations at the overhead, etc.) Take note of the variety of solutions and encourage children to test solutions which seem questionable.

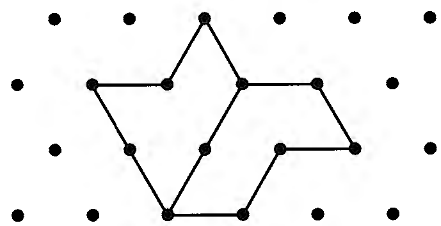
Activity 2

Distribute fresh copies of Blackline 153. Ask the teams to examine each shape on this blackline once more and work on the following problem. Share solutions and thinking as above.

TEACHER *See if you can find some ways to split each shape into two regions that have the same area but are not congruent. Record your solutions on geoboard paper.*

Here are possible answers for three of the shapes:

2)  4) 

9) 

This problem stands in contrast with the previous one. Together, the two problems illustrate that congruent regions necessarily have the same area. Regions that have the same area, however, are not necessarily congruent.

Teacher Tips

Use the children's observations in Activity 1 as an opportunity to review (or introduce) concepts or vocabulary.

Homework

Ask the children to work the problems on Blackline 155, Congruence and Area Homework (shown on following page). Answers may be recorded on geoboard paper. Provide some time during the next class for children to share their work on these problems.

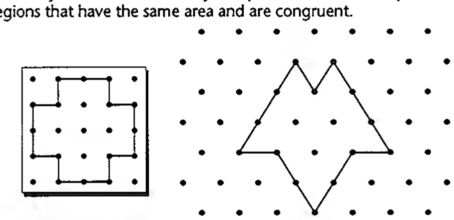
Reminder

What sequence starters on Blackines 117–125 are yet to be explored? Choose one of these starters (or one of your own) and examine some pattern possibilities.

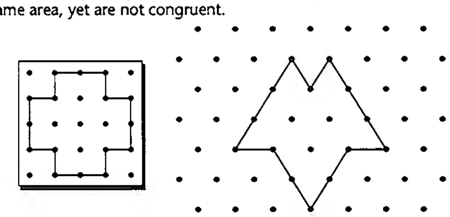
16 Congruence & Area (continued)

Opening Eyes to Mathematics **Congruence and Area Homework** **Blackline–155**

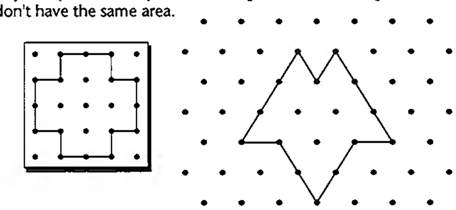
1. See if you can find some ways to split each of these shapes into two regions that have the same area and are congruent.



2. Here are the same shapes as in Problem 1. Now, see if you can find some ways to split each shape into two regions that have the same area, yet are not congruent.



3. Here are the same shapes once more. See if you can find some ways to split each shape into two regions that are congruent but don't have the same area.



17 Operations on Numbers, Part I

You Will Need

- Chapter 5, Addition and Subtraction, and Chapter 6, Multiplication and Division
- Volume 3, Appendix B: Operations on Numbers

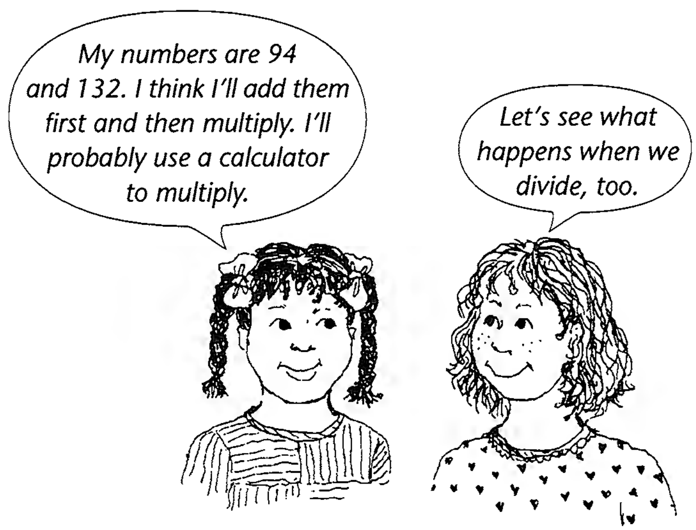
  for each child
- materials for modeling number operations and performing calculations: base ten pieces, grid paper, calculators, etc.
- Number Card Packets (optional)

Your Lesson

This lesson is an expanded version of what was described in Lesson A, Daily Computation. It can be a time for practicing computation, making decisions about calculating options and mathematical exploration.

Have the children choose two arbitrary numbers. For example, the children might select the last two digits in the year along with their favorite athlete's jersey number. Any two numbers can be used. Number Cards are a good source of examples and, of course, you can choose numbers to suit particular mathematical purposes.

Invite the children to perform different arithmetical operations (e.g., add, subtract, multiply, divide, average) on the two numbers. Leave it to the children to choose the calculating option(s) for performing these operations.



Encourage the children to evaluate the options they have chosen. How do they feel about their choices? Were the choices appropriate to the calculations? Would it have helped to make a sketch or model for a particularly troublesome calculation?

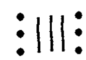
Ask the children to explore further. They can pursue some "What If's?" of their own or examine some posed by you. Here are some examples.

What if you added your numbers in the opposite order? Would the sum be the same? Why or why not? What happens if you subtract (multiply, divide) them in the opposite order?

The following dialogue illustrates some of the mathematical discovery and discussion that can occur during this time.

GEENA *The difference between 63 and 27 is 36. I made a sketch.*

$$\begin{array}{r} 63 \\ -\,27 \\ \hline 3 \\ 30 \\ +\;3 \\ \hline 36 \end{array}$$



ROB *You're right. I put 63 – 27 in the calculator and got 36.*

LISA *Wait a minute. I used a calculator, too, but mine shows –36.*

GEENA *Did you do it like Rob?*

LISA *Well, I put the 27 in first. I went 27 – 63 and pressed the equals button. See it gives –36. Why is that?*

JIM *I don't know. I'm confused.*

GEENA *I guess that's like playing a game. If your score is 63 and mine is 27, then I'm 36 points behind.*

LISA *Yes, and I'm 36 points ahead.*

Note the role of the calculator in this dialogue. Lisa's observation of the –36 has caused the children to wrestle with the non-commutativity of subtraction.

The ending comments in the dialogue also point to a subtle aspect of the difference model for subtraction. This model is commonly used in comparison situations such as Geena describes with her game: Geena's score is 63 and Lisa's is 27. How do these scores compare?

Subtraction can be used to communicate this comparison in two different ways.

63 – 27 = 36 (From Geena's view, she is ahead by 36 points.)

27 – 63 = –36 (From Lisa's view, she is behind by 36 points.)

In practice, the choice between these two ways is governed by the information to be communicated.

The above dialogue and comments reflect one way the class might proceed. It's important to recognize that other twists may occur, depending on the "What If?" being explored. We urge you to read two other scenarios (involving division) described in Appendix B in this volume.

Teacher Tips

As you can tell from the discussion (and from Appendix B), this lesson appears simple in nature, but is potentially quite complex. This is okay. After all, mathematics is a complex subject. We urge you to discuss the complexities as they occur—in fact, if the issues aren't raised by the children, you can raise them with the questions you ask. For example, you might ask what happens if one divides by zero or initiate a discussion about how to interpret answers generated by a calculator.

However, it is not necessary to bring closure to the discussion or to consider every detail at this point. Also, children needn't always be exact in their answers, particularly with fractions and decimals. This can be an opportunity for children to strengthen their estimation skills.

The discussions are ongoing and will resume in later grades. For now, what happens in your class can help children develop their number and operation sense, recognize there is more to learn, and prepare for similar work in *Visual Mathematics.*

Journal Entry

Here are two possibilities:

Select two of the numbers you used in this lesson and describe one of the operations you performed on these numbers. Support your answer with a sketch. Why do you feel your answer is reasonable?

Think about one of the calculating methods a friend used during this lesson. Describe this method and apply it to another calculation.

Published Piece

Your children might enjoy creating a bby book that describes how they operated on numbers today.

Homework

The main activity of this lesson can be done at home with a family member or friend. Ask each child to select two numbers and perform various operations on them. If possible, send the base ten pieces home in case they're needed. Also, have grid paper available for use.

18 Analyzing Graphs

You Will Need

- Chapter 10, Data Analysis and Graphing
- selected graphs found in newspapers, magazines, textbooks, student publications from different grade levels, etc. (found or brought in by you or your children)
- transparencies of the above graphs
- copies of the above graphs for each team of two children

Your Lesson

One of the challenges of the Information Age is to communicate and process information in helpful and economical ways. Graphs are a visual way of doing this, as is evidenced in newspapers such as *USA Today*. Graphs are commonly used to summarize and display statistical information and trends.

In this lesson, your children are asked to analyze graphs taken from different sources. Conduct a large-group discussion about one of the graphs you've collected.

- What is the graph claiming?
- Is it possible to tell how the data was collected?
- Did the data come from a sample? Was the sample random? Was it stratified? (See Teacher Tips)
- How was the graph constructed?
- What is the graph *not* telling us?
- Does the graph display a good use or a misuse of statistics?
- What events or situations might have prompted this graph?
- Is the information presented in a clear manner? Or would it have been better to display it in a different way?
- What further questions might be asked about the information?

You might discuss other graphs in the same way. Alternatively, ask the children to work in pairs and write brief descriptions about some graphs, addressing questions like those above. Consider, too, giving each child a copy of a graph to be analyzed in their journal.

Here are some other general questions for discussion (or for journal responses):

- If you were working for a newspaper, what information might you want to gather? How might you display that information?
- Why do people use graphs? What purposes do graphs serve?

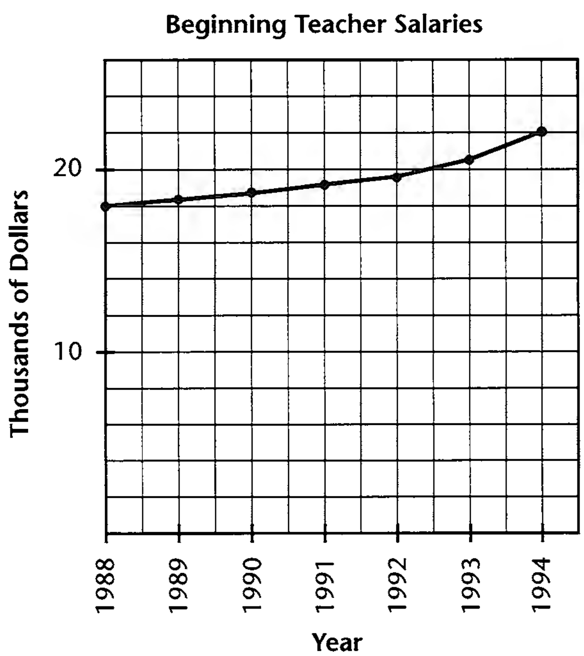
Teacher Tips

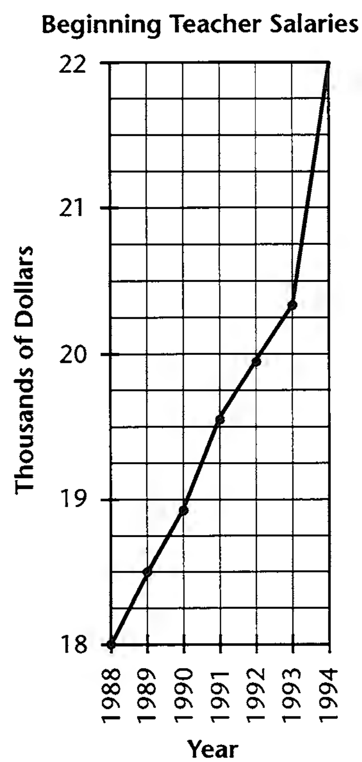
Consider asking the children to find a different way to present the information on one of the graphs. For example, if the original graph is a pie chart, perhaps the children can display the same information in a bar graph.

18 Analyzing Graphs (continued)

Misuses of statistics are often related to sampling procedures that are not random. For example, one would question any inference about the preferences of the entire school that is based on a sample drawn from just one grade (see Opinion Polls, Lessons 33 and 34 of this volume). Sometimes, to ensure representation from all parts of a population (such as grade levels within a school or annual incomes of different professions), the population is divided into appropriate subcategories. Random samples within each subcategory are then made and compiled. This type of procedure is called "stratified sampling".

Other misuses of statistics can occur when data is drawn from small samples or presented in a distorted manner. How do you feel about the graphs below? Both display essentially the same information, yet which one might an advocate for higher beginning salaries present to the public? Which graph might an opponent of higher beginning salaries present in support of their argument?





Darrell Huff's *How to Lie with Statistics* (W. W. Norton & Company, Inc.) is a good source of information about misuses of statistics.

Daily Computation

Ask your children to solve this problem: Tickets to a play cost $2 and the play was sold out last Saturday. The play was held in a theater that had 42 rows of seats with 32 seats in each row. How much money was made in ticket sales for last Saturday's performance?

19 Softball Tournament

You Will Need

- materials for solving problems and displaying results
- a transparency of Blackline 156 (Softball Tournament)
- a copy of Blackline 156 and 157 (Softball Tournament) for each team of two children
- a copy of Blackline 158 (Discussion Card War Tournament) for each child

Your Lesson

Your children have undoubtedly participated in (or at least are familiar with) tournaments of different sorts. These tournaments might have been athletic (softball, NBA playoffs) or scholastic (college bowl types) in nature. Have you ever attempted to schedule one of these events? If so, you may have found out this can be a challenging problem.

This lesson focuses on the problem of scheduling a tournament. The activities draw upon your children's ability to organize and interpret information. The lesson can link naturally with Lesson 18, Analyzing Graphs.

Activity 1

Ask the children to form pairs and work on Problem 1 of Blackline 156.

Opening Eyes to Mathematics | **Softball Tournament** | **Blackline–156**

1. Five teams, A, B, C, D and E are going to play a round-robin softball tournament, where each team plays every other team exactly once. How many games will be played in this tournament? Create a display which shows the opponents in each game.

2. Suppose a sixth team, F, joins the five teams in Problem 1, for a round-robin tournament. How many games will be played in this six-team tournament? Create a display which shows the opponents in each game.

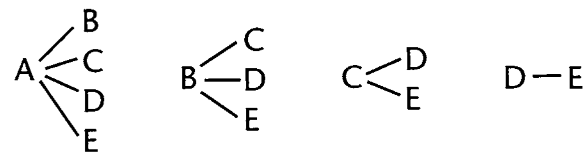
3. Continue on. How many games will be played in a 10-team round-robin, where each team plays every other team exactly once? How about if there were 15 teams?

19 Softball Tournament (continued)

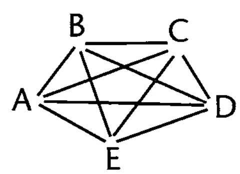
Provide time for partners to share their results. Some may list the opponents in a somewhat random way. Others may attempt to organize the schedule of games, as illustrated here.

a) A – B
A – C
A – D
A – E
B – C
B – D
B – E
C – D
C – E
D – E

b)



Sometimes a graph is used to display the games. Construct the following diagram on the overhead, explaining that each line connects two opponents.



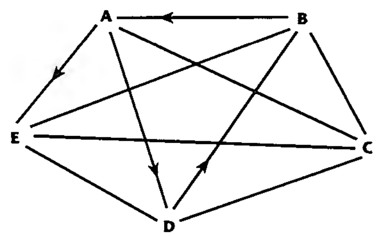
Continue the lesson by asking the children to investigate Problems 2 and 3 on Blackline 156. What happens if the number of teams continue to increase? Invite the children to predict the number of games that will be played. Can the children support a general conclusion about the number of games?

Activity 2

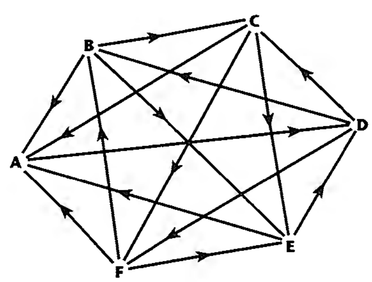
You may wish to introduce "directed graphs" as a way of reporting the victor in each game. Part 1 of Blackline 157 shows the results of a tournament in progress. The arrow on each line points to the winner of the game.

Opening Eyes to Mathematics | Softball Tournament | Blackline–157

Part 1



Part 2



Display a transparency of Part 1 and discuss the graph with your class. Who won the game between A and B? Name some other winners. How many games remain to be played? Suppose D beats C, how could that be shown on the graph?

The directed graph of Part 2, Blackline 157, displays the results of a completed 6-team tournament. Ask your children to analyze this graph and summarize all they can deduce about the tournament.

The homework on the next page provides a follow-up to these activities. Ask the children to attempt this homework and report their results for discussion at the next class.

Teacher Tips

The underlying spirit of this lesson has much to do with organization. For that reason, it's important to provide time for children to consider for themselves how they might organize the schedule of games. It is also important for children to see how others (including you) would organize the same information. One young girl predicted the total number of games in this way:

I just listed the teams with a blank like this:

A ____

B ____

C ____

D ____

E ____

I can imagine 4 teams in the blank with A, 3 in the blank with B, 2 with C, 1 with D, and 0 with E since E has been paired with the others already. That's 4 + 3 + 2 + 1 or 10 games.

Discuss the different ways of organizing that emerge during the lesson. How do the children feel about these methods? What are the advantages and disadvantages of each? Questions such as these can help children recognize they have options when it comes to organizing information.

Your children might enjoy following a favorite team (e.g., a professional or high school team) throughout a tournament. Information about that team can be gathered and posted daily.

Homework

Distribute individual sets of Multiplication/Division Discussion Cards and copies of Blackline 158 (see following page). Ask each child to play Discussion Card War with a family member or friend and then answer the questions on the blackline.

Daily Computation

Children are generally interested in the performance of local teams or players. Consider using game scores or individual statistics as part of Daily Computation problems.

Pattern Reminder

Keep the pattern possibilities going with one of the remaining sequence starters on Blacklines 117–125.

Opening Eyes to Mathematics — Discussion Card War Tournament — Blackline–158

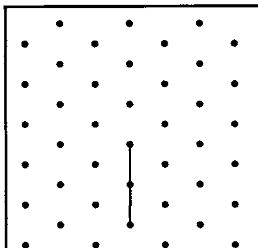
1. Suppose four people are going to compete in a Discussion Card War tournament, where each person plays every other person exactly once.

How many games will be played in this tournament? Create a display which shows the opponents in each game.

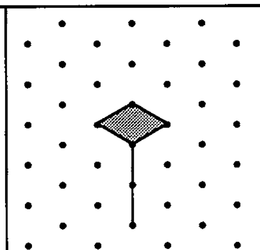
2. If each member of our class played one round of Discussion Card War with every other member, how many games would be played? Explain.

Note

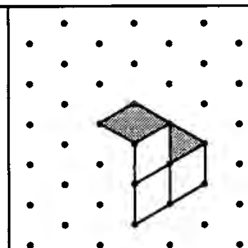
In Lesson 20, your children will be asked to sketch cube towers on Blackline 154, Isometric Dot Paper, *turned sideways*. To prepare for this, you might spend some time demonstrating how to make such sketches and then have the children practice. We have found the following sequence of steps helpful.



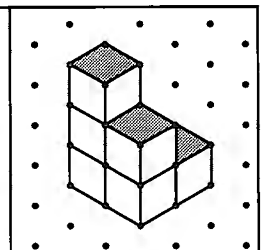
Establish edge line between the front and the right side.



Shade in top square of this stack.



Sketch right side, again shading in top square in stack.



Sketch front side, shading in top square in stack.

20 Cube Towers, Part I

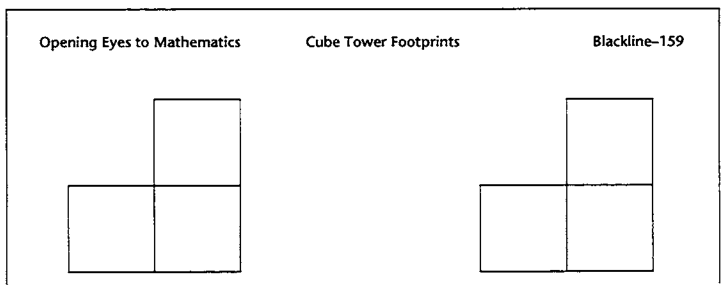
You Will Need

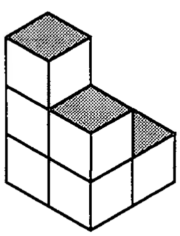
- Chapter 11, Geometry, and Chapter 2, Sorting
- a transparency of Blackline 159 (Cube Tower Footprints)
  for each pair of children
- about 60 wooden cubes (or hex-a-links)
- 2 copies of Blackline 159 (Cube Tower Footprints)
- copies of Blackline 154 (Isometric Dot Paper)
- chart paper and markers
- 2 loops of yarn

Your Lesson

Children generally enjoy making "buildings" from items such as blocks, Legos, cards, etc. Such activity appeals not only to a child's creativity but can also strengthen their spatial awareness. This lesson engages children in activities that do both. In addition, children are asked to relate three-dimensional models (called cube towers) with two-dimensional views (front, side and top) of these models.

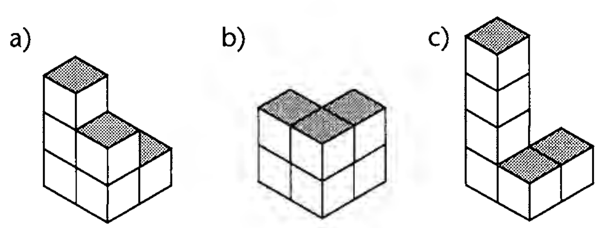
Activity 1





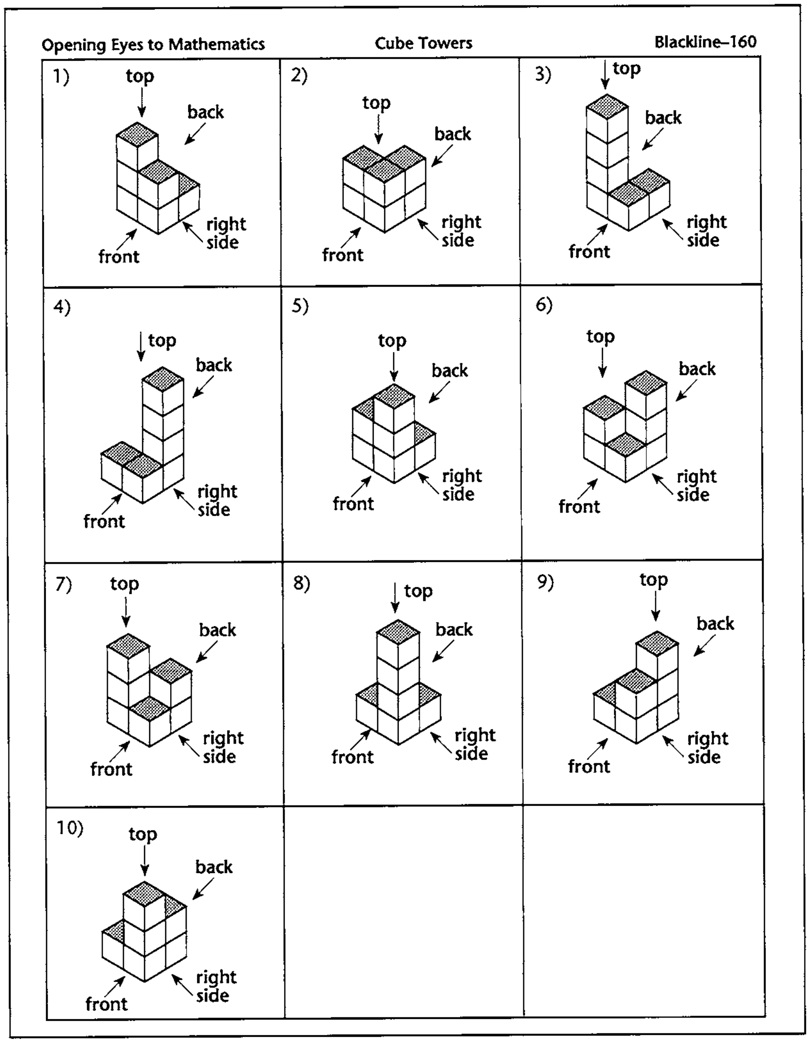
Display a transparency of Blackline 159 and ask the class to imagine that each diagram is like a "footprint" of the bottom of a cube tower. Build the tower pictured to the left on the first footprint and ask each child to do the same.

Point out that this tower uses just 6 cubes and that neighboring cubes always share entire faces (with no gaps or overlaps). Ask the children to now build a different tower of 6 cubes (keeping the restriction about neighboring cubes sharing entire faces) on the second footprint. Several different towers will likely be created. Observing some of the results, perhaps by taking a "field trip" to neighboring tables, would be helpful for the children.

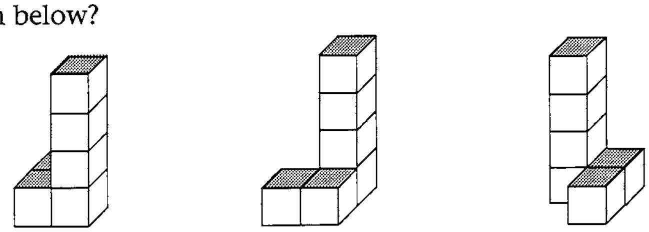


It is now natural to pose this problem: How many different towers of 6 cubes are there which have this footprint? Work with a partner and build each possibility and leave each one standing.

Ten different towers are possible under the restrictions, some of which are shown on the previous page. (The entire collection is pictured below on Blackline 160, Cube Towers.)



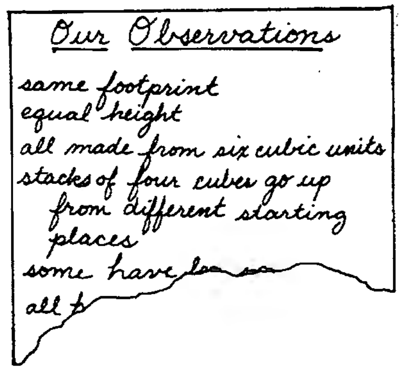
Notice how the teams proceed with this task. Do they use trial-and-error or follow a plan? What decisions do they make, if any? Be sure they address these questions: How do they know when all possible towers have been made? What do they think about the towers shown below?



Are these towers the same or different?

Activity 2

Ask each team to select two of its towers and make sketches of these towers on Blackline 154, Isometric Dot Paper. (See Note at end of Lesson 19. Be sure to have the students turn the dot paper sideways for their sketches.) Have the teams attach their sketches and answer these questions on the paper: How are the towers alike? How are they different? When ready, have teams discuss their charts with neighboring teams.



Continue the activity by having each team sort their towers into two sets according to some criterion. Give them two pieces of yarn for this purpose and ask them to give each set a title that describes how it was formed. (You might have teams conceal their title and conduct a round-robin whereby other teams attempt to identify the sorting criterion used at each table.)

Teacher Tips

As with any sorting exercise, disagreements may arise if teams attempt to use attributes that aren't clearly defined as sorting criteria. Also, note that some teams may select sorting procedures that require overlapping sets and these may need to be discussed further. (Please see Chapter 2 of the Teaching Reference Manual, page 9, for further information.)

Homework

Ask each child to make a sketch of a tower on isometric dot paper. What information (other than a picture) would they give a friend to enable that person to build the chosen tower?

Daily Computation

Ask your children to solve this problem: Last weekend, Jim's family traveled 278 miles in their car. The car used 12.4 gallons of gasoline during this trip. About how many miles per gallon did the car get?

21 Cube Towers, Part II

You Will Need

- Chapter 11, Geometry, and Chapter 2, Sorting
- transparencies of Blackline 160 (Cube Towers) and Blackline 161, 2 pages, (Cube Tower Recording Sheet and Answer Sheet)
- materials for making displays, as needed

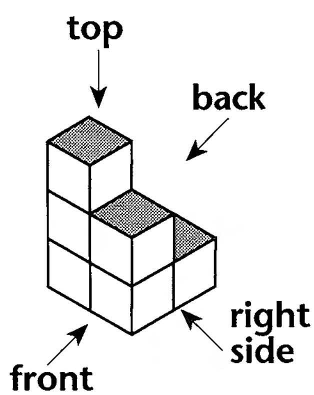
for each pair of children

- about 60 wooden cubes (or hex-a-links)
- copies of Blackline 160 (Cube Towers), Blackline 161 (Cube Towers Recording Sheet, page 1), Blackline 162 (Cube Towers Building Mat), Blacklines 163, 2 pages, and 164 (Cube Towers Activity Sheet and Extension)

Your Lesson

In Part I of this lesson, your children constructed and sorted all the different cube towers shown on Blackline 160 (see Lesson 20, Cube Towers, Part I). Here these towers are used to motivate explorations involving different two-dimensional views (i.e., front, side and top) of models. These views are sometimes called projective views of a model.

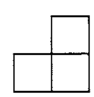
You might begin by showing several representations of rooms or buildings. These could include blueprints, sketches or computer drawings. If possible, invite an architect to talk with the children about that profession.



Activity 1

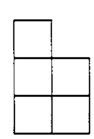
Place the children in teams of two and give each team a copy of the Building Mat on Blackline 162 (see next page). Then display the first tower shown on Blackline 160 (pictured to the left) and ask the teams to build this tower on their mat. Invite suggestions for how an architect might make a "blueprint" of it.

Discuss making sketches of top, side and front views of the tower. For example, the top view is obtained by looking straight down at the tower from directly above it. Ask the teams to do this now and make a sketch of the top view on Blackline 161 (Cube Towers Recording Sheet—see next page). Have a volunteer sketch this view at the overhead on a transparency of Blackline 161.



top view

Now ask the teams to look directly at the front of the tower and make a sketch of the front view on Blackline 161. Once again, invite someone to sketch this view at the overhead.



front view

Opening Eyes to Mathematics | Cube Towers Building Mat | Blackline–162

BACK

LEFT SIDE

RIGHT SIDE

FRONT

Opening Eyes to Mathematics | Cube Towers Recording Sheet page 1 | Blackline–161

Tower 1

top view	front view	left side view	right side view	back view

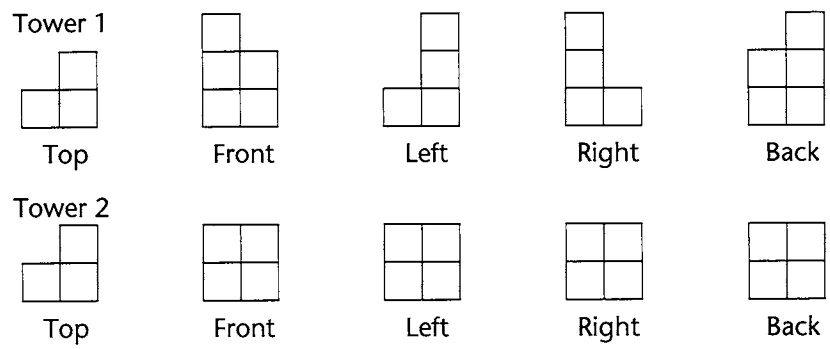
Tower 2

Continue by asking the teams to sketch the left side, right side and back views of the tower. Tell them it is very important to position themselves directly in front of each view in order to make an accurate sketch of that view. Ask the children to share their results at the overhead and to discuss their methods and thoughts.

Activity 2

Repeat Activity 1 for the second tower on Blackline 160. Each view may be recorded on Blackline 161.

The different views for Towers 1 and 2 are pictured on the following page and on Blackline 161, Cube Towers Recording Sheet, Answers. Display a transparency of this blackline and invite observations about these views.



"All the views of Tower 2, except the top, look the same."

"The tops of both towers are the same. That's because the towers had the same footprints."

"The left and right views of Tower 1 are reversed. So are the front and back views. Why is that? They don't look that way for Tower 2."

In particular, note that the front views of these two towers are different. Ask the teams to examine the front view of each tower on Blackline 160 and classify these towers according to front views. Notice the strategies used. For example, some teams may examine each tower after building it. Others may attempt to imagine the front view of a tower by looking at its picture on Blackline 160. Invite the teams to make a display of their classification on chart paper. Teams can decide how to do this.

Continue classifying. Ask the teams to sort the towers according to top views. Discuss their work. Why are all the top views the same? Next, sort the towers according to right side views. Discuss. Would a classification according to left side views be any different from that of right side views? Why or why not?

Here are some additional activities:

Activity 3

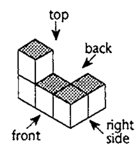
Ask each child to choose and sketch the front and right side views of a tower and give the sketch to their partner. Can the partner build the chosen tower using the sketch?

Ask the teams to work the problems on both pages of Blackline 163 (Cube Tower Activity Sheet).

Opening Eyes to Mathematics | Cube Towers Activity Sheet page 1 | Blackline–163

1. Use the fewest cubes possible to build each model shown below. Be sure neighboring cubes share full faces (without gaps or overlaps). Sketch (in the space provided) the top, right side and front view of each of your models.

a)

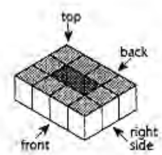


top view:

front view:

right side view:

b)

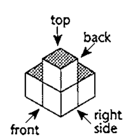


top view:

front view:

right side view:

c)



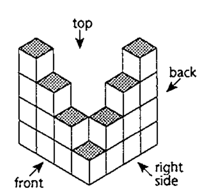
top view:

front view:

right side view:

Opening Eyes to Mathematics | Cube Towers Activity Sheet page 2 | Blackline–163

d)

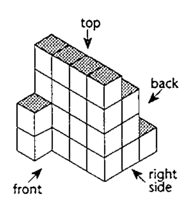


top view:

front view:

right side view:

e)



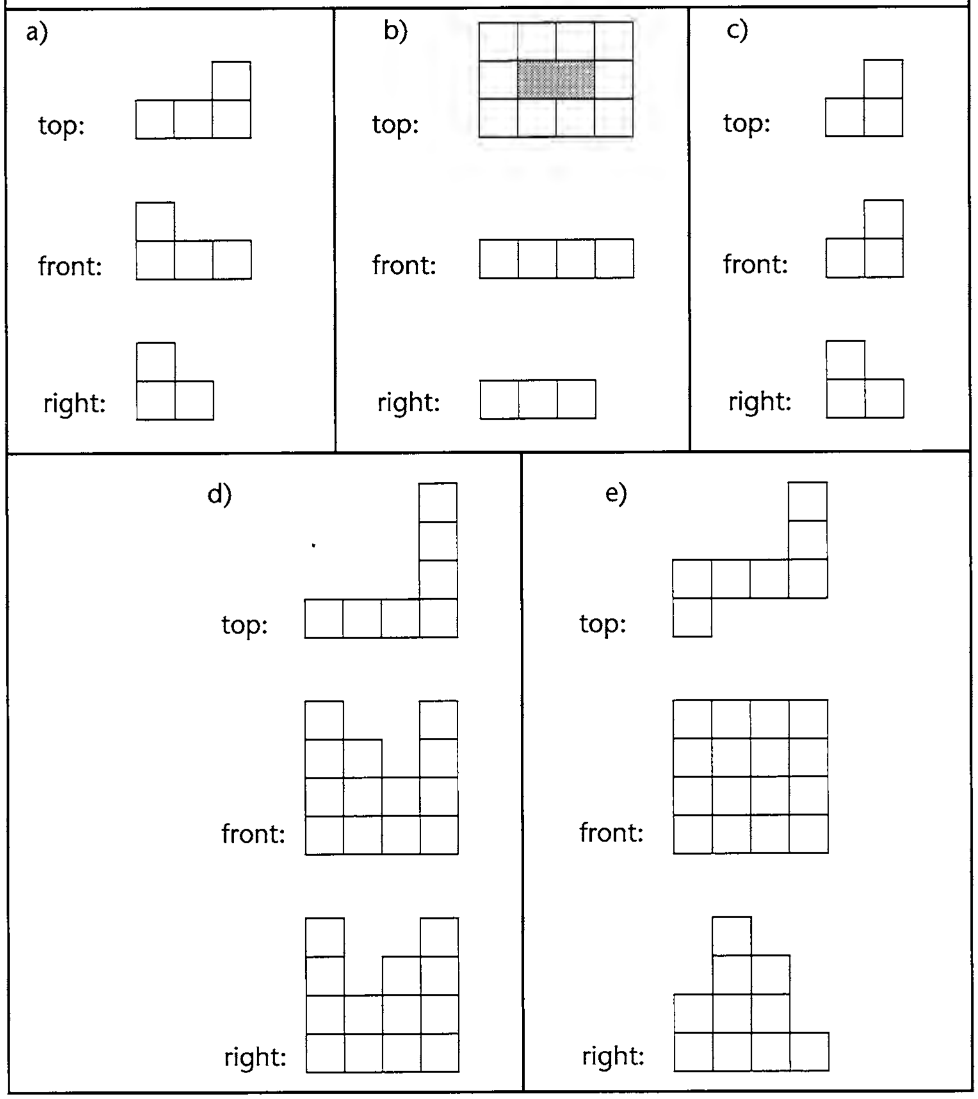
top view:

front view:

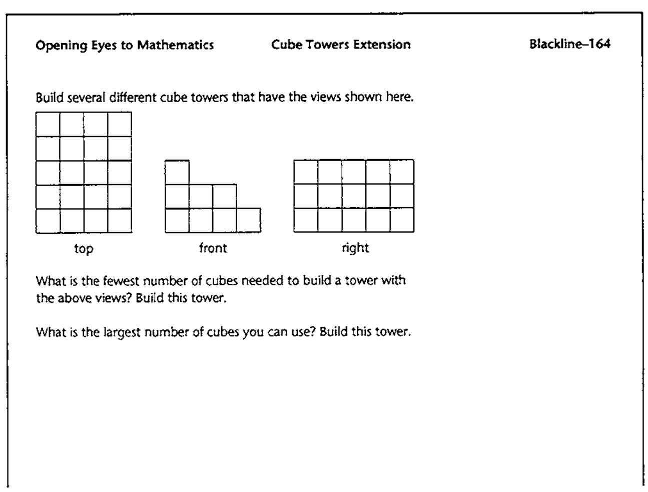
right side view:

Invite the children to pose some related questions of their own.

Answers to Blackline 163:



A extension of this lesson is offered on Blackline 164, Cube Tower Extension.



Opening Eyes to Mathematics — Cube Towers Extension — Blackline-164

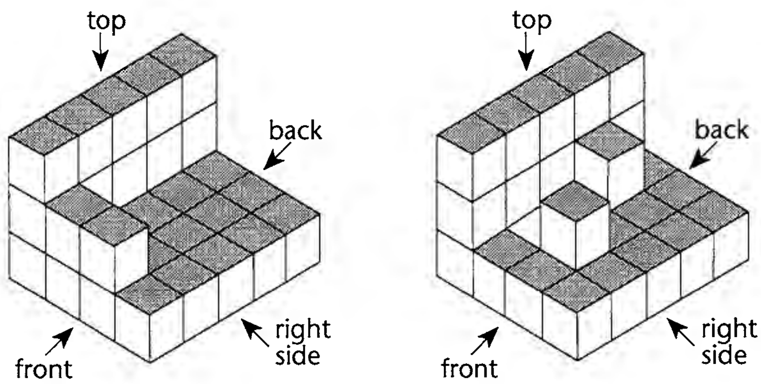
Build several different cube towers that have the views shown here.

top — front — right

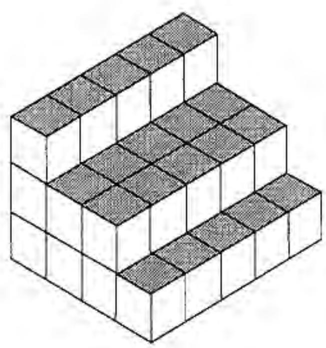
What is the fewest number of cubes needed to build a tower with the above views? Build this tower.

What is the largest number of cubes you can use? Build this tower.

Answer to Blackline 164: Several possibilities exist. Here are two towers that use the fewest number of cubes:



Here is a tower that uses the most cubes:



Journal Entry

Build a tower of your choice. Then write directions that would instruct a friend how to build your tower.

22 Primes & Composites, Part I

You Will Need

- Chapter 6, Multiplication and Division
- Calendar Extravaganza: Today's Array (pages 120–125)
- chart paper and marking pens
- overhead tile available

for each child

- copies of Blackline 8 (Base Ten Grid Paper) and Blackline 25 (1-cm Grid Paper)
- square tile, linear units and calculators

Your Lesson

Rectangular arrays are used throughout Opening Eyes because they can be used as models for concepts such as multiplication, division, factors, prime numbers and composite numbers. In the following sequence of activities, the children are invited to explore these concepts further.

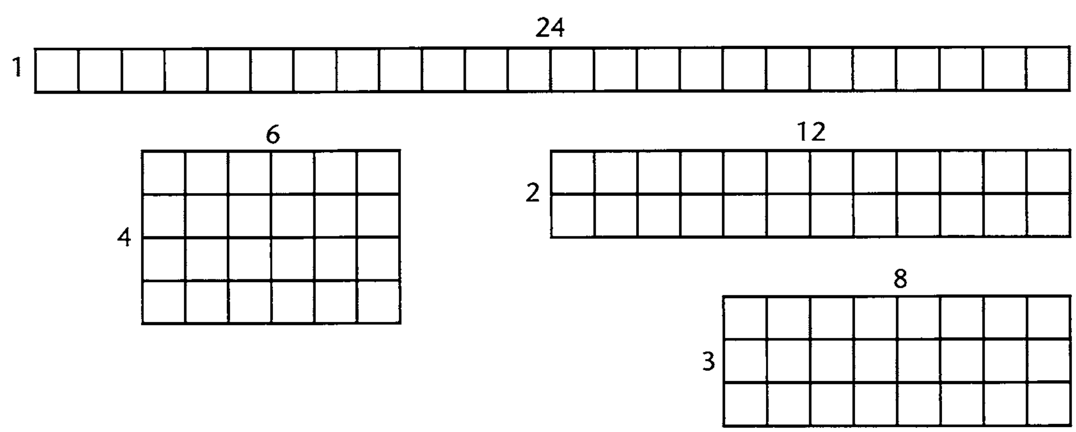
Activity 1

Use the setting of Today's Array to begin this activity.

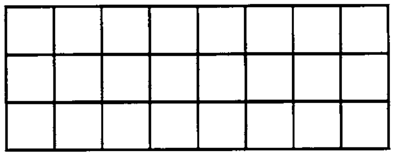
"Today is the 24th day of the month!"

Distribute tile and have the children form rectangular arrays that use 24 tile each. How many different arrays are possible? What are the dimensions of each array? Discuss the children's results.

Note: Children may count the number of arrays in different ways and report various totals as a result. It is helpful to leave this decision to them at this point. If one is restricted to using only whole tile and also disregards orientation, however, exactly 4 arrays are possible:

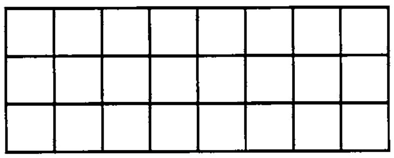


Select one of these arrays and, with the children working in small groups, ask them to prepare a list of observations about it. In particular, ask for some number statements that can be "seen" in the array. Compile a class chart of these observations.

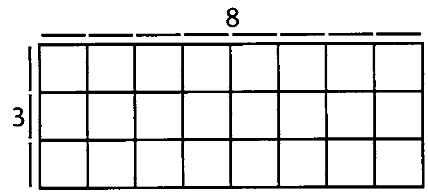


There are 3 rows.
Each row has 8 tile.
8 + 8 + 8 = 24.
3 x 8 = 24.
The perimeter is 22 linear units.

Invite volunteers to describe how they "see" some of their observations. For example, a 3 by 8 array models $3 \times 8 = 24$. How do the children "see" the 3, 8 and 24 in this array? Shown below are two possible responses.



"I see 3 rows of 8 tile each. That's 24 tile altogether."



"This is a 3 by 8 array. Its dimensions are 3 and 8 linear units. Its area is 24 square units."

This is a chance to review such terms as area, perimeter and factors. The children can also identify some hidden arrays that are pictured and write some related multiplication and division stories.

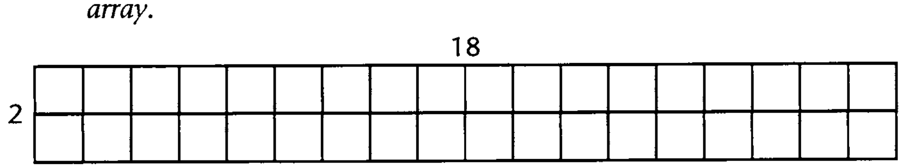
Activity 2

If one is restricted to using only whole numbers of tile and disregards orientation, exactly 4 different rectangular arrays can be formed that use 24 tile each. (See illustration on the previous page.)

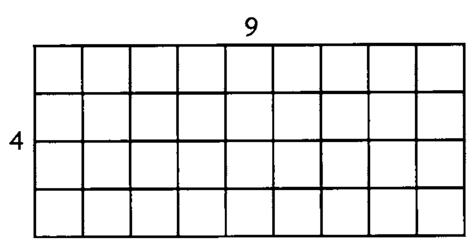
Under these restrictions, have the children work in groups to investigate this problem: How many different rectangular arrays can be formed that use 36 tile each? Make a sketch of these arrays on centimeter grid paper and report the dimensions of each.

Encourage any attempts by children to bring a knowledge of multiplication or of number relationships to their work.

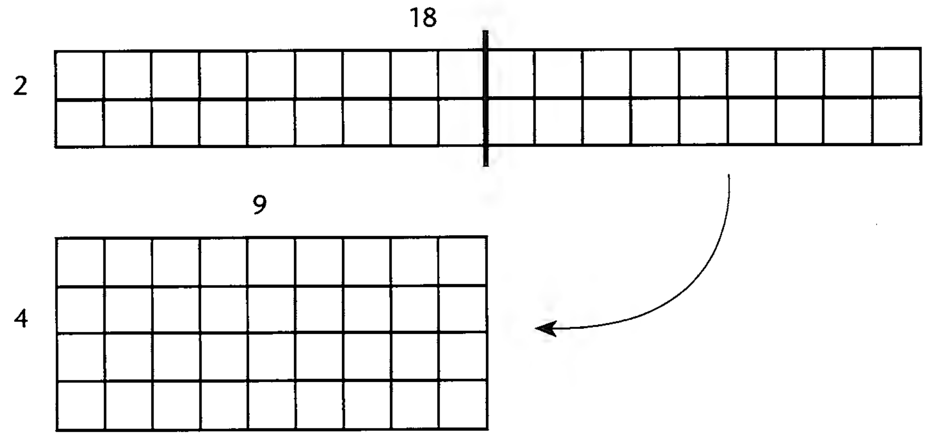
JIM *36 is even so there will be an array with 2 rows. I can make a 2 by 18 array.*



KIM *I remember that 4×9 is 36, so there will be a 4 by 9 array.*



MARSHA *You can also split a 2 by 18 to make a 4 by 9.*



Results can be recorded on chart paper. Conduct a show-and-tell discussion of these results and the thinking behind them, and share with the class. Five arrays are possible: 1 by 36, 2 by 18, 3 by 12, 4 by 9 and 6 by 6.

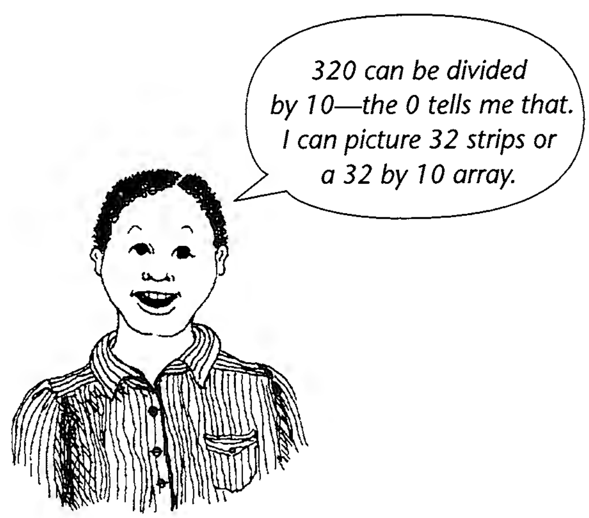
Activity 3

Ask the groups to address these problems:

a) How many different rectangular arrays can be formed that use 13 tile each? Make a sketch of each array and report its dimensions.

b) Do as in Part a), only this time imagine forming arrays that use 84 tile each, 320 tile each, a number of tile of your choice.

There is nothing special about these numbers, except one is prime (13) and some are large enough to cause children to reflect about them mentally. Look for strategies that may emerge and have the children share them. Some children may use trial-and-error completely. Others may apply related knowledge of multiplication. Some may also want to use a calculator for the purpose of testing possible divisors. Encourage attempts to make predictions before arrays are actually formed (or sketched).



22 Primes & Composites, Part I (continued)

This can be an opportunity to discuss prime and composite numbers (see page 59 of the Teaching Reference Manual). It may also be appropriate to review (or introduce) different ways of expressing the relationship between a number and its factors. For example, it is customary to note that 3 is a factor of 12 because $3 \times 4 = 12$. This relationship can also be described by "12 is divisible by 3", "3 is a divisor of 12" and "12 is a multiple of 3".

Teacher Tips

Here are the possible arrays for the numbers given in Activity 2:

13: 1 by 13

84: 1 by 84, 2 by 42, 3 by 28, 4 by 21, 7 by 12

320: 1 by 320, 2 by 160, 4 by 80, 5 by 64, 8 by 40, 10 by 32, 16 by 20

It is not always easy to know when one has found all the different rectangular arrays that are possible in this activity. This is because some divisors of a number are hard to spot. In general, a combination of strategies that draws upon knowledge of multiplication and number relationships and possibly uses trial-and-error (perhaps by testing with a calculator) is helpful.

Homework

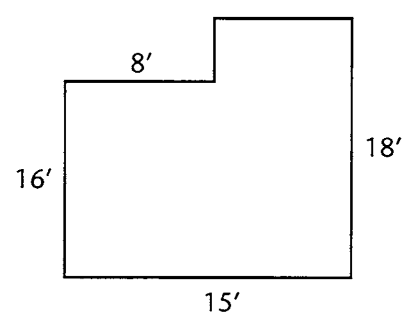
Select a number of tile between 70 and 80. How many different rectangular arrays can be formed using your number? Sketch each of these arrays, along with their dimensions, and describe the reasoning you used to find them.

Journal Entry

Describe something about numbers you found interesting in today's activities.

Daily Computation

Determine the perimeter and area of the following shape. (Note: There is a right angle at each corner.)



Pattern Reminder

What Pattern Possibilities do your children have for one of the remaining sequence starters on Blacklines 117–125?

23 *Primes & Composites, Part II*

You Will Need

- Chapter 6, Multiplication and Division
- Calendar Extravaganza: Today's Array (pages 120–125)
- overhead tile available

for each child

- square tile and calculators
- chart paper and markers
- copies of Blackline 8 (Base Ten Grid Paper) and Blackline 25 (1-cm Grid Paper)

Your Lesson

In Lesson 22, your children found that exactly 4 rectangular arrays can be formed with 24 tile (disregarding orientation and using only whole tile); however, only 1 rectangular array can be formed with 13 tile. They also determined the different arrays that can be formed with other collections of tile. Invite them to explore this situation further, perhaps as described in the following activities.

Activity 1

Ask your children to imagine that you have a collection of tile in your hand which satisfies both of these conditions:

The number of tile is between 5 and 20.

The tile can be arranged in a rectangular array in exactly one way.

Pose this question: How many tile might be in your hand?

Have the children work in groups and search for possible answers to this question, and then build the corresponding arrays. Discuss their answers, recording sketches of them on chart paper. Any prime number between 5 and 20 is a possibility. While some children may recognize this directly, invite discussion of other observations and look for opportunities to use related multiplication knowledge.

JAMAAL *I know you aren't holding an even number of tile.*

TEACHER *Why is that?*

ROSALIE *It's always possible to make an array that has one dimension of 1—like 1 by 7. You could be holding 7 tile.*

JAMAAL *Right—and an even number of tile can be put into another array with two rows. So you can't be holding, say, 8 tile.*

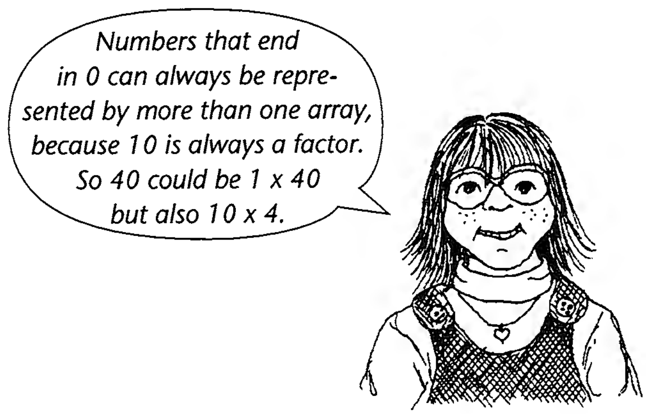
LUIS *Wait a minute! What about 2 — that's an even number but you can only make a 1 by 2.*

JAMAAL *You're right, but isn't that the only even number? Every other one has more than one array.*

Repeat this activity, only this time tell the children the number of tile in your hand is between 20 and 30, between 50 and 70, greater than 100.

The number of tile in each possible collection must be prime. Note that the corresponding dimensions are 1 and the number itself. Note, too, there is no "formula" for generating prime numbers, so the children will have to develop helpful strategies for finding answers.

Invite groups to explain the thinking they brought to the problem. This can motivate some interesting discussion of number relationships. Did they use pictures of arrays to assist them? Or mental arithmetic? Or perhaps they visualized patterns from their work with hundred's matrices or number pattern songs? Or did they use a calculator to systematically test for divisors of a number?



Invite the children to make some predictions based on their previous observations. Do they have any general observations to offer (and support)? For example, how do they feel about a collection of 250 tile? Can this collection be arranged in a rectangular array in just one way? How about 1258 tile? etc.

Activity 2

This is essentially a repeat of Activity 1, only this time ask your children to conduct a search for numbers (of tile) that can be arranged into rectangular arrays in exactly 2 ways. You might proceed in this way:

Ask your children to imagine that you have a collection of tile in your hand which satisfies both of these conditions:

The number of tile is between 5 and 20.

The tile can be arranged in a rectangular array in exactly two ways.

Pose this question: How many tile might be in your hand?

With the children in groups, have them search for possible answers to this question and to build the corresponding arrays. Discuss their answers, recording sketches of them on chart paper. Also record the dimensions and related multiplication statement associated with each array.

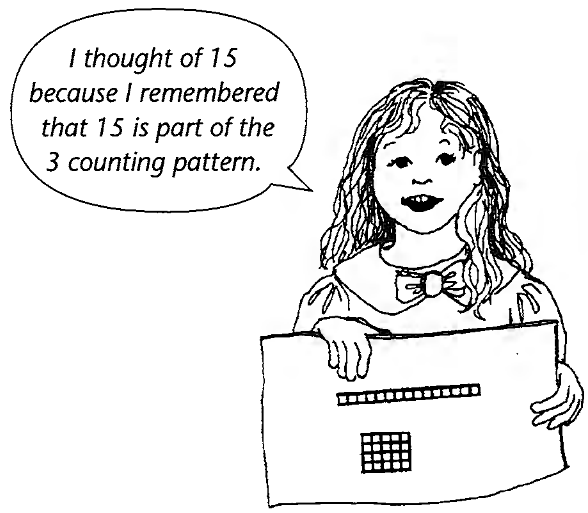
AMANDA *You might have 14 tile. We could build a 1 by 14 and a 2 by 7 using 14 tile. There are no others.*

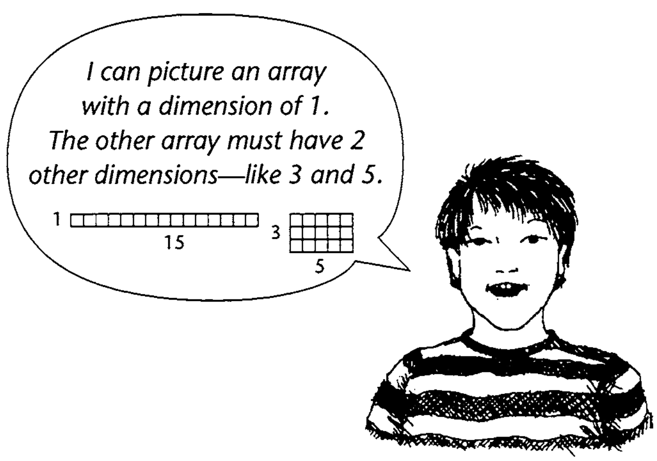
IRA *Look! You could have 9 tile, 1 by 9 and 3 by 3.*

LESLIE *I'm not sure how to find an example.*

Continue by asking the children to find other examples of numbers (of tile) that can be formed into rectangular arrays in exactly two ways. At some point, you might invite them to offer examples that are between 50 and 100; or greater than 250; etc., and make sketches of these arrays on base ten grid paper. Ask each group to prepare a report of their work that includes sketches of results and any general observations they have.

Any number that is the product of two different prime factors will "work". Some children may recognize this and generate examples by choosing two prime numbers and multiplying them. Others may conduct the search in a trial-and-error manner or with the help of other relationships they observe. Calculators should be available as needed. In any case, encourage attempts to make predictions before arrays are actually formed (or sketched).





Activity 3

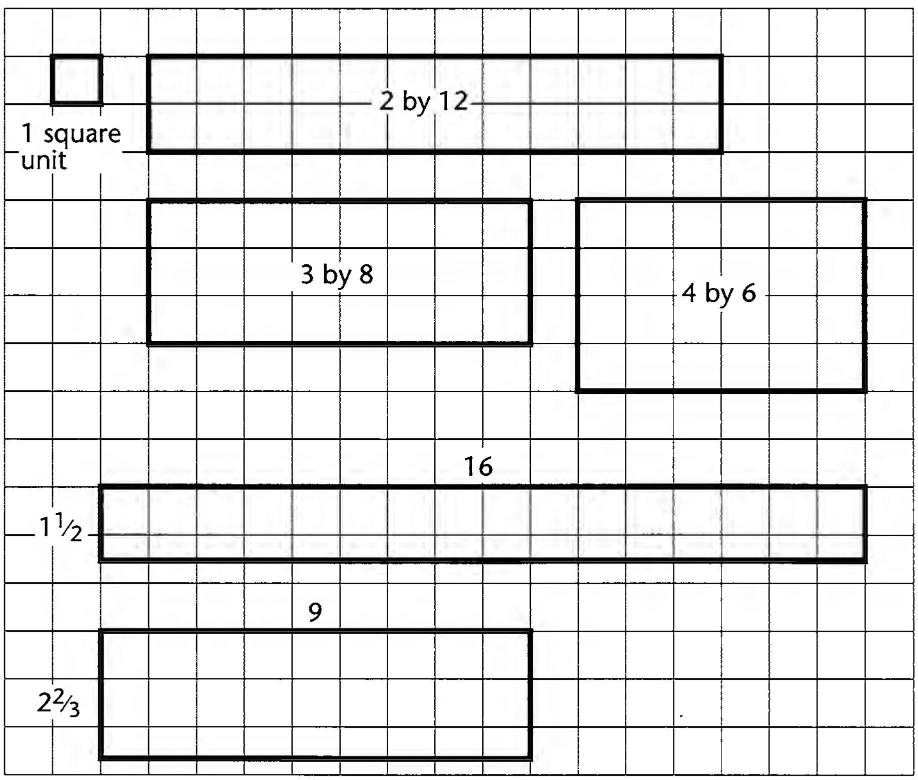
Invite your children to pose additional problems, related to the above activities, that they would like to explore.

This is a chance for children to become problem-posers, though you may have to offer some general help. Changing the parameters of a problem or negating the conditions can often lead to interesting mathematical investigations. Here are some possibilities:

What numbers (of tile) can be arranged into rectangular arrays in exactly four (or some other number) ways?

What if tile were not used and rectangles were made on square dot paper? Then, how many rectangles can be made (with dots at the corners) that have an area of 24? (Note: There are more than four, even if one disregards orientation.)

Suppose rectangles are sketched on graph paper. If one is not restricted to whole number dimensions, how many rectangles having area 24 can be drawn?



Homework

Work with a friend or family member and try to find at least two answers to this puzzle: a) I am the number of whole tile that can be put into a rectangular array in exactly two different ways; b) I am between 100 and 200.

Make sketches of the arrays (and their dimensions) for each number and explain the thinking you used to find the numbers.

24 *Ancient Numeration Systems, Part I*

You Will Need

- Chapter 4, Place Value, and Chapter 6, Multiplication
- Volume 3, Appendix C: The Duplation Method of Multiplication
- transparencies of Blacklines 165 and 166 (Ancient Numeration Systems)
- a class chart with the information shown on Blackline 167 (Egyptian Numeration System)

for each child

- a copy of Blackline 166 (Ancient Numeration Systems) and Blackline 168 (Egyptian Multiplication)
- calculators, chart paper, markers available

Your Lesson

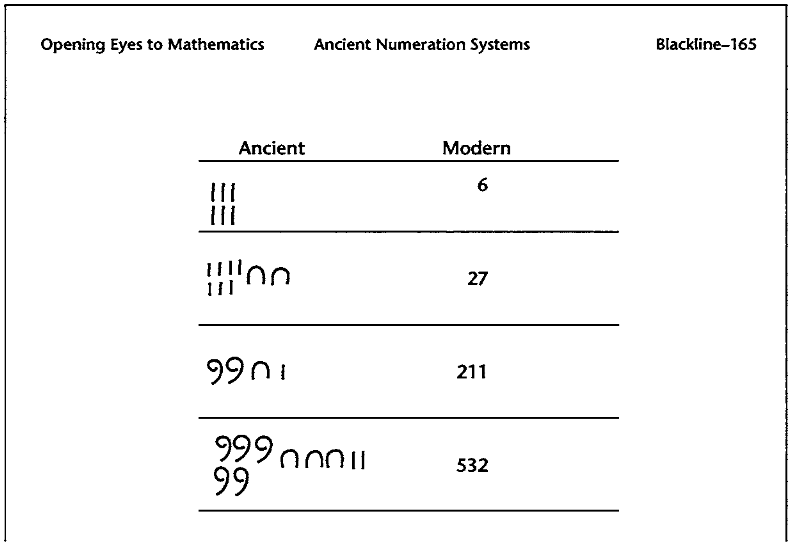
It is important for children to become familiar with the base ten numeration system. Previous explorations of base ten and other bases were conducted with this goal in mind. From a historical perspective, however, place value concepts and the use of positional notation emerged slowly. The concept of zero was a particularly difficult one, and it wasn't until the Middle Ages that zero was first used in the way we use it today. It is not surprising, then, when our children experience the same difficulties with place value concepts as their ancestors.

It may help children appreciate their struggles with numerical concepts and, in turn, deepen their understanding of number, by learning how people worked with numbers in the past. In so doing, children also have the opportunity to view mathematics in a warm, human way, and see its relationship to the rest of their world and culture.

Activity 1

In this activity, children imagine they are archeologists trying to decipher newly discovered numerical tables. Certain clues are available and the challenge is to "crack the code". In so doing, the features of the ancient Egyptian numeration system are uncovered and discussed.

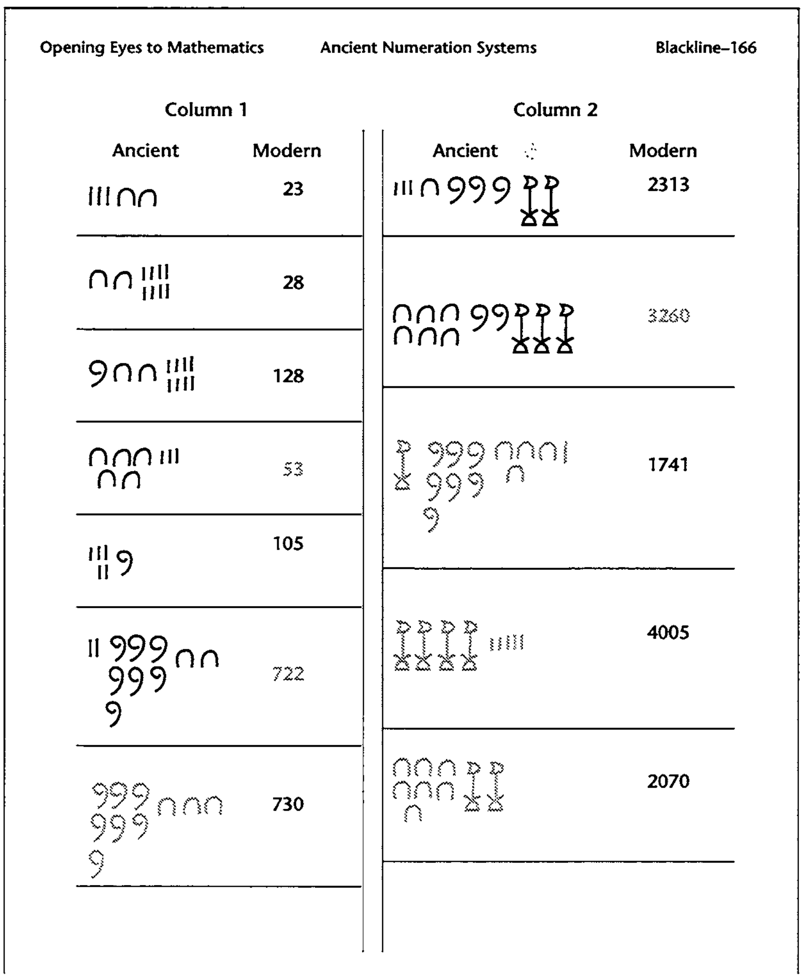
Display on the overhead the entries in the left hand column of Blackline 165 (shown on the following page) and ask the children to imagine these are numerals from an ancient society. Tell the class some information is known about these numerals and reveal the first three entries in the right-hand column. Ask the children to predict the number (532) which corresponds to the fourth numeral in the left column. Discuss these predictions.



Opening Eyes to Mathematics — Ancient Numeration Systems — Blackline–165

Ancient	Modern
	6
	27
	211
	532

Divide the class into small groups. Challenge the groups to learn more about this ancient numeration system by completing the table on Blackline 166. (For your benefit, we show below the completed table. The answers to be supplied by the children are shown in gray.)



Opening Eyes to Mathematics — Ancient Numeration Systems — Blackline–166

Column 1 Ancient	Column 1 Modern	Column 2 Ancient	Column 2 Modern
	23		2313
	28		3260
	128		1741
	53		4005
	105		2070
	722		
	730		

Have the groups create a key for this numeration system and answer these questions: What observations do you have about this ancient numeration system? How is this system like our own? How is it different? Provide time for groups to share their work.

GROUP 1 *We decided that | stands for 1, ∩ for 10, and 9 for 100. Looks like they used base 10, too.*

GROUP 2 *There's a lot more writing here. If you want to show 90 you have to write nine ∩'s.*

GROUP 3 *We agree about the writing; but it's easy—just put enough of the 1's, 10's and 100's together.*

99∩∩||| ⟶ 200 + 20 + 3 = 223

|||∩∩99 ⟶ 3 + 20 + 200 = 223

9|||∩∩9 ⟶ 100 + 3 + 20 + 100 = 223

Tell the children that this system was used in ancient Egypt. The main characteristics of this system are summarized on Blackline 167.

Opening Eyes to Mathematics — Egyptian Numeration Systems — Blackline–167

Heiroglyphic Numerals		Modern Numerals
|	(staff)	1
∩	(heelbone)	10
9	(scroll)	100
	(lotus flower)	1,000
	(bent finger)	10,000
	(tadpole)	100,000
	(astonished man)	1,000,000

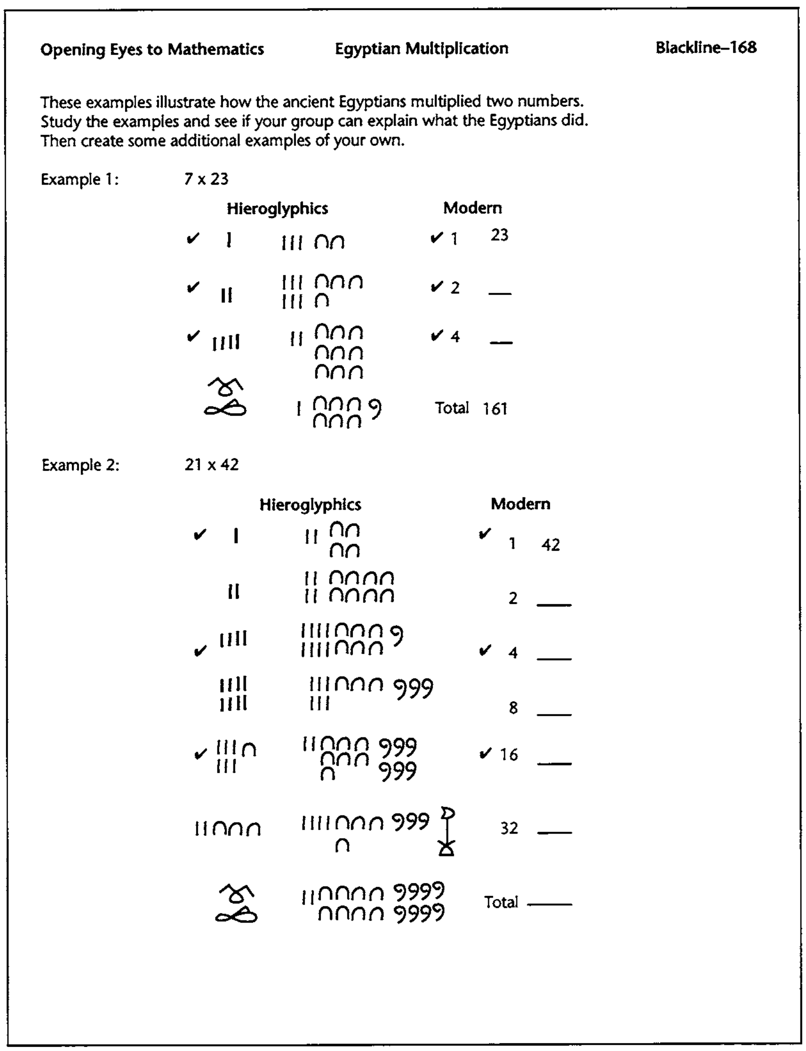
- Base 10
- Additive system 99∩∩||| ⟶ 200 + 20 + 3 = 223
  Add the value of the symbols together.
- No symbol for zero.

Post these characteristics on a class chart to discuss them further.

Have the groups select a few numbers and create the corresponding Egyptian numerals. These numbers might include today's date, the age of each group member, the last four digits of a telephone number, the area of the classroom, etc. Groups can also consider the reverse problem: Create some Egyptian numerals and ask a neighboring group to determine the corresponding number in our system.

Activity 2

The ancient Egyptians had a surprising way of multiplying two numbers called the Duplation Method (or the Method of Doubling). You and your children will enjoy figuring out how it works by studying the examples on Blackline 168. (Some clues are presented in the dialogue on the next page.)


Opening Eyes to Mathematics — Egyptian Multiplication — Blackline–168

These examples illustrate how the ancient Egyptians multiplied two numbers. Study the examples and see if your group can explain what the Egyptians did. Then create some additional examples of your own.

Example 1: 7 x 23

Hieroglyphics / Modern

✔ 1	23
✔ 2	—
✔ 4	—
Total	161

Example 2: 21 x 42

Hieroglyphics / Modern

✔ 1	42
2	___
✔ 4	___
8	___
✔ 16	___
32	___
Total	___

We suggest you offer Blackline 168 to the class without any introductory examples. Have the children work in groups and provide time for them to decipher the method on their own. As appropriate, encourage the groups to consider their task as if solving a puzzle: looking for clues, testing hypotheses, creating other examples, etc. Some rich mathematical discussion is likely to occur during this work period. Invite the groups to capture their thinking and prepare a report for sharing.

GROUP 1 *Both columns are doubling. But we think only the checked numbers are used. We added the right-hand numbers in the checked rows to get the answer.*

✔	1	23
✔	2	46
✔	4	92
	Total	161

✔	1	42
	~~2~~	~~84~~
✔	4	168
	~~8~~	~~336~~
✔	16	672
	~~32~~	~~1344~~
	Total	882

GROUP 2 *Seems like the checked numbers on the left add to one of the numbers being multiplied. The other number starts the right column.*

These add to 21

✔	1	42
	~~2~~	~~84~~
✔	4	168
	~~8~~	~~336~~
✔	16	672
	~~32~~	~~1344~~
	Total	882

GROUP 3 *We looked at the checked numbers, too. In the second one, there are twenty-one 42's being added.*

1	42	1 x 42
~~2~~	~~84~~	+
4	168	4 x 42
~~8~~	~~336~~	+
16	672	16 x 42
~~32~~	~~1344~~	
Total	882	21 x 42

GROUP 4 *We tried this method for 28 × 43. We knew the answer was 1204 because we used a calculator. We think the Egyptians would do this:*

These make 28

	~~1~~	~~43~~	
	~~2~~	~~86~~	
✔	4	172	4 x 43
			+
✔	8	344	8 x 43
			+
✔	16	688	16 x 43
	~~32~~	~~1376~~	
	Total	1204	

28 x 43

Ask the children to suggest some "What if's?" for further investigation and then pick some to explore. Of course, you can make suggestions of your own as well! Here are some possibilities:

What if....

- other multiplication problems were tried? Would the Egyptian Method always work? Explain.
- each group created a hieroglyphic example of the Duplation Method. Could the neighboring group figure out the example?
- division were tried? How might the Duplation Method be adapted to divide one number by another?
- in Example 2 of Blackline 168, we tried 21×42 the Egyptian way by putting 21 as the top number in the second column instead of 42? Will the method still work?

Teacher Tips

The examples on Blackline 168 are similar to ones found in the Rhind Papyrus, one of the oldest mathematical documents in existence. This papyrus dates back to approximately 1700 B.C. It is a collection of 87 problems (and their solutions) that reveals much about ancient Egyptian mathematics. Please refer to A. B. Chace's book, *The Rhind Mathematical Papyrus*, for added information (see listing in Appendix C).

Appendix C also describes how the Duplation Method can be modeled visually and applied to division.

People have used several methods for determining products and quotients throughout history (see references in Appendix C). These methods are reminders that our "standard" algorithms are relatively new and quite arbitrary. Your children may be interested in conducting research about some of these methods and the historical context associated with them.

The children might also enjoy some general reading about life in ancient Egypt.

Journal Entry

How do you feel about the ancient Egyptian numeration system? In your opinion, what are some advantages to this system? What are some disadvantages?

***Reminder**

For Lesson 25, ask your children to bring from home various scales, such as those used to weigh food, people, letters, babies, etc.

25 Standard Units of Weight

You Will Need

- Chapter 7, Measurement
- various scales such as ones used to weigh food, people, letters, babies, etc. Include a metric scale as well (possibly found in the school library). Perhaps your children can bring some of these scales from home.
- sample weights to be used as benchmarks for estimation activities. These might include:
  - 2-, 4-, 5- or 10-lb. bags of sugar
  - 5-, 10- or 20-lb. bags of flour
  - several 1-lb. bags of rice, dirt, gravel, etc.
  - several bags of material packaged in metric units
  - 12-ounce cans of pop
  - sets of weights—both metric and English units
- copies of Blackline 88 (Guess and Check Record Sheet) for each small group

Opening Eyes to Mathematics — Guess and Check — Blackline 88

Name________________ GUESS AND CHECK

Item	Unit of Measure	Guess	Check

Your Lesson

The overall structure of Volume 3, Lesson 8 (Standard Units of Length) may be adapted to this lesson. The general goals of the two lessons are the same and the activities are similar. The lesson also links with Volume 1, Contact Lesson 55, Non-Standard vs. Standard Units of Weight.

Discuss the scales you have gathered. How are they used? Do they work in any special way? Your children can describe the scales brought from home (though we've had to be mindful of special scales that children shouldn't handle). The children can also describe instances where they or others have used these scales.

Give each group of children a benchmark weight, such as a 2-pound bag of sugar. Ask them to find some items in the room that weigh approximately 6 pounds.

Have groups try first to identify these items without measuring. Instead, encourage the teams to establish a "sense of a pound" with the help of their benchmarks. This sense, along with some estimation strategies, can aid them in their search. Actual measurements may then be made as checks. Repeat, seeking objects of other weights (try to vary these weights to include light and heavy ones).

Discuss such questions as these: What strategies did the groups use in their search? How do they feel about these strategies? Given their awareness of the benchmark, was it hard or easy for them to distinguish among different weights? If it was hard at first, did it get easier with time? How precise does one need to be when weighing objects? How does this depend on the object being weighed and the purpose for weighing?

Ask the groups to work on the following tasks, recording results on Blackline 88:

- Choose some items of interest to weigh.
- Select an appropriate unit of weight and estimate the weight of each item.
- Find the actual weight of the item. Discuss your results. How do the actual weights compare with your estimates? What might explain any differences?

As part of the above activity, encourage different groups to measure the same item, and to do this with different units of weight. How do their results compare? For example, how do pounds and ounces compare? What about pounds and kilograms?

This lesson may be extended with projects such as these:

- How does recycling work in your area? How much does recycling cans pay? How many pounds of cans would need to be recycled to generate enough money to buy scales for your room?
- What is the average amount of waste thrown away from food trays during lunch in your school? How can this be estimated and checked? How can your class estimate and determine its average daily contribution to this waste? Develop strategies for doing this and for sharing the results with the entire school!

Teacher Tips

Consider posting a question by your calendar each week (see Teacher Tips, Lesson 8) that focuses on weight. For example: Before Calendar Time today, pick up Curtis' bookbag. What do we have in this room that has approximately the same weight as his bag?

Ask parent volunteers to fill ziplocks for establishing benchmark weights. This can be done either at school or at home.

Journal Entry

How did you think about the weights of the items chosen by your group? What did you know? On what did you base your decision regarding which unit of weight to use?

26 Pattern Generalizations

You Will Need

- Chapter 3, Patterns
- overhead tile

for each child

- square tile and toothpicks available
- Blacklines 169–176 (Pattern Generalizations)
- chart paper, markers and other materials for making reports

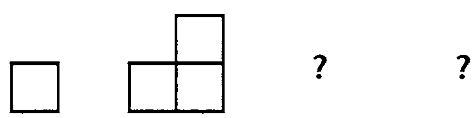
Your Lesson

Generalizations are an important part of mathematics and problem solving. They reflect a high level of thinking, capture the essence of an idea or problem, and often apply to a variety of situations. In this lesson, your children are invited to extend sequences of arrangements with the help of visual thinking and with the goal of forming some generalizations about each sequence. The sequences and activities are similar to those found in Lesson 1, Pattern Possibilities, of this volume.

Activity 1

Build the first few arrangements of a sequence at the overhead and have the children do the same at their desks. Invite volunteers to share some pattern possibilities for extending the sequence and select one of the suggested possibilities for discussion. In the following dialogue, one of the sequence starters from Lesson 1 is used as an illustration, but any starter sequence would work.

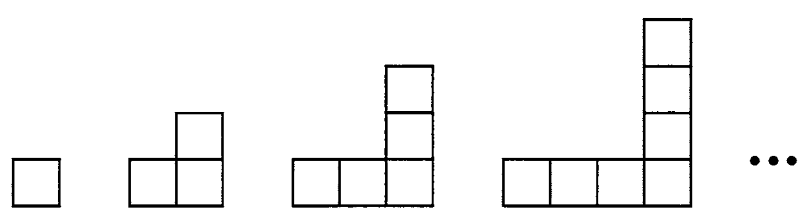
TEACHER *Look at these two tile arrangements. Then build the 3rd and 4th arrangements so all the arrangements form a pattern that can be extended.*



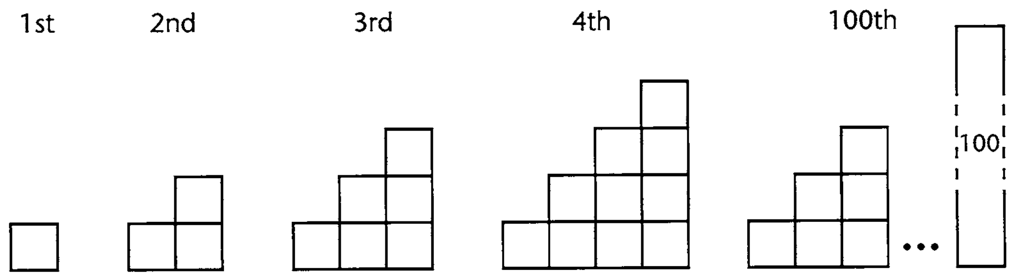
arrangement: 1st 2nd 3rd 4th

JOSE *Here's how I did it. In my pattern, each arrangement will always have 1 tile in the corner and 2 "arms". Each arrangement has 1 more tile in each arm than the one before it. Just keep on adding a tile to each arm. The 4th one uses 7 tile, so the 5th one will need 9.*

1st 2nd 3rd 4th



AHMED *I see stacks of tile in my pattern. Each arrangement has one more stack than the one before it. The first stack has 1 tile, the second 2, the third 3, and so on. When do you stop the stacks? When you have as many as the arrangement number!*

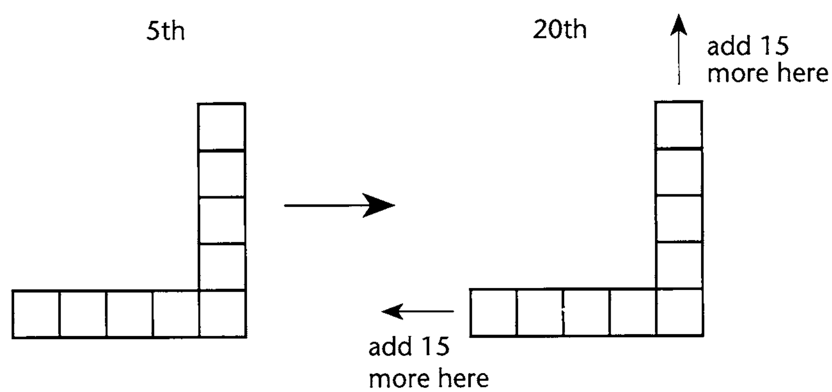


TEACHER *Let's examine Jose's pattern possibility for now (referring to Jose's arrangements). When he described his pattern, he said each arrangement had 1 more tile in each arm than the previous arrangement.*

Imagine building more arrangements using his pattern. With your partner, build the 5th arrangement and then discuss what the 20th arrangement would look like. How many tile are needed to make the 20th arrangement?

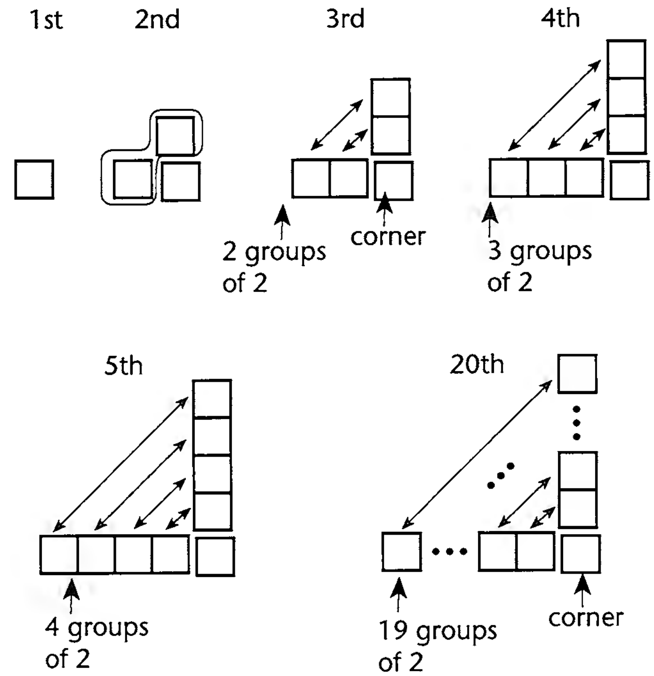
(The children work in teams of two and then share their thoughts with the entire class.)

IRIS *We added 2 tile to the 4th arrangement—one in each arm—to make the 5th one. You can keep on doing this. The 20th one will need 15 more tile in each arm than the 5th one has. Since the 5th one has 9 tile, the 20th will have 9 + 15 + 15 or 39 tile.*

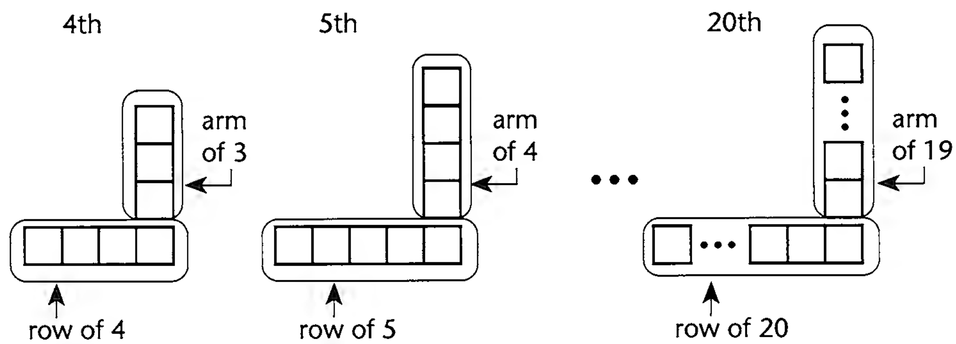


26 Pattern Generalizations (continued)

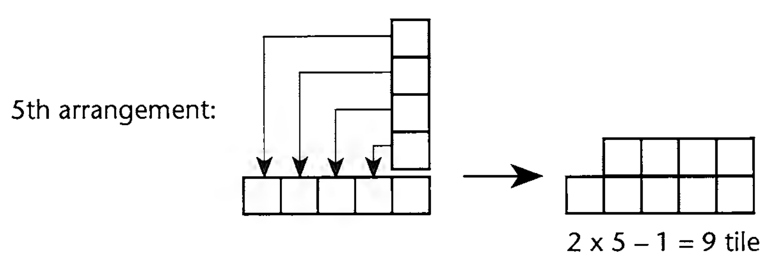
LUIS *We agree with Iris, but we noticed something else. Each arrangement has the tile in the corner. The arms of the 5th one has 4 groups of 2. The arms of the 20th one will have 19 groups of 2. So the 20th arrangement has 19(2) + 1 or 39 tile.*



ANITA *I see something else in Jose's arrangements. There's a row of tile on the bottom and then an arm going up. The bottom row has one more tile than the arm. So the 20th arrangement will have a row of 20 tile and an arm of 19. That's 39 tile.*



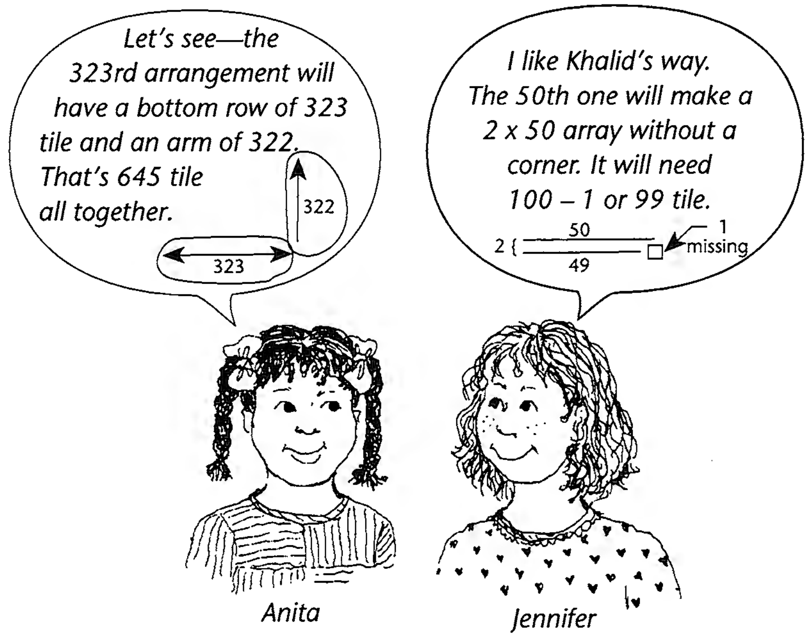
KHALID *If you turn Anita's arms sideways, you get an array with a tile missing. The 20th one will be a 2 by 20 array with a missing corner. That will be 2 × 20 – 1 or 39 tile. Looks like one dimension will always be 2. The other dimension is as long as the bottom row.*



Notice that four different ways of determining the number of tile in the 20th arrangement were presented. Children often perceive the same sequence of arrangements in validly different ways. Each child's view should be acknowledged and appropriately discussed. Note, too, that each child has begun referring to part or all of the sequence in general terms.

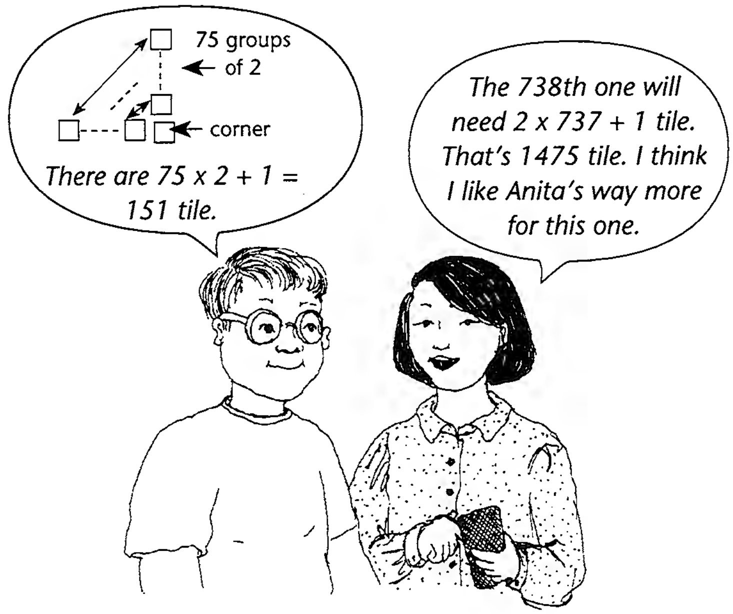
The class continues to predict about the arrangements according to Jose's pattern possibility. What does the 50th arrangement look like? the 323rd arrangement? How many tile will be needed in each case?

The children can do this in an open way, using any view of the arrangements that is comfortable.



It is also valuable to ask for predictions based on a particular child's view. We urge you to try this from time to time.

TEACHER *What would the 76th arrangement look like according to Luis' view? How many tile would be needed for this arrangement? How about the 738th arrangement?*



Notice the second child's preference in the illustration. It is important to note such preferences, as they occur, and to discuss the relative merits of each view.

TEACHER *Let's think about Anita's view. Does it have any advantages? Does it have any disadvantages?*

MYRA *Well, I like to add and I can picture Anita's view. That makes it easy for me.*

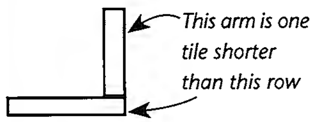
ROLAND *I can double some numbers pretty quickly. I can also see the array in Khalid's method. So I would probably use it for arrangements like the 50th one, but Anita's for other ones.*

Children won't always have the same feelings about these matters and that's okay. What is easy for one child may be hard for another. It is not necessary that every child become expert with each view. We examine different views in these activities with the hope of increasing children's options for analyzing a situation and abilities to consider the appropriateness of each option. We also hope that children who are struggling might piggyback on the ideas of others.

Following the discussion, ask the children to work in small groups and compose a list of observations (i.e., generalities) that are always true about this sequence. Some may even suggest a formula for the number of tile needed to build each arrangement. Shown here are possible lists for Anita's and Khalid's views, respectively.

Anita's view

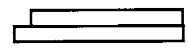
The bottom row has one more tile than the arm.
The number of tile in the bottom is the same as the arrangement.
Each arrangement looks like



The number of tile is
$n + (n + 1)$.
n is the arangement number.

Khalid's view

We can always make an array with a missing corner.
One dimension of the array is always 2.



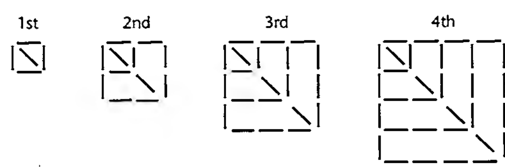
The bottom dimension is the arrangement number.
The number of tile is $2n - 1$.
n is the arrangement

Activity 2

Repeat Activity 1 using other starter sequences. Blacklines 169—175 show additional starters, together with questions calling for extensions and generalities about each starter. These starters use square tile or toothpicks as illustrated.

Opening Eyes to Mathematics **Pattern Generalizations** **Blackline–173**

Build the four arrangements pictured here:



a) Look for a pattern in the first four arrangements that might suggest what the 5th arrangement looks like. Describe your pattern and use it to build the 5th arrangement.

b) Imagine building more arrangements using your pattern. Describe how you would determine the number of toothpicks needed to make the 17th arrangement.

c) Describe how you would determine the number of toothpicks needed to make the 53rd arrangement.

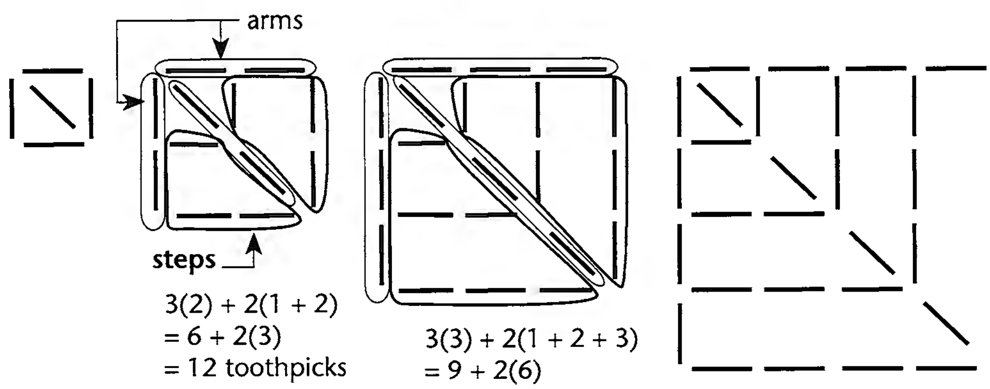
d) Make a list of observations that seem to be true for all the arrangements made with your pattern.

You may wish to examine one of these starters (see illustration below) at this time and then space the others throughout the coming months. We also encourage you and your children to generate starters of your own.

Blackline 173
1st view:

Each arrangement has 3 "arms" and 2 "steps". The arms are as long as the arrangement number. The steps start at 1 and go to the arrangement number. The 17th arrangement uses lots of toothpicks! We needed a calculator to figure out how many.

1st 2nd 3rd 4th

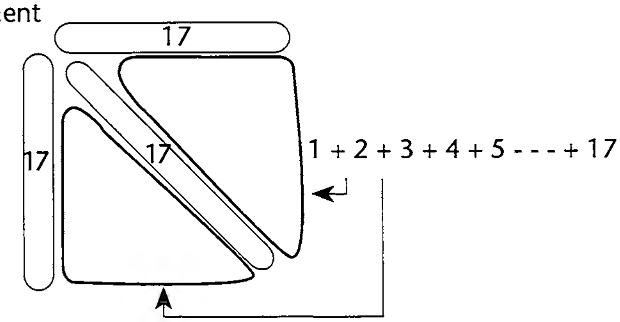


3(2) + 2(1 + 2)
= 6 + 2(3)
= 12 toothpicks

3(3) + 2(1 + 2 + 3)
= 9 + 2(6)
= 21 toothpicks

3(4) + 2(1 + 2 + 3 + 4)
= 12 + 2(10)
= 32 toothpicks

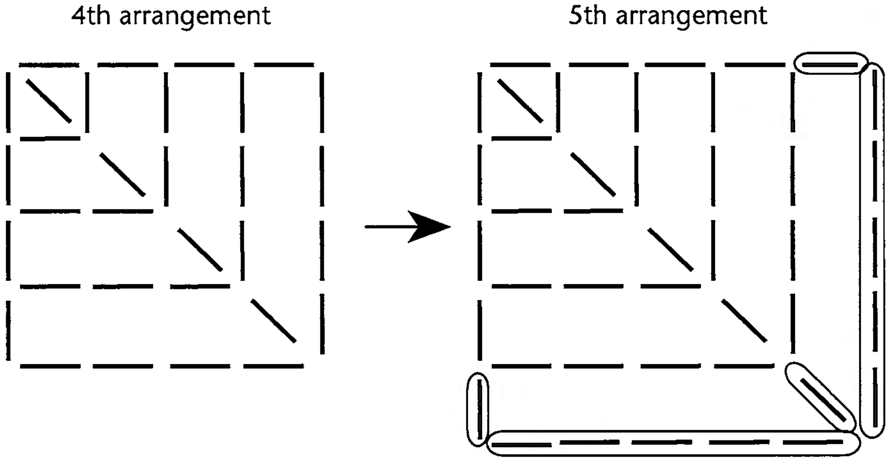
17th arrangement



3(17) + 2(1 + 2 + 3 + 4 + 5 - - - + 17)
= 51 + 2(153)
= 357 toothpicks

Each arrangement can be built from the one before it. Just add a toothpick on the top, on the left, and along the slant. Then make the bottom and right side have as many toothpicks as the top.

Blackline 173 2nd view:



Teacher Tips

Notice the importance of the first few arrangements in the sequences of this lesson. Children can use the visual clues found with the earlier, specific arrangements to form a mental picture about all arrangements in a sequence. This picture enables one to characterize the entire sequence either in a few sentences or with a generic diagram.

27 Revisiting Counting Patterns

You Will Need

- Chapter 6, Multiplication and Division
- overhead tile
- chart paper and markers

  for each child
- units from the base ten area pieces
- Blackline 8 (Base Ten Grid Paper), Blackline 141 (Square Dot Paper) and calculators available
- a copy of Blackline 176 (Pattern Generalizations)
- a copy of Blackline 177 (Sammy's Strange Allowance), (see Home work)

Your Lesson

This lesson is a continuation of Lesson 26, Pattern Generalizations, that revisits some of the counting patterns your children explored in Volumes 1 and 2, and Musical Array-ngements. The lesson initially focuses on visual models of odd and even numbers. Children can then use these models to construct properties of these numbers and to reflect about different counting patterns. Please note that these models are also explored in Unit II, Activity 2, of the *Math and the Mind's Eye* project and in *Visual Mathematics, Course I.*

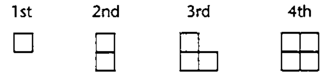
Activity 1

Divide the class into small groups and ask the groups to work the problems on Blackline 176. Follow the general procedures for Activity 1, Lesson 25, Pattern Generalizations, of this volume.

Opening Eyes to Mathematics **Pattern Generalizations** **Blackline–176**

Build the four arrangements pictured here:

1st 2nd 3rd 4th



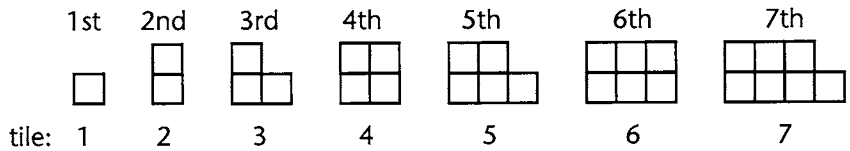
a) Look for a pattern in the first four arrangements that might suggest what the 5th arrangement looks like. Describe your pattern and use it to build the 5th arrangement.

b) Imagine building more arrangements using your pattern. Describe how you would determine the number of tile needed to make the 17th arrangement.

c) Describe how you would determine the number of tile needed to make the 53rd arrangement.

d) Make a list of observations that seem to be true for all the arrangements made with your pattern.

Be sure to discuss the following pattern possibility for the sequence on the blackline.



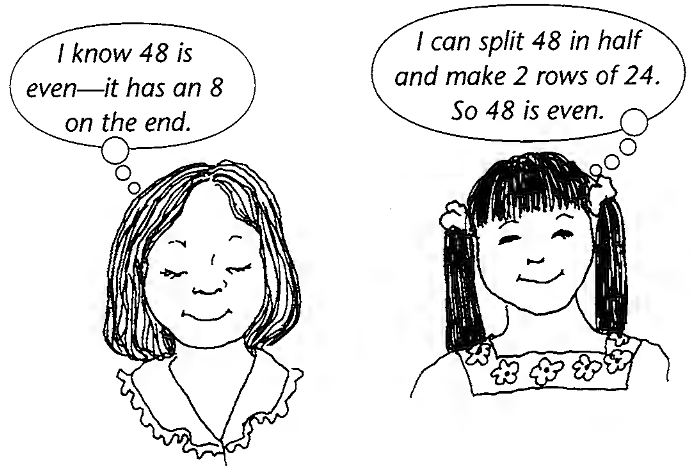
"There is 1 tile in the first arrangement. After that, each arrangement has 1 more tile than the previous one. Arrange the tile so as to form a rectangular array having one dimension 2. Sometimes there will be a tile left over."

This pattern possibility is a natural one and probably will be suggested by one of the groups. If necessary, you can bring it up for discussion. Either way, ask the groups to build the first seven arrangements as pictured above.

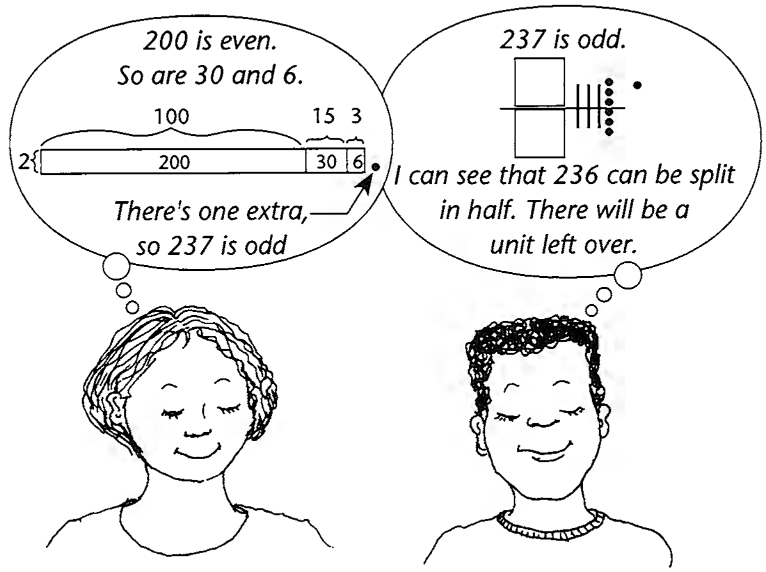
This sequence models odd (1, 3, 5, 7, etc.) and even (2, 4, 6, 8, etc.) numbers. Identify these numbers and point out how the names *odd* and *even* are suggested by the arrangements in the sequence. Even amounts of tile can be arranged in a rectangular array having one dimension 2; an odd number of tile can be arranged in a similar array with an extra tile left over.

Provide time to explore and discuss problems such as these:

- Are the following numbers odd or even? 48, 37, 83, 246, other numbers (suggested by the children)



- Ask the children for some examples of odd numbers between 224 and 239.

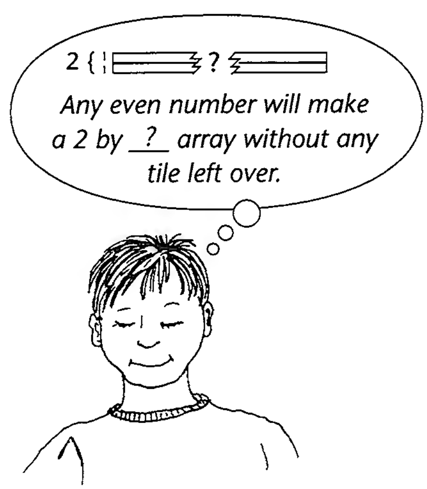


- Odd or even? A number larger than 700 that has a units digit of 3; a number with a units digit of 6.

JANICE *Any number of 10's is even. The same for 100's and 1000's. So when you add these, the total is even. But if there are 3 units added in, the number will be odd, because odd + even is odd.*

ALLEN *I can picture a number with a 6 on the end. The number will be even because the 6 is even and so is the rest of the number. Look at 836. 800, 30, and 6 can be split in two piles. Each pile has 400 + 15 + 3 or 418 in it.*

- Ask the children to imagine an even number. Invite them to describe their mental picture of this number. Compile a list of general observations the children have about this number. Do the same for an odd number.



27 Revisiting Counting Patterns (continued)

The illustrations indicate some of the methods and models children may use to think about odd and even numbers. They also describe how children might recall rules, discover relationships, or propose hypotheses about properties of these numbers. Consider ways to encourage the children to explore their observations further. (Please see Teacher Tips.)

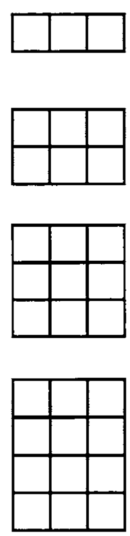
> Jeremy's hypothesis: Any bunch of even numbers that are added will give an even number.
>
> This is what Jeremy thinks is true about adding even numbers. How do you feel about his hypothesis? With your partner, see if you can find some way of finding out if his statement is true all the time.

Activity 2

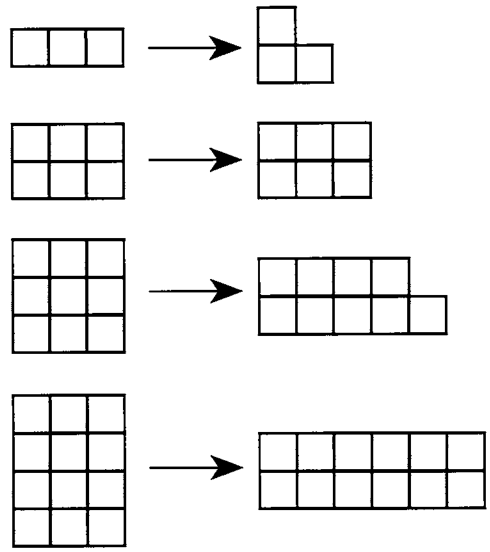
The 2 counting pattern consists of all multiples of 2, and, as such, is identical with the entire set of even numbers. What about the 3 counting pattern? Does it have any special properties? How is it related to the 2 counting pattern? This activity explores some of these questions.

Build the following rectangular arrays at the overhead and label their areas. These arrays model the first four positive numbers in the three counting pattern. Ask the children to share what they recall about this pattern.

1st
2nd
3rd
4th



Call attention to the areas of these arrays. These areas (3, 6, 9 and 12) are alternately odd and even. Ask a volunteer to demonstrate this by rearranging the arrays to show parts of the odd-even model described in Activity 1.



Ask the children to explore these problems in small groups:

- What are the 8th and 9th numbers in the 3 counting pattern? Are they odd or even? Explain.
- Decide which numbers between 170 and 190 are part of the 3 counting pattern. How did you make your decision? Are these numbers odd or even?
- Choose two numbers whose difference is 30 and write them down. Find the numbers in the 3 counting pattern that are between your two numbers. Why do the 3 counting pattern numbers still alternate between odd and even?

Activity 3

The 2 counting pattern consists entirely of even numbers. The 3 counting pattern alternates between odd and even numbers. Ask the children to search for other counting patterns which have all even numbers. How about others which alternate between odd and even? How about some counting patterns that have all odd numbers?

Teacher Tips

Below is a list of properties that may come up for discussion, together with some supporting illustrations. They are presented here for your reference. Do not feel that all (or any) of these properties must be discussed at this time. If it seems appropriate within the context of a discussion to explore some of them, give the children the opportunity to do so on their own.

- The sum (or difference) of two even numbers is an even number.
- The sum (or difference) of two odd numbers is an even number.
- The sum (or difference) of an odd and an even number is an odd number.

even	even	← the result is even.
even	odd	← the total is odd.

- Any number with a units digit of 0, 2, 4, 6 or 8 is an even number. If the units digit is 1, 3, 5, 7 or 9, then the number is odd.
- Picture a number—each 10, 100, 1000, etc., can be split in two. The only place in question is the units place. If that digit can be split in two (i.e., if it is 0, 2, 4, 6 or 8), then the whole number will be even; otherwise it will be odd.
- An even number is divisible by 2, with 0 remainder.

Homework

Ask your children to analyze Sammy's Strange Allowance (Blackline 177). Allow time for the children to share their thinking about this problem.

Opening Eyes to Mathematics **Sammy's Strange Allowance** **Blackline–177**

Sammy received a strange allowance. He got a nickel each day of the week except Thursday and Sunday. On those two days only, he got a dime. During which month this year will he receive the most money? How much money will he get during that month?

Draw a model or picture that illustrates your thinking. Write some sentences to describe why you chose the month you did. Show any equations that support your thinking.

During which month will Sammy receive the least amount of money? Why?

How much money will Sammy receive during the entire year from his allowance?

Journal Entry

- Why do you think the numbers in the 3 counting pattern go back and forth between odd and even?
- Think of the counting pattern for a number between 41 and 49. Are the numbers in that counting pattern all even? all odd? going back and forth between odd and even? Explain your answer.

Published Piece

(Optional) Create a display that shows all possible combinations of *odd* whole numbers that total 15.

Daily Computation

What are some numbers that belong to both the 3 and the 5 counting patterns? What are some numbers that are part of the 3, 4 and 6 counting patterns?

28 *Problem Solving—Football Scores, Part I*

You Will Need

- a local newspaper containing football scores of interest to the class
- a transparency showing the following information:

Touchdown: 6 points
Field Goal: 4 point
No other types of scores can occur

- chart paper, markers, grid paper and calculators available
- a copy of Blackline 178 (Football Scores) for each child

Your Lesson

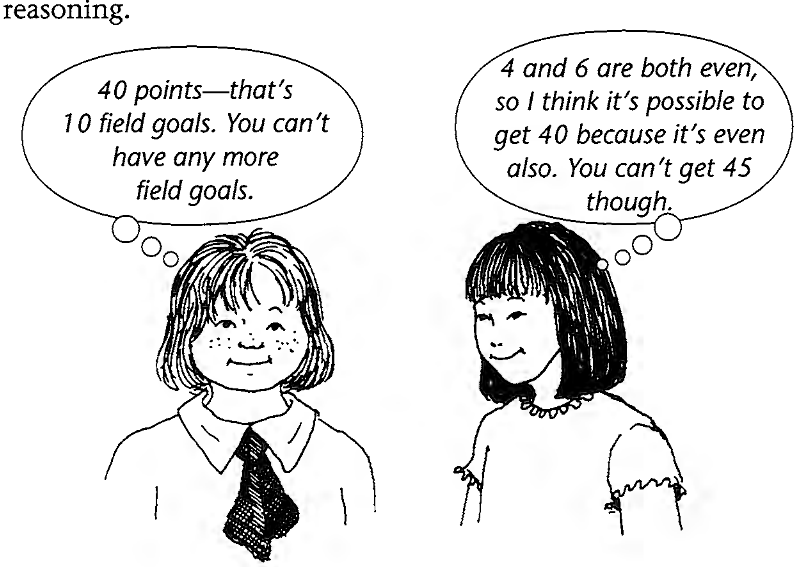
Interest in football runs high in our part of the country, so we find this game offers a nice context for problem solving (Please see Teacher Tips). The activities of both parts of this lesson revisit those conducted in Contact Lesson 40 (Volume 1) and extend some of the numerical relationships discussed in Lesson 27 of this volume, Revisiting Counting Patterns. These activities are likely to take several days and, as you will see, provide plenty of computational practice.

Activity 1

After a brief discussion of the results of recent football games, ask the class to imagine a special football game in which only touchdowns and field goals can be made. Further, in this game, each touchdown is worth 6 points and each field goal 4 points (see transparency above).

Ask the children to explore this problem in small groups: In this special football game, is it possible for a team to score a total of 40 points? If so, how many ways are there of achieving this total?

Encourage the groups to identify clues that might help them solve this problem. When ready, have the groups share their results and reasoning.



It is likely that groups will find more than one solution to the problem. Continue the search until all possibilities have been found. How can one be sure when all solutions have been identified?

Points	Points	Points	Points
10 FG = 40	7 FG = 28	4 FG = 16	1 FG = 4
0 FG = 0	2 TD = 12	4 TD = 24	6 TD = 36
Total 40	Total 40	Total 40	Total 40

Now pose these problems to the groups (answers are in parentheses):

A total of 40 points can be achieved in four different ways. How many different ways can each of the following point totals be achieved?

a) 70 points. (There are 6 different ways.)
b) 73 points. (This total can't be achieved.)
c) 124 points. (There are 11 different ways.)

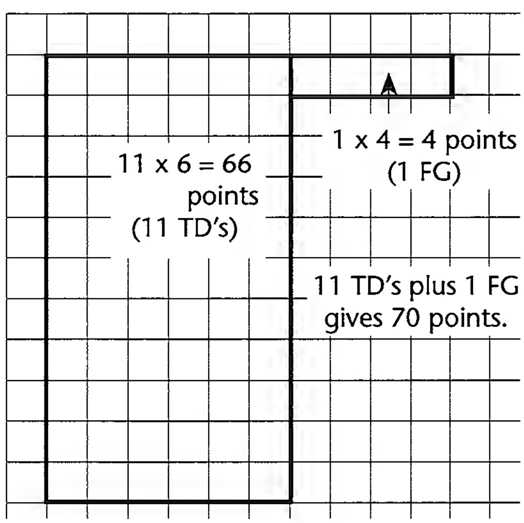
While some groups may use only trial-and-error to answer these questions, others might have some initial observations and strategies to report. If so, it might be appropriate to discuss them at this time.

LEON *We don't think 73 can be done. Perhaps because 73 is odd.*

PENNY *A touchdown and field goal are both even. That should give even totals only—remember, we learned an even plus an even gives an even number.*

CHICO *We noticed that 3 field goals could be traded for 2 touchdowns. Both ways give 12 points.*

MARCIA *Here's a picture of one way to get 70.*



TEACHER *How can we be sure when we've found all the solutions?*

The last question asked by the teacher in the above dialogue is an important one—encourage groups to be mindful of it as they work. Discuss any suggestions children have for answering it. For example, some might reason that any solution for 70 points must have less than 12 touchdowns and subsequently test all the possibilities less than 12.

Activity 2

Distribute copies of Blackline 178 and have the children fill in 6 points for each touchdown and 4 for each field goal. Then, in groups, have them answer the questions on the blackline.

Opening Eyes to Mathematics **Football Scores** **Blackline–178**

Name ____________________

Each touchdown is worth ______ points.

Each field goal is worth ______ points.

1. How many different ways, if any, can a team score a total of 92 points? List the ways.

2. Think of a point total larger than 100 that can be made. List the different ways of making this total.

3. If possible, give at least two examples of point totals larger than 100 that can't be made. Explain the reasoning you used to arrive at your answer.

4. What observations or hypotheses do you have about point totals that can be made?

5. What observations or hypotheses do you have about point totals that can't be made?

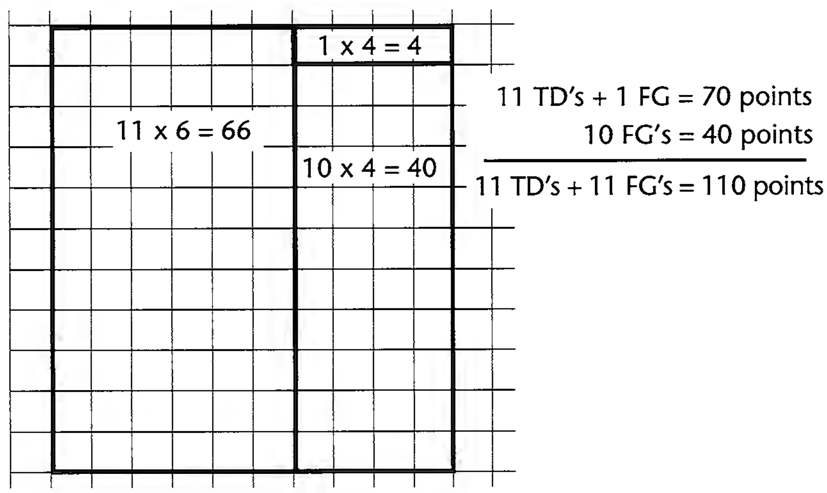
Note the strategies and thinking used by the groups. Some may continue to use only trial-and-error. Others may apply some of the information and patterns discussed earlier or discover some new relationships that are helpful.

GROUP 1 *We chose 120 as our total in Question 2 (Blackline 178). We knew that 30 field goals would work since $30 \times 4 = 120$. Then we traded in 3 field goals for 2 touchdowns each time until we got to 20 touchdowns. You can make a table.*

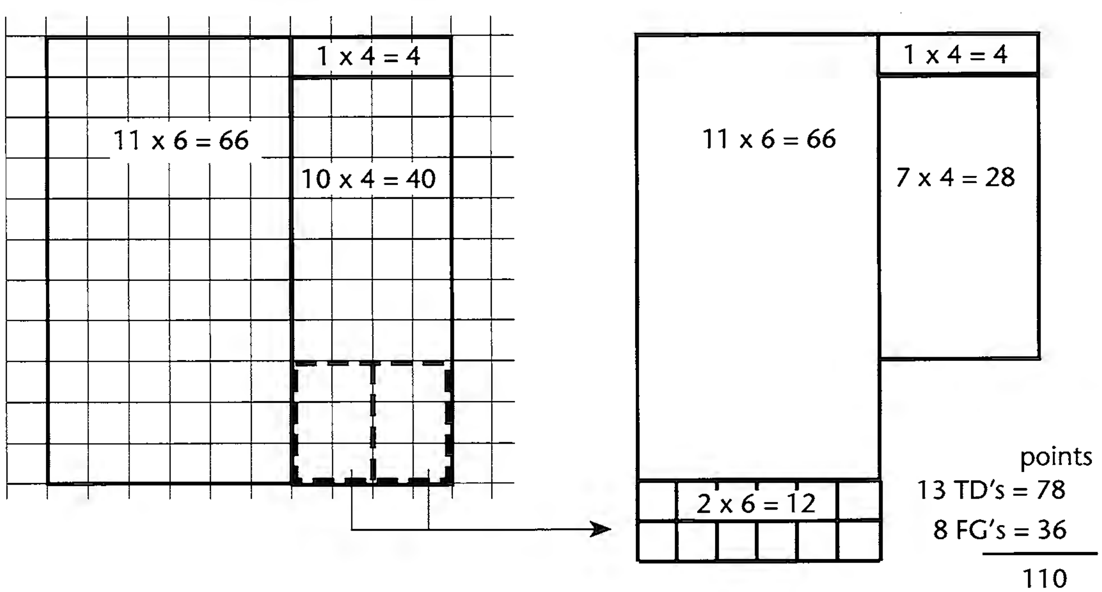
TD	FG
0	30
2	27
4	24
6	21
8	18
20	0
.	.

There are 21 different ways to make 120 points.

GROUP 2 *We thought of 110 as our total in Question 2. We made 110 by putting answers for 70 and 40 together. Those answers came from before. This picture shows our totals.*



The picture helped find another way.



In this case, where a touchdown is worth 6 points and a field goal 4 points, it is not possible to achieve any odd point total (there is also one even point total that can't be achieved). As described above, this can be explained by referring to addition properties related to odd and even numbers.

Teacher Tips

You may prefer to use a context other than football for this lesson. For example, a setting that uses 4¢ and 6¢ stamps works well. How could one send a letter requiring 40¢ postage, choosing only from 4¢ and 6¢ stamps?

Homework

Remind the children that, in the above game, a point total of 40 can be achieved in exactly four different ways. Encourage them to work with a friend or family member on this problem:

Find a point total, other than 40, that can also be achieved in exactly four different ways. Describe the reasoning you used to solve this problem.

Journal Entry

Describe a friend's strategy for doing the problems of this lesson that you like. What did you like about the strategy?

29 Problem Solving—Football Scores, Part II

You Will Need

- chart paper, markers, grid paper, and calculators available
- several copies of Blackline 178 (Football Scores) for each child

Your Lesson

In Lesson 28 of this volume, your children examined a special football game in which touchdowns were worth 6 points each and field goals were 4 points each. In this game, it is not possible to achieve any odd point total. You might introduce this part of the lesson by having the children share some of their homework results from Lesson 28.

The activities of this lesson examine similar games, where the point values of a touchdown and a field goal are changed.

Distribute copies of Blackline 178 and divide the class into three sets of small groups. Ask the groups in each set to fill in the following touchdown and field goal values:

	Value of each touchdown	Value of each field goal
Set A	8 points	4 points
Set B	9 points	6 points
Set C	7 points	5 points

Now ask the groups to address the questions on Blackline 178 and to prepare a summary report of their work on chart paper (or in a book). Allow plenty of time and opportunity for groups to explore and take "field trips" to discuss their progress with one another.

Your role in this investigation is very special. You will need to be a facilitator, deciding when, if at all, to intervene if a group is struggling. Questions such as these may be helpful: Would it help to look at smaller point totals first? Would it help to look at the patterns discussed in Lesson 28? Why do you feel a particular total can't be achieved? How did you decide that all solutions had been found? Will that line of reasoning always work?

This is also an opportunity for you to be an observer. In fact, you may wish to use a checklist as an assessment tool. How are the children doing relative to such things as asking questions, making decisions, looking for connections, devising strategies, etc.?

Mathematically, this investigation can motivate discussion of common factors and other numerical relationships. Notice that 2 is a common factor of 4 and 6 since it divides both numbers. Consequently, any combination of 4's and 6's (as in Lesson 28) yields point totals that must be multiples of 2. Similarly, 4 is a common factor of 4 and 8 and, for the groups in Set A, any possible point total must be a multiple of 4. It will be interesting to see how your children convey these ideas in their reports.

Some pairs of natural numbers, such as 4 and 9, do not have any common factor other than 1. In later grades, the children will learn that such pairs are called relatively prime numbers. In the chart below, the pairs marked with asterisks are relatively prime numbers.

Numbers	Common Factors
5 and 9 **	1
5 and 10	1 and 5
10 and 60	1, 2, 5 and 10
7 and 11 **	1
10 and 27 **	1

The groups in Set C were assigned relatively prime values (5 and 7) for field goals and touchdowns. In this case, some small point totals can't be achieved (such as 13). But beyond 23, every point total can be achieved.

Teacher Tips

Here is some background information. The first question asked in Lesson 28 called for solutions to this equation:

$4x + 6y = 40$, where x represents the number of field goals and y represents the number of touchdowns.

One solution is $x = 10$ and $y = 0$ since $4(10) + 6(0) = 40$.

Another is $x = 7$ and $y = 2$ since $4(7) + 6(2) = 40$.

In this problem, x and y must be positive integers (or natural numbers). With this restriction, the equation $4x + 6y = 40$ is an example of a Diophantine Equation. This is because the search for (integer) solutions to this equation dates back to an ancient Greek mathematician named Diophantus. This person was a principal contributor to the development of number theory. In fact, several of his contributions are often found in elementary school mathematics curricula today. You and your children might enjoy doing some library research about Diophantus and some of the problems he studied.

Homework

Give each child a fresh copy of Blackline 178 and ask them to select values for touchdowns and field goals that haven't been used yet. Then, with a friend or family member, ask the children to answer the questions on Blackline 178 using their chosen values.

Published Piece

(Optional) You might have the children compile the results of their group work in this lesson, along with those from the homework, into a book for publication.

Pattern Reminder

Consider asking your children to complete one of the remaining Pattern Generalizations (Blacklines 169–176).

30 Shifting the Unit, Part I

You Will Need

- Chapter 7, Measurement
- transparencies of Blacklines 179–181 (Shifting the Unit)
- copies of Blacklines 179–181 (Shifting the Unit)
- geoboard and rubber band for each child

Introduction

We're sure the following statement has meaning for you:

Jennifer has 3 dollars in her purse.

In this sentence, the "3" is used to report a measurement. It describes an amount of money in terms of a unit (dollar) that is familiar to people who possess a knowledge of our monetary system. That unit, in fact, gives the sentence meaning. Read the statement again, only this time leave out the word "dollars":

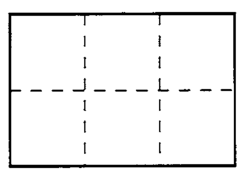
Jennifer has 3 in her purse.

Whenever one wishes to communicate measurements, it is important to be aware of the corresponding units of measure. Sometimes these units are clearly understood from the context of a situation; if not, then the units must be specified.

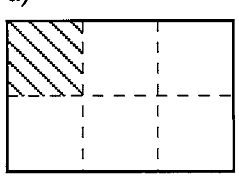
The choice of a unit of measure is arbitrary, though it needs to fit with what is being measured. Because of this, different numbers may be reported for the same measure.

Jennifer has 3 dollars (300 cents, 30 dimes, etc.) in her purse.

As another example, the illustration below shows three ways of reporting the area of a particular rectangle. In each part of the illustration, the shaded region represents 1 unit of area.

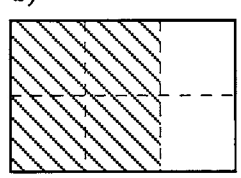


a)



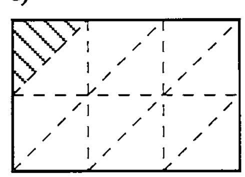
area: 6

b)



area: 1½

c)



area: 12

Note: Typically, square units are used to measure areas. It is possible, however, to use units of other shapes as shown in c) above.

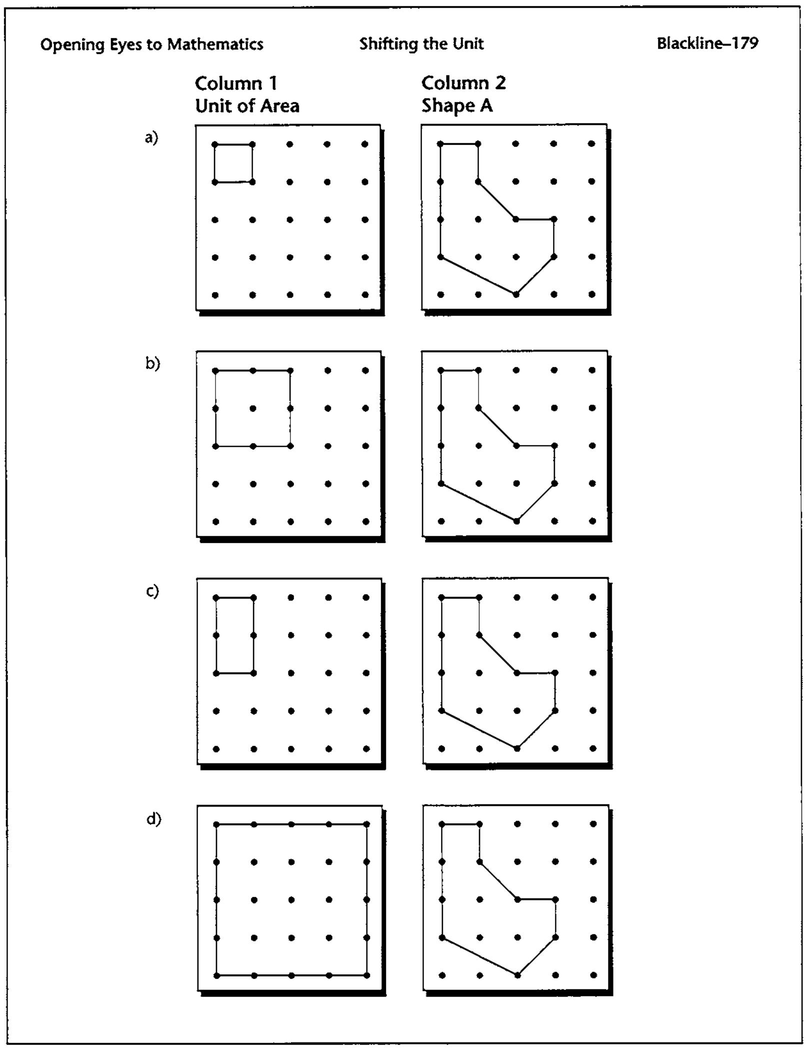
30 Shifting the Unit, Part I (continued)

Lessons 30–32 of this volume explore the mathematical consequences of changing the unit of measure in a problem. The activities of these lessons are introductory in nature and are extended later in *Visual Mathematics*.

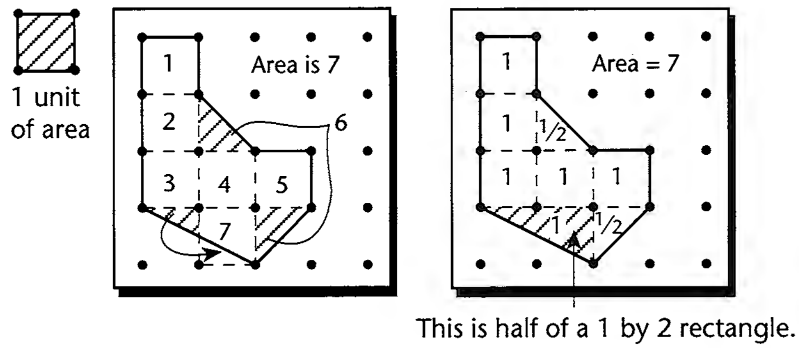
Your Lesson

In this lesson, children use different units of area to determine the area of a given shape. They examine the effects of changing the unit on the reported area of a shape.

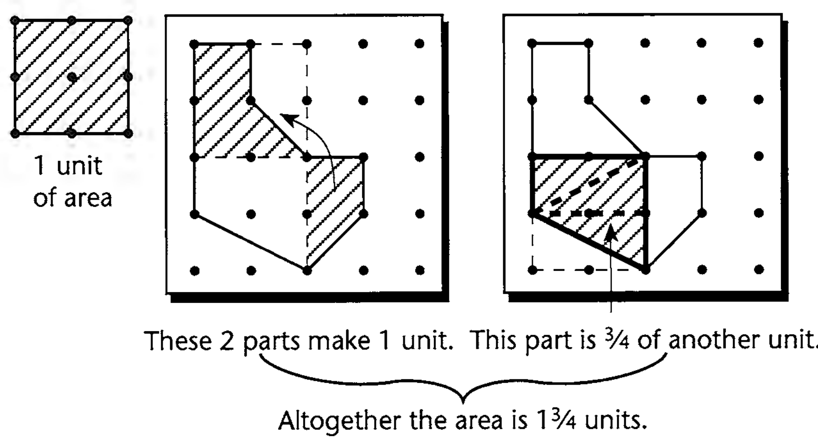
Distribute geoboards, rubber bands and copies of Blackline 179 to each team of two children. Referring to Part a) of this blackline, ask one team member to build the square shown in Column 1 on their board and the other to build Shape A.



Display a transparency of this blackline on the overhead, revealing just Part a). Tell the class the square in Column 1 has an area of 1 square unit. Knowing this, ask the teams to determine the area of Shape A. Invite volunteers to share some different ways of determining this area.

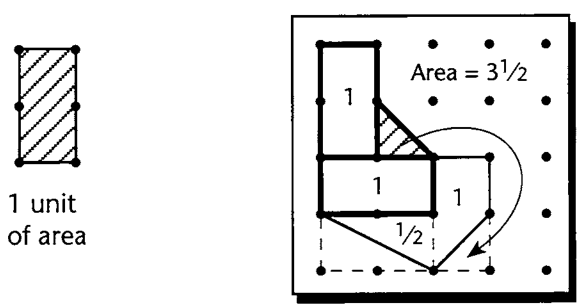


Reveal Part b) of the transparency. Ask the appropriate team members to build the new square shown in Column 1; Shape A remains the same on the other geoboard. Ask the class to imagine that this new square is now the unit of area—that is, it has an area of 1 square unit. Assuming this, have the teams determine the area of Shape A once more. Teams are likely to have several ways of doing this (one possibility is shown below), so allow for an appropriate amount of sharing time.

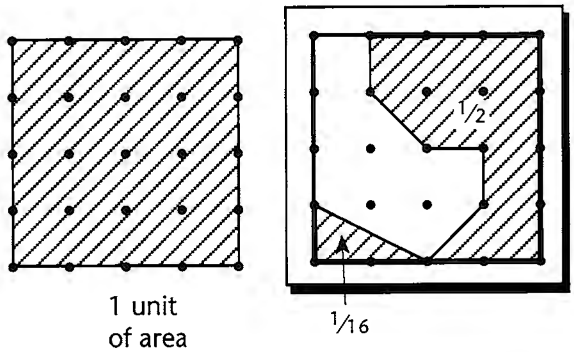


Repeat for Parts c) and d) of the blackline, changing the unit of area to be the corresponding shapes shown in Column 1.

Part c)

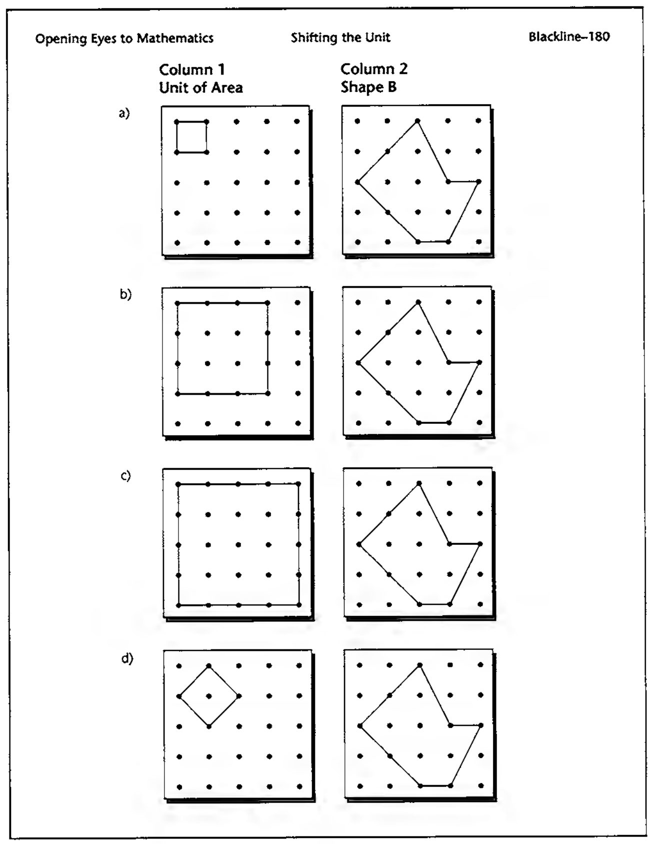


Part d)

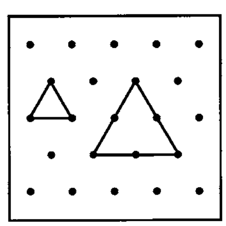


The area of Shape A is
$1 - \frac{1}{2} - \frac{1}{16} = \frac{1}{2} - \frac{1}{16}$
$= \frac{8}{16} - \frac{1}{16}$
$= \frac{7}{16}$

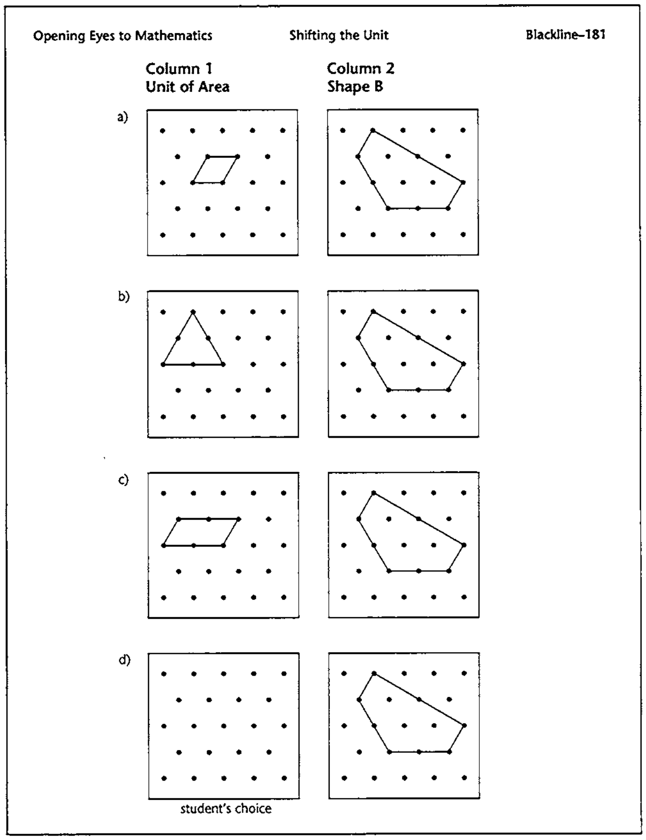
Blacklines 180 and 181 contain problems similar to those of Blackline 179. Ask the teams to find the areas of the shapes in Column 2 of these blacklines, using the corresponding units in Column 1.



The shapes on Blackline 181 are formed on isometric grid paper, where the dots are arranged in a grid based on equilateral triangles.



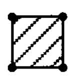
equilateral triangles

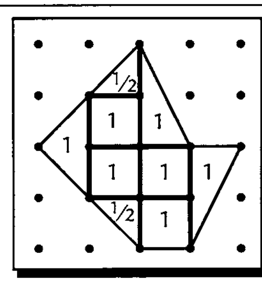


Note and discuss any strategies that children bring to these exercises. Encourage any movement of the teams towards making connections among the parts of the blacklines. Here are sample solutions for some parts of these blacklines.

Blackline 180

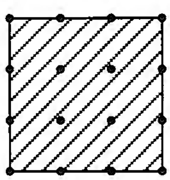
Part a)

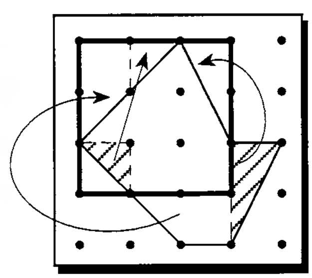




The area of Shape B is 8.

Part b)



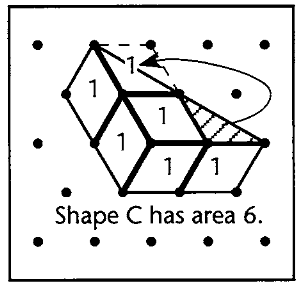


Shape B can be rearranged to fill 8/9 of a unit. It's area is 8/9.

Part c)
Look at the unit in Part a). It takes 8 of these to make shape B. But it takes 16 of them to make the new unit in Part c). So B has ½ of what is needed to make a whole unit. It's area is ½.

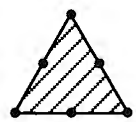
Blackline 181

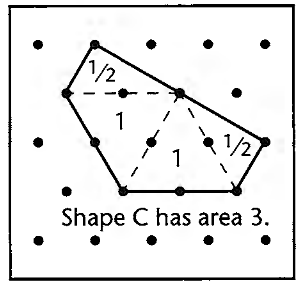
Part a)



Part b)

Solution 1:





Solution 2: "It takes 2 of the old unit—Part a)—to make the new, so you'll need ½ as many of the new to fill Shape C."

Part c)

Shape C has area 3. This unit is the same size as the one in Part b).

Provide further independent practice, as needed, by having the children repeat this activity for other shapes and area units. Have the children make up problems for their friends to try.

Teacher Tips

The activities of this lesson are geoboard versions of those done with pattern blocks in Volume 2, Contact Lessons 151–153. You may wish to revisit those lessons.

Journal Entry

Briefly describe something about area that you learned during this lesson.

Assessment

Ask each child to select an example from this lesson that was difficult for them in some way. Then have each child provide the following information about that example: A restatement of the problem in the example; related diagrams or sketches; a brief description of the difficulty; an explanation of how the problem was solved.

31 *Shifting the Unit, Part II*

You Will Need

- Chapter 4, Numeration: Place Value
- overhead base four area pieces
- transparency of Blackline 182 (Base Four Area Pieces)

for each group of four children

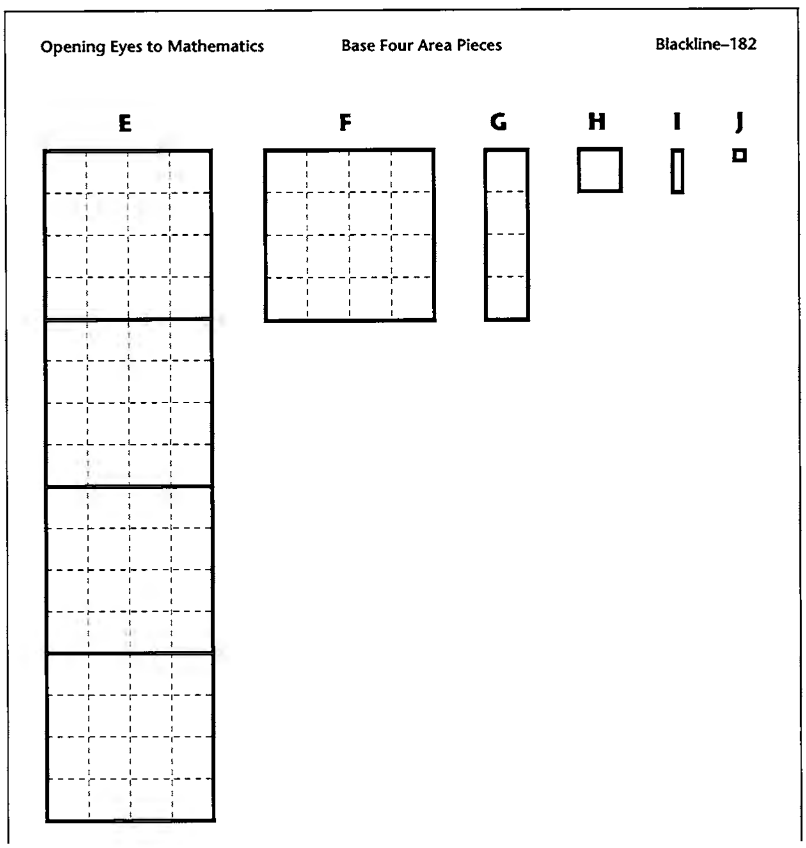
- copies of Blackline 182 (Base Four Area Pieces)
- a set of base four area pieces (from Lesson 4, Volume 3)

Your Lesson

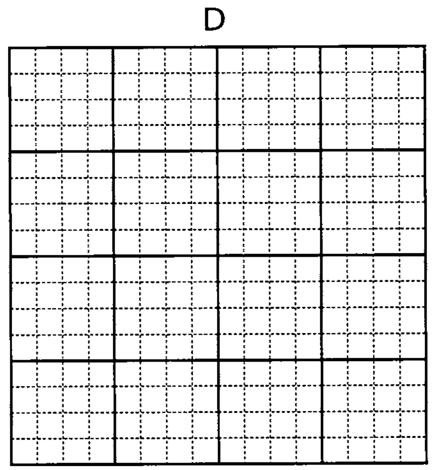
In Lesson 30, Volume 3, your children investigated the effects of changing the unit of area on the reported area of a shape. This lesson is a continuation of that investigation, only this time focusing on areas of base four collections. The activities of the lesson provide added experiences with place value models, positional notation and fraction arithmetic.

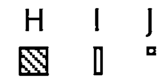
Activity 1

Give each group of four children a set of base four area pieces and a copy of Blackline 182. This blackline shows six different base four pieces in order of decreasing size. Display a transparency of this blackline at the overhead and review the structure of these pieces with your class. What patterns are exhibited?



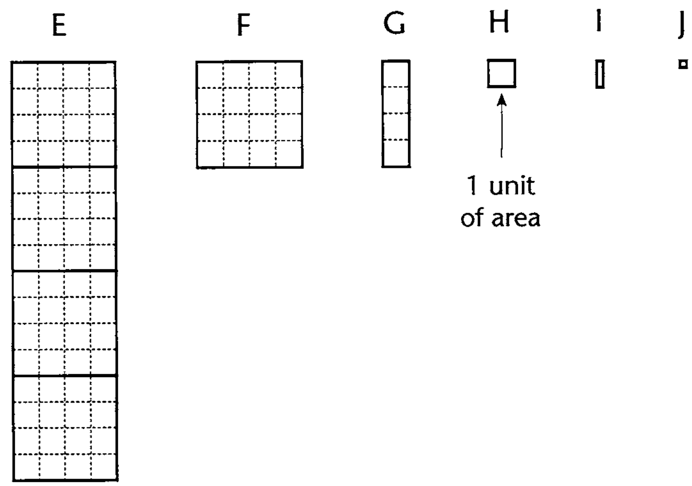
What would piece D (the next largest piece) look like?





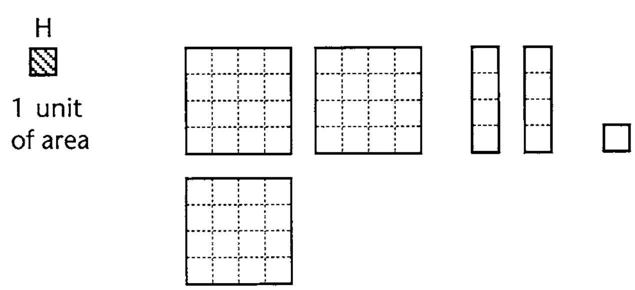
The area of H is 4 times that of I and 16 times that of J.

Point to piece H and tell the class to assume this piece has area 1.



Now ask the groups to address this problem, making a sketch of their answers on chart paper: Form the minimal collection for 57 units of area using base four pieces. If appropriate, have them also use positional notation to report their collection as a base four numeral (see Teacher Tips).

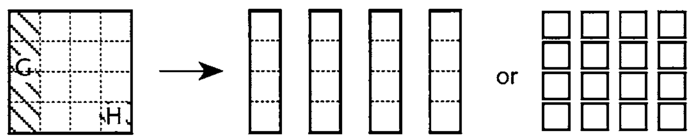
Conduct a show-and-tell discussion of the children's work. Identifying the pieces by the letters shown on Blackline 181, the required collection consists of 3 F's, 2 G's and 1 H. It may be reported as 321_{four}. In this context, 57 and 321_{four} are the base ten and base four names for the area of the collection, respectively.



Base ten name for area: 57
Base four name for area: 321_{four}

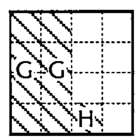
Have the groups keep their collection in front of them. The previous illustration reports the total area of this collection assuming piece H is the unit. Suppose, however, piece F is selected as the unit of area (i.e., the area of piece F is 1 square unit). Have the groups determine the area of their collection using this new unit, summarizing their results on their charts.

GROUP 1 *We thought about the growing pattern in the pieces. It takes 4 G's to make an F; so the area of G is ¼. It takes 16 H's to make an F, so H is 1⁄16.*

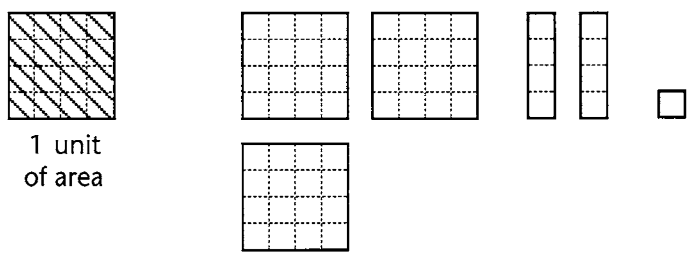


1 F splits into 4 G's or 16 H's

GROUP 2 *We knew the area was more than 3, since there are 3 F's to begin with. We laid the 2 G's and the H on top of one of the F's and saw that 9⁄16 of the F was covered.*



Altogether the area is 3 9⁄16.



Base ten name for the total area: $3 + \frac{2}{4} + \frac{1}{16} = 3\frac{9}{16}$

Base four name: $3{\odot}21_{\text{four}}$

Invite observations from the children about the results thus far.

ALPHONSO *Looks like the pieces are the same, but the area is a smaller number.*

TEACHER *Why do you think that is?*

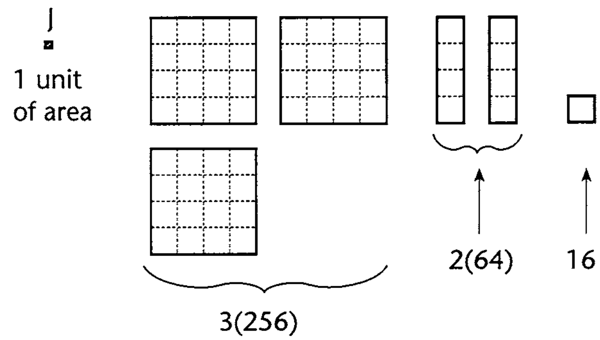
BILL *I'm not sure.*

JANE *The unit is bigger—so it takes fewer units to build up the collection.*

SONJA *What if we made another piece the unit? What would the area of our collection be?*

Activity 2

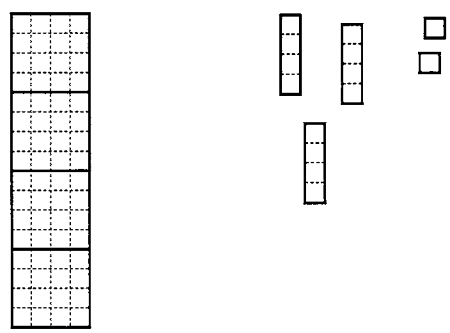
Suppose piece J is selected as the unit of area. Using this new unit, have the groups determine the area of the collection of 3 F's, 2 G's and 1 H.



Base ten name: $3(256) + 2(64) + 16 = 902$
Base four name: 32100_{four}

Examine other collections in the same way. What would their areas be using different pieces as the unit of area? What kind of relationships do the children notice? Here is an example.

Example 1: What is the area of this collection: 1 E, 3 G's and 2 H's?



There are several answers to this question, depending on which piece is chosen as the unit of area. Some of these answers are reported in the following chart.

	unit			
area	H	F	G	E
base ten name	78	$4\frac{14}{16}$ or $4\frac{7}{8}$	$19\frac{1}{2}$	$1 + \frac{3}{16} + \frac{2}{64} = 1\frac{14}{64} = 1\frac{7}{32}$
base four name	1032_{four}	$10{\odot}32_{four}$	$103{\odot}2_{four}$	$1{\odot}032_{four}$

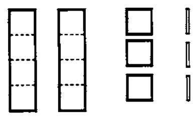
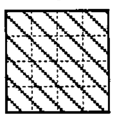
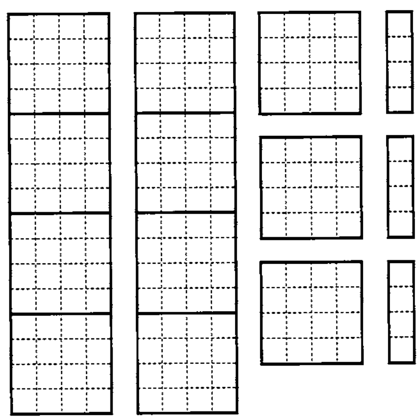
Activity 3

Ask the teams to form the minimal collection of base four pieces that have the areas listed below. You may also wish to have the teams represent each area by a base four numeral. (Note: Leave the choice of unit up to the teams.)

a) 136 area units
b) $11\frac{3}{4}$ area units
c) 300 area units
d) areas of their choice.

The diagrams below show some solutions for parts b) and c):

Part b)

unit	area 11¾ (base ten)	collection	base four numeral
Solution 1 H		2 G's, 3 H's, 3 I's	$23{\odot}3_{four}$
Solution 2 F 		2 E's, 3 F's, 3 G's	$23{\odot}3_{four}$

Part c)

unit	area (base ten)	collection	base four numeral
Solution 1 H	300 or 1(256) + 2(16) + 3(4)	1 D, 0 E's, 2 F's, 3 G's, 0 H's	10230_{four}
Solution 2 I	300 or 1(256) + 2(16) + 3(4)	1 E, 0 F's, 2 G's, 3 H's, 0 I's	10230_{four}

Activity 4

Ask the teams to make up problems like those above. Problems may then be exchanged among teams.

Activity 5

Repeat the above activities using a different base.

31 Shifting the Unit, Part II (continued)

Teacher Tips

If you wish, these activities can be done without referring to base four notation. Children can still gain helpful experience with place value models, areas, fractions and changing units.

Working with base four notation can strengthen an understanding of positional notation and reveal characteristic features of our numeration system. Look at Example 1, Activity 2, once more and notice what happens when the unit is changed from H to F:

The area of the new unit (F) is 16 times that of the old (H).

Consequently, the new reported area of the collection ($4\frac{14}{16}$) is $\frac{1}{16}$ of the old reported area (78). In fact, $\frac{78}{16} = 4\frac{14}{16}$.

In turn, this is reflected in the corresponding base four descriptions of the area:

		unit	
		H	F
area	base ten	58	$4\frac{14}{16}$
	base four	1032_{four}	$10\odot32_{four}$

Note the shift of the $\odot$ mark two places to the left. This shift corresponds to the division by 16 that occurred in the area. A similar shift of two places occurs in base ten whenever one divides by 100.

Finally, notice the base four numerals reported in the solutions for Part b), Activity 3, on the previous page. They're the same for each solution, regardless of the unit chosen. This is to be expected, given the relative sizes of the pieces and the nature of positional notation. $23\odot3_{four}$ always represents $2(16) + 3(4) + 3(\frac{1}{4})$, regardless of the models for 16, 4 and $\frac{1}{4}$. Similar statements are true in base ten. For example, 233 always represents $2(100) + 3(10) + 3(1)$.

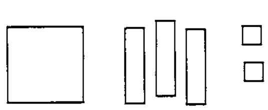
Journal Entry

(Or Assessment) You may wish to draw upon problems such as the following for journal entries, homework or portfolio assessment:

1) Form a collection of base four area pieces and choose a unit of area. What is the area of your collection? Make a sketch of your collection and describe how you determined its area. Use positional notation to describe your collection as a base four numeral. Select a new unit of area. What is the new area? Explain your thinking.

2) Choose a base. Form some collections of area pieces for your base that have a total area of 83 units (or some other number). Describe your procedure for doing this. How many different collections are there? How can you be sure when you have them all?

3) Select a smaller base than the one above. Using this new base, is it possible to form a greater number of different collections with an area of 83 units than before? Why or why not?



4) Show the children a generic sketch of some mats, strips and units (an example is show at the left). Ask them to report the total area of this collection using different bases and different units of area.

32 *Shifting the Unit, Part III*

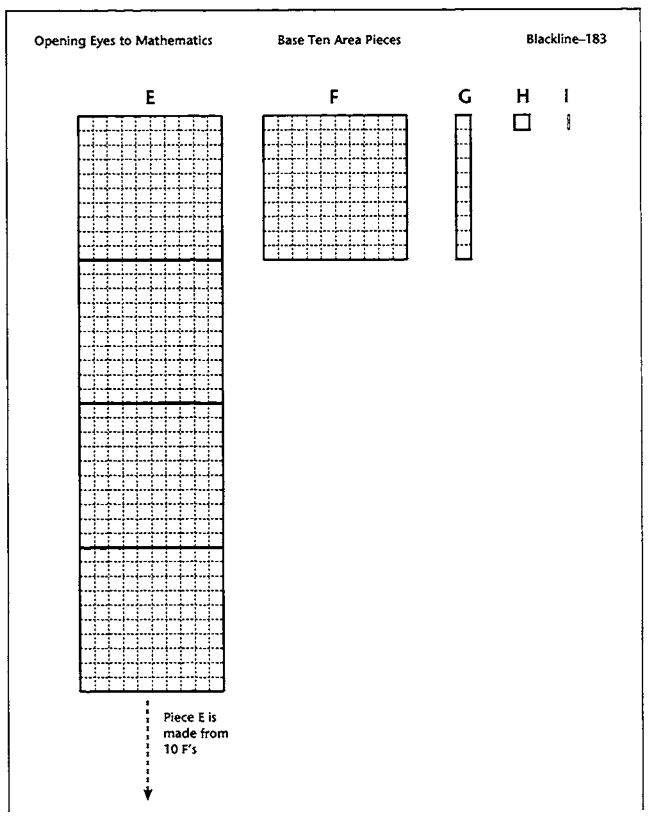
You Will Need

- Chapter 4, Place Value, and Chapter 6, Multiplication and Division
- transparencies of Blackline 183 (Base Ten Area Pieces) and Blackline 184 (Shifting Units of Length)
- class chart showing pieces E, F, G, H and I in the order shown on Blackline 183 (Base Ten Area Pieces)
- base ten area pieces and linear pieces for each small group of children
- Blackline 183 (Base Ten Area Pieces), 1 copy per child, and Blackline 184 (Shifting Units of Length), 2 copies per child

Your Lesson

Well, after all the base four work in Lesson 31, Volume 3, it's time to return to base ten! First, let's look back at familiar patterns in base ten and examine some consequences of changing the unit.

Activity 1



Display a chart of base ten area pieces as shown on Blackline 183, though keep pieces E and I covered for the moment. Uncover pieces E and I after the children have had a chance to describe them.

Distribute base ten area pieces, base ten linear pieces, and Blackline 183 to each group. Ask the class to assume that piece H has area 1. With this assumption, have the groups determine the area and perimeter of each piece shown on the blackline. Make a list of any patterns or numerical relationships modeled by the pieces.

Encourage the search here—try to promote the spirit associated with the Number Study Chart (Lesson C of this volume). When ready, conduct a sharing time during which volunteers describe some of their observations. These observations may be added to a blank piece of chart paper.

	E	F	G	H	I	
area	1000	100	10	1	$\frac{1}{10}$	square units
perimeter	220	40	22	4	2.2	linear units

$1000 = 10 \times 100$ $10 = \frac{1000}{100}$ $\frac{50}{100} + .5 = 1$

$1000 = 10 \times 10 \times 10$ $.3 + .3 + .4 = 1$

$1000 = 10^3$ $\frac{300}{10} + \frac{300}{10} + \frac{400}{10} = 100$

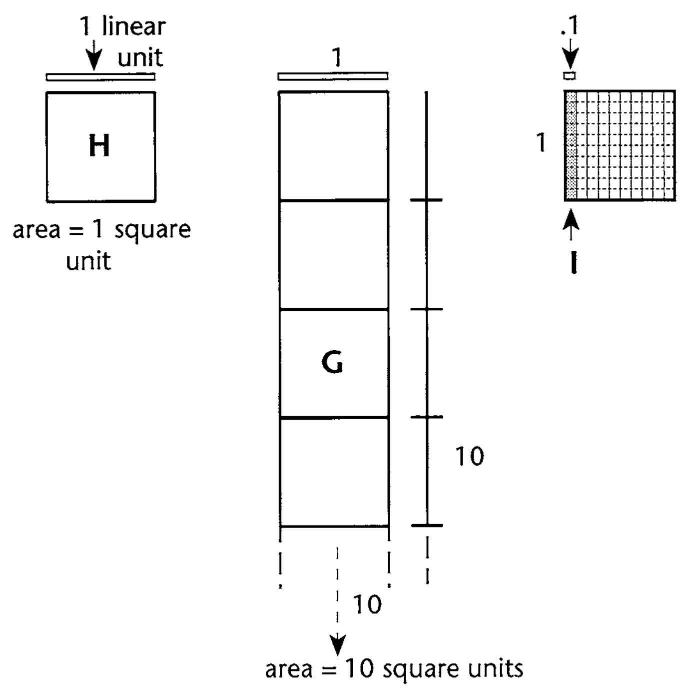
1 = 10 tenths $\frac{500}{100} + 5.00 = 10$

$100 = 10^2$ Area of H is $\frac{1}{10}$ of strip G, $\frac{1}{100}$ of mat F, $\frac{1}{1000}$ of strip-mat E, $\frac{1}{10000}$ of mat-mat D $\frac{30}{10} + \frac{30}{10} + \frac{40}{10} = 10$

etc.

Discuss the perimeter of each shape. How did the children make their decisions here? Sketch linear pieces on the chart to facilitate the discussion. What is the length of each linear piece? In particular, what are the lengths of the edges of piece I?

The perimeters reported in the above illustration assume the edge of piece H has a length of 1. The unit of length is generally taken to be the edge of the square whose area is 1. (Please see note at the end of the lesson.)



Activity 2

Display a transparency of Blackline 183 and pose these problems for investigation: Suppose we select piece F to be a new unit of area. So the area of F is 1 square unit. Determine the area and perimeter of each piece shown on Blackline 183.

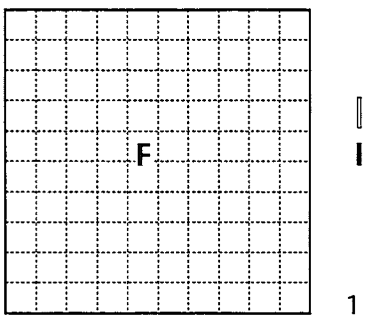
Allow plenty of time for investigation and discussion.

ALLISON *It still takes 10 mats to make a strip-mat. So the area of the strip-mat (piece E) must be 10 now.*

RAMON *What about G? I guess that area must be $\frac{1}{10}$.*

RUBY *That makes sense—10 G's make up the new unit. So do 100 H's—each H must be $\frac{1}{100}$.*

AMY *How many I's make up the unit then? 1000?*



1 square unit

	E	F	G	H	I
area	10	1	$\frac{1}{10}$=.1	$\frac{1}{100}$=.01	$\frac{1}{1000}$=.001

MARK *I'm confused.*

RAMON *What about perimeter? Is the perimeter of F still 40 like before?*

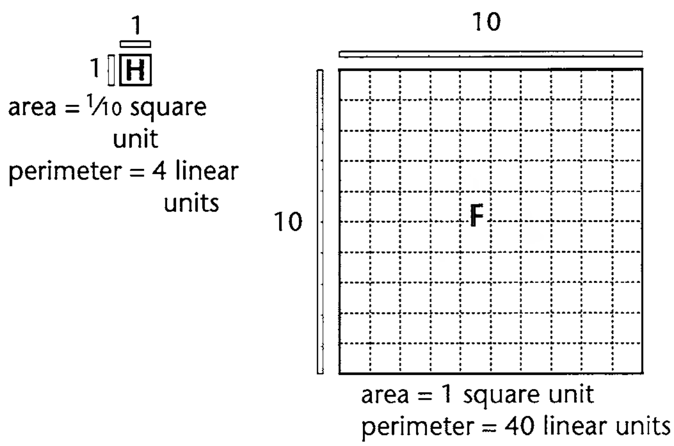
ALLISON *Right. The areas change but the perimeters don't.*

AMY *I see 1 unit of area; but what is 1 linear unit? If it's like before, then the perimeter of H is 4.*

RUBY *Maybe we should let the edge of F be 1 linear unit. That will make the perimeter of F be 4.*

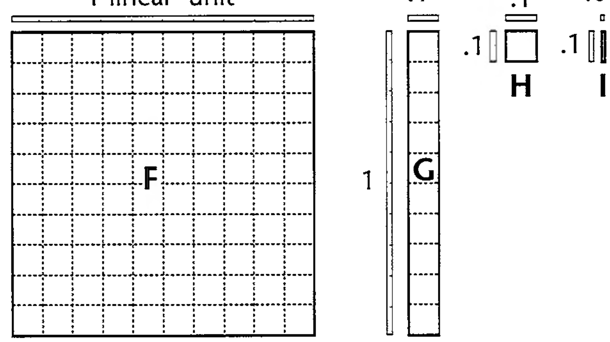
RAMON *We could try it both ways!*

ALLISON *I think I agree with Ruby. This picture looks strange to me:*



(Please see note at end of lesson.)

The chart below shows the perimeters using Ruby's idea.



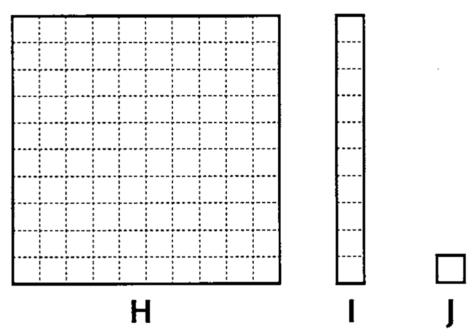
	E	F	G	H	I
area square units	10	1	$\frac{1}{10}=.1$	$\frac{1}{100}=.01$	$\frac{1}{1000}=.001$
perimeter linear units	22	4	2.2	.4	.22

Continue the lesson with activities such as these:

Activity 3

Ask the groups to imagine piece J—the next smallest piece after I in the sequence. What would this piece look like? How is it related to the other pieces?

JIM *The pattern continues. J would be a square and it would take 10 of them to make an I.*



Assume that square J is the unit of area and the edge of J is the unit of length. Using this assumption, have the groups determine the area and perimeter of each piece shown on Blackline 183. Have the groups share their work and invite observations about the results.

1 linear unit

1 square unit

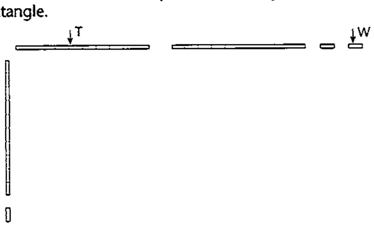
	E	F	G	H	I	J
area square units	100000	10000	1000	100	10	1
perimeter linear units	2200	400	220	40	22	4

Activity 4

Distribute Blackline 184 and discuss the diagram at the top. This diagram shows base ten linear pieces arranged to form the outline of a rectangle. Linear piece T is therefore 10 times as long as piece W.

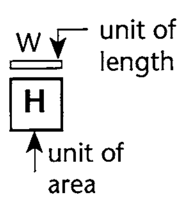
Opening Eyes to Mathematics | **Shifting Units of Length** | **Blackline–184**

In this picture, base ten linear pieces are arranged to form the outline of a rectangle.



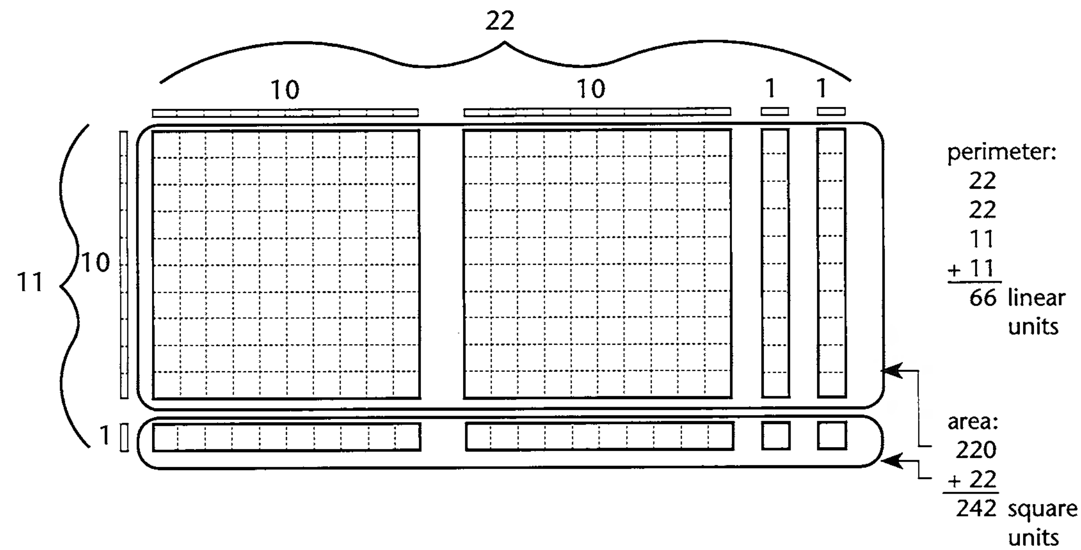
1. Which linear piece is the unit of length?
2. What is the perimeter of the outlined rectangle? Write a number statement that shows how you determined the perimeter.
3. What is the area of the outlined rectangle?
4. Make a sketch of the rectangle and show how to use the Distributive Law to find the area.
5. Write a story that could be modeled by the rectangle.

Assume that linear piece W is the unit of length and the corresponding square (labelled H below) is the unit of area.

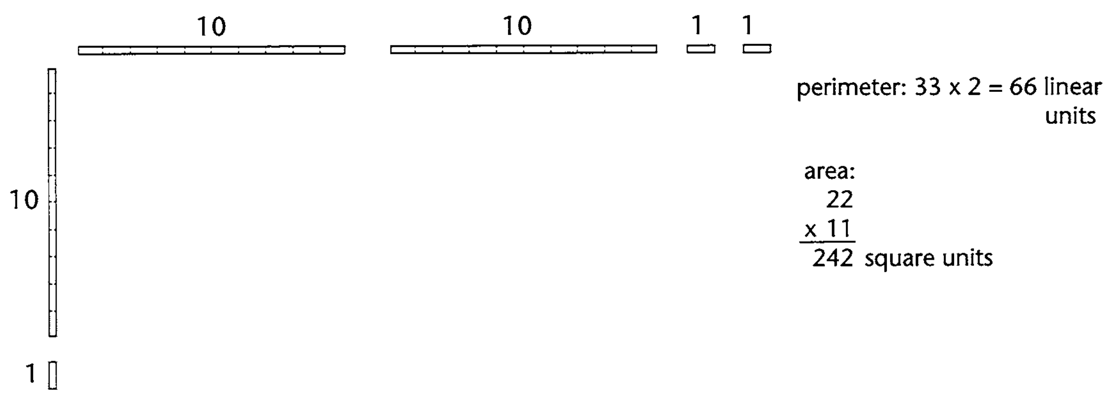


Using these assumptions, have the children work in small groups and answer the questions on Blackline 184.

GROUP 1 *We completed the rectangle and made a sketch to help us.*



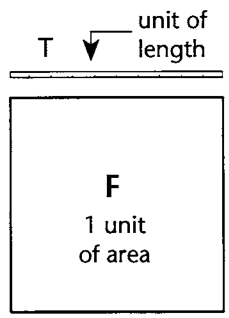
GROUP 2 *We could see half of the rectangle outlined, so the perimeter is 33 × 2 or 66 linear units. We used a calculator and multiplied 22 × 11 to get the area.*

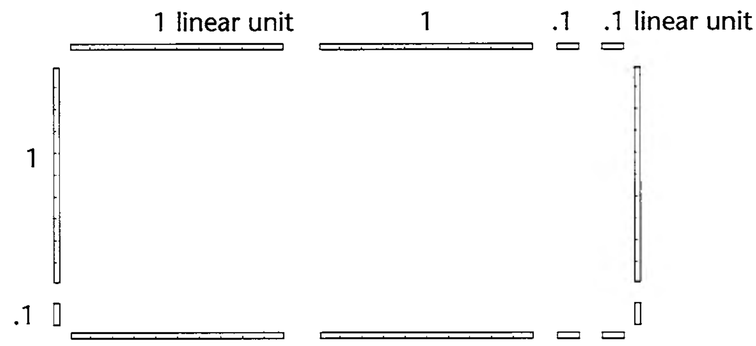


GROUP 3 *The rectangle will be made of 22 elevens. Ten 22's make 220. Add another one and you get 242 square units for the area.*

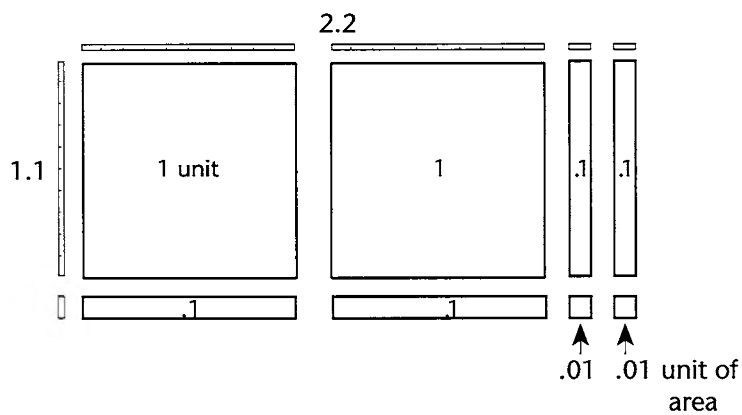
Activity 5

Distribute clean copies of Blackline 184 and repeat the above activity, only this time ask the class to imagine linear piece T is the unit of length and the corresponding square (labeled F below) is the unit of area.





perimeter = 3 + 3 + .3 + .3 = 6.6 linear units



area = 2.2 x 1.1
= 2 + .1 + .1 + .1 + .1 + .01 + .01
= 2 + .4 + .02
= 2.42 square units

Notice how the change in unit suddenly generates decimal multiplication. This illustrates the power of the area model of multiplication.

As time permits, repeat this activity beginning with outlines of other rectangles.

This activity is continued in *Visual Mathematics*.

Published Piece

(Optional) You may wish to have your children compile their work from Activity 2 into a published piece. This piece can include:

- Statements of the problems being solved.
- Sketches and description of strategies.
- Using as models the various rectangles that were made, write multiplication and division stories to fit.
- A list of patterns or clues which might help one visualize the boundaries or interiors of the rectangles.
- Number statements (and sketches) which illustrate the Distributive Property of Multiplication.
- "What if's" suggested by the children.

Daily Computation

This can be an opportunity for your children to think about operating with decimals or larger numbers. How would the children do computations such as these?

a. $1.23 + 1.5$ b. $12340 - 785$

c. $12.3 - 7.06$ d. 14×2.6

Pattern Reminder

Have your children investigate one of the remaining Pattern Generalizations found on blacklines 168–176.

Note

The issues raised in the dialogue of Activity 2 may not come up in your class, so you can decide whether to raise them. As mentioned before, it would be typical in this situation to select the unit of length to be the edge of square F (the corresponding unit of area). The reason for doing this is to preserve this area formula:

$$\text{Area of a rectangle} = \text{base} \times \text{height}.$$

Lengths and areas of rectangles measure different quantities; but they are standardly related by the above formula. This relationship is a consequence of our choosing a square to be the unit of area and then selecting the edge of that square to be the unit of length. (Allison's last comment in the dialogue does refer to something strange, unless we want to give up the base times height formula.)

33 Probability—Opinion Polls, Part I

You Will Need

- Chapters 9, Probability, and 10, Data Analysis and Graphing
- materials for graphing or preparing displays available

for each child

- a copy of Blackline 185 (Survey Form), completed as described below
- (recommended) copies of Blacklines 186 and 187 (Sample Surveys)

Your Lesson

"Seven out of 10 people prefer Brand X over Brand Y!"

"Are you in favor of the current school bond issue?"

Advertisements and surveys with statements or questions like these are quite common. The study of probability and statistics can help people respond to these situations in an informed way. Part of this study involves analyzing data and becoming familiar with the nature of chance. This lesson seeks to do some of this within the context of a two-question opinion poll. It assumes your children have had the graphing experiences in the Contact lessons of Volume 2.

Ask each child to respond "Yes" or "No" to each of the following questions:

Question 1: Do you like (Brand X)? (Yes or No)

Question 2: Do you like (Brand Y)? (Yes or No)

Replace "Brand X" and "Brand Y" with appropriate items of interest to your class. (We have included sample data sets on Blacklines 186 and 187, shown on following page, and these sets will be used as illustrations in the discussion below.) One way to gather information is to post Blackline 185, Survey Form (with the questions filled in) on a bulletin board, and invite the children to mark their responses sometime prior to this lesson.

Opening Eyes to Mathematics **Survey Form** **Blackline–185**

Please Answer "Yes" or "No" to each question.

Question 1:

Question 2:

Child	Question 1	Question 2
1		
2		
3		
4		
5		
6		

After the data has been collected, distribute a copy of the completed blackline to each child. Invite the children to make some general first observations about the results.

Opening Eyes to Mathematics — Sample Survey Form — Blackline–186

Please Answer "Yes" or "No" to each question.

Question 1: Do you like regular cola?

Question 2: Do you like diet cola?

Child	Question 1	Question 2
1	Yes	Yes
2	Yes	No
3	Yes	No
4	No	No
5	No	Yes
6	Yes	Yes
7	Yes	Yes
8	Yes	Yes
9	Yes	Yes
10	No	Yes
11	No	No
12	Yes	Yes
13	Yes	No
14	Yes	Yes
15	No	Yes
16	Yes	Yes
17	Yes	No
18	No	Yes
19	Yes	Yes
20	No	Yes
21	Yes	Yes
22	Yes	No
23	Yes	Yes
24	Yes	Yes
25	Yes	Yes
26	Yes	Yes
27	Yes	Yes
28	Yes	Yes
29	Yes	No
30	Yes	Yes

Opening Eyes to Mathematics — Sample Survey B — Blackline–187

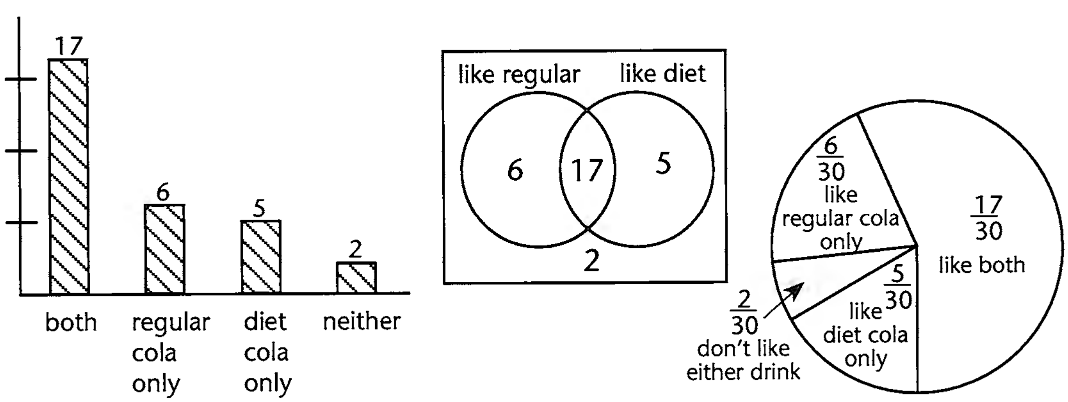
Please Answer "Yes" or "No" to each question.

Question 1: Do you like regular cola?

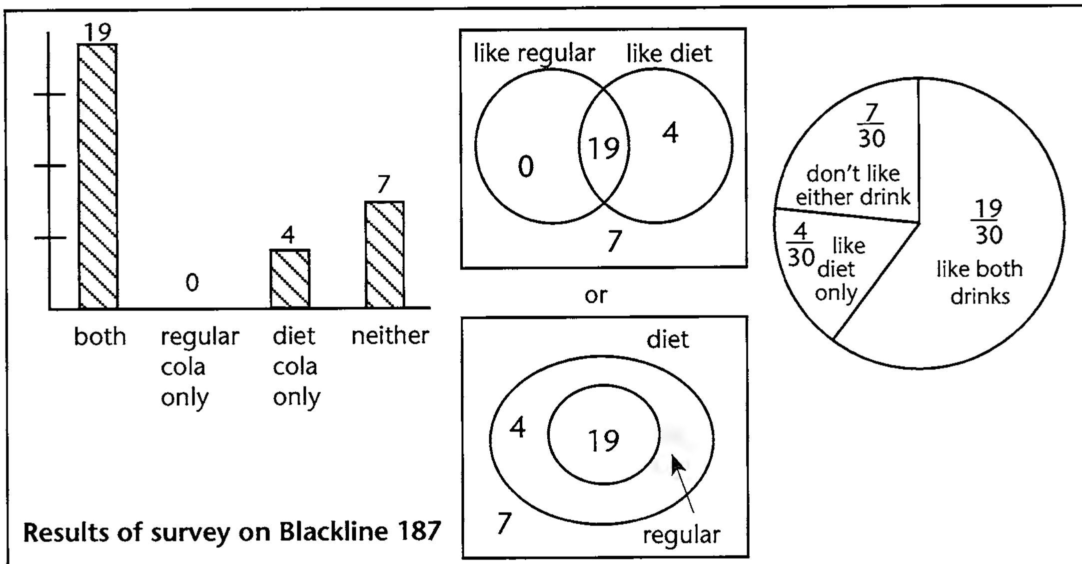
Question 2: Do you like diet cola?

Child	Question 1	Question 2
1	Yes	Yes
2	No	No
3	Yes	Yes
4	Yes	Yes
5	No	Yes
6	Yes	Yes
7	Yes	Yes
8	No	No
9	Yes	Yes
10	No	Yes
11	Yes	Yes
12	Yes	Yes
13	No	No
14	Yes	Yes
15	Yes	Yes
16	Yes	Yes
17	Yes	Yes
18	Yes	Yes
19	No	Yes
20	Yes	Yes
21	No	No
22	Yes	Yes
23	Yes	Yes
24	No	No
25	Yes	Yes
26	Yes	Yes
27	No	No
28	No	No
29	Yes	Yes
30	No	Yes

Divide the class into three groups for the purpose of analyzing the data more closely. Ask the first group to prepare a bar graph that summarizes the results of the survey. Have them attach observations about the data to their graphs. Ask the second and third groups to do the same, only organizing the results in a Venn Diagram and a pie graph, respectively.



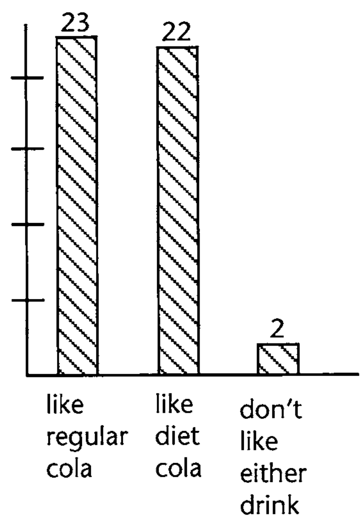
Results of survey on Blackline 186



Results of survey on Blackline 187

Conduct a show-and-tell discussion of these analyses. This will illustrate how a given data set can be represented in several ways.

Note with care the use of language and the representation of data in this discussion, for there is a risk of miscommunication. For example, look at the following graph, which represents the data from Blackline 186. If there are only 30 children in the class, why does the graph show more than 30 responses? There must be some double-counting. Notice that those who like both drinks have been counted in each of the first two bars. In fact, children who like regular cola include those who like only regular cola (and not diet) and those who like both.



The discussion can motivate questions related to experimental probability. Pose this situation to the class:

TEACHER *(referring to Blackline 186) Suppose a new child is going to join our class tomorrow. Based on the results of our survey, what are the chances that this child will like regular cola?*

ISIAH *If the new person is like us, I think there would be a pretty good chance. Way over half the people in our class like regular.*

ROSALIE *I agree. I think around ¾ of our class said "Yes" to regular, so I think there's a real good chance the new person will too.*

APRIL *The new person is sure to like one of the drinks. Only two of us said they didn't like either drink.*

LUC *Yes, but there is still a chance the person may not like either one.*

ALPHONSO *I have an idea! Maybe it will depend on whether the new person is a boy or a girl.*

Technically, the data above suggests an experimental probability of $23/30$ that the new person will like regular cola. This assumes the person is similar to a random person selected from the class. Notice, however, it is not necessary to insist on this fraction (though you may want to discuss it). It is equally helpful to speak in terms of estimates as suggested in the dialogue. It is also helpful to discuss such questions as these (answers in parentheses refer to the data on Blackline 186):

What are the chances the new person will like both drinks? (About even.)

How do the chances the new person will like both drinks compare to the chances the person will like regular? (In this case, there is a greater chance of liking regular.)

What are the chances the new person will like at least one of the two drinks? (Very good—$^{28}/_{30}$ of the class liked at least one drink.)

How confident do you feel about your answers?

We encourage you and your children to add similar questions to this list.

The preferences of children responding to Blackline 186 are different from those expressed on Blackline 187 in several respects. For example, every child who likes diet cola also likes regular cola in the latter survey, but not in the former. In all likelihood, your class survey will exhibit responses different from those on these blacklines. These differences often emerge from sampling procedures or from factors related to chance. It's important to examine their differences and discuss possible explanations for them. We urge you, therefore, to repeat this lesson using data sets that exhibit different kinds of responses (perhaps the one from your class and the ones on Blacklines 186 and 187).

Save the data sets used in this lesson for Lesson 34, Opinion Polls, Part II.

Technology

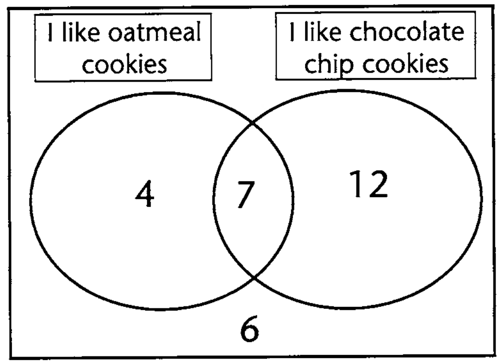
You may be equipped to conduct this survey with the help of a spreadsheet or data base. Data can be entered and possibilities for graphically displaying results can be investigated.

Journal Entry

Briefly describe one thing that pleased you mathematically about this lesson.

Homework

Make copies of the following Venn Diagram for your class.



Tell the children that this diagram shows the results of a survey taken in another class and pose this problem:

If a new child were to join the class that was surveyed, how would you describe the chances that this child would like oatmeal cookies only? Explain your thinking.

Ask another person the same question, perhaps a family member or friend. How did they respond? What was their thinking?

34 Probability—Opinion Polls, Part II

You Will Need

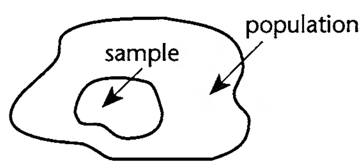
- Chapters 9, Probability, and 10, Data Analysis and Graphing
- materials for graphing or preparing displays available
- Blackline 188 (Pie Chart)

for each child

- a copy of the complete class survey from Part I
- fresh copies of Blackline 185 (Survey Form), completed as described below

Introduction

The data from the survey conducted in Lesson 33 of this volume were used to make predictions about the likely preferences (for Brands X and Y) of a new child about to enter the class. Using data from a sample of people in this way reflects a typical statistical procedure. Usually, the sample is used to make inferences about a larger population, as in the case when television networks use a small percentage of results to "project" the outcome of an election.



To confidently make inferences about a population on the basis of information drawn from a sample, care must be taken to ensure that the sample is not biased. It is important, for example, that samples be randomly formed so as to be *representative* of the population. Randomness is a notion directly related to chance, and true randomness is very hard to achieve. Sometimes tables, or computer-generated lists, of random numbers are used to create random samples.

The size of the sample is also important. If the sample is too small, then it will likely not yield enough information. On the other hand, it may be impractical (because of time or money) to gather very large samples. In practice, finding a satisfactory balance between "large enough" versus "too large" generally requires much analysis.

Your Lesson

How typical are the results of your class survey from Lesson 33? Is this sample of opinions "representative" of those of other children? Could these results be confidently used to predict the opinions of a larger group, such as the whole grade or the entire school?

Invite discussion of these questions. Some children may feel that others in their grade are "like them" and would, therefore, have the same preferences. They may not be so sure about the whole school, however. (Of course, if your class is multi-age, you may get another response.) This is an opportunity discuss the general nature of random sampling.

Divide the class into small groups and ask each group to suggest a sampling procedure that would permit a comparison of the class' preferences with those of the entire school. What seem to be their thoughts about randomness? How large a sample would they need? Have the group record their work.

> ***Sampling Procedures—Group A***
>
> *Get a list of children in each classroom. Then choose 5 names from each class and ask those children the questions.*

> ***Sampling Procedures—Group B***
>
> *We think it would be ok to ask the first 50 children we meet in the hall.*

Have the groups share their proposals with the entire class. The accompanying discussion need not be technical. How do the children feel about each proposal? Do they think the proposal will generate a sample of students that is representative of the entire school? How confident are the children of their reasoning? In general, what appears to be the children's understanding of randomness at this point? How large do they feel their samples need to be?

BILL *(referring to Group A above) I think it's important that kids sampled in each class include some boys and some girls.*

RENEE *(referring to Group B above) Perhaps Group B needs to be careful. Maybe the first 50 students will be mostly third graders.*

It is now time to refine and implement one or more of the proposals. How this occurs will depend on the nature of the above discussion. Perhaps the class will come to a consensus about what proposal(s) to try. Or it might be more appropriate to send the proposals back to the groups for further work and subsequent testing. In any case, we're sure your children will want to "survey the school" using their proposal(s). (Blackline 185, complete with survey questions, may be used to record responses.) Once the survey(s) are complete, have the groups make bar graphs, Venn Diagrams and pie charts (Blackline 188) of the results.

How closely do the preferences of the class match those of the school? Have the groups compare the graphs of these surveys and discuss any similarities or differences that exist. Can any inferences be confidently made?

Teacher Tips

We have included Blackline 188 for the purpose of making pie charts that describe the results of the school survey. Here is an opportunity for your children to estimate fractional parts.

Technology

As in Lesson 33 of this volume, results of the school survey may be entered on a spreadsheet and displayed graphically in various ways. Also, you may wish to explore ways to create a random sample of your school population using your technology's capability to generate random numbers. Here is a possibility:

Suppose there are 900 children in your school and they are listed by name in alphabetical order. Further, suppose these names are numbered from 1 to 900 as they appear on the list. A random sample of 75 children can be formed by having your technology generate 75 random numbers less than 901. The corresponding names on the list identifies the children to be sampled.

Published Piece

(Optional) You may wish to have your children compile their work on the major parts of this lesson into one publication, such as a bulletin board or poster display.

Journal Entry

What does "random" mean to you? Give an example of something that is, in your opinion, random. Give an example of something that, to you, is not random.

35 Problem Solving—Counting & Organization

You Will Need

- materials for solving problems and displaying results
- a copy of Blacklines 189 (Arrangements), 191 (Paths) and 192 (Tracing Designs) for each child
- a copy of Blackline 190 (Group Assessment) for each small group

Introduction

Counting—one of the first mathematical processes we learned as children. Can you remember, in fact, a time when you didn't know how to count? Yet we also know this process is not innate and can be quite complicated.

Sometimes a count is easily obtained and one is sure of the answer. Perhaps it is small and can be checked by counting one at a time, or perhaps it can be determined by a direct arithmetic computation.

- How many children are in your class this year?
- If each child brings 2 cookies to class, how many cookies would there be altogether?

On the other hand, large or otherwise less obvious counts can be hard to come by. How can one be sure the count is complete and that no double-counting has occurred?

- How many different shapes having perimeter 12 can be created on a geoboard?
- How many games will be played in a 10 team round-robin softball tournament, where each team plays every other team exactly once?
- How many different ways are there to divide a geoboard into two congruent halves?

Many applications in probability, statistics, social sciences, and other fields draw upon strategies for counting large collections. In fact, "Combinatorics" is an area of mathematics devoted to the study of such strategies.

Your Lesson

As in Lesson 19, Softball Tournament, of this volume, the activities of this lesson draw upon your children's ability to analyze a problem and organize information. The activities are introductory in nature and are best explored in an informal way.

Activity 1

Divide the class into groups of four and pose this problem for investigation: Susan, Charlene, Kurt and Stephanie are trying to decide how to line up single file for lunch. In how many different ways can they line up?

Give the children plenty of opportunity to wrestle with the challenge of organizing an accurate count of the different ways. They may have to spend some time developing an understanding of the problem. Perhaps they will combine some initial listing of lineups with some modeling of the situation (e.g., acting it out as if they were the four children, cutting and pasting squares that represent the children, or lining up colored cubes or pattern blocks.)

Some groups may take steps to organize their work; others may continue in a random manner. Consider letting the organization come from the children (please see Teacher Tips). If it seems appropriate to facilitate the process, you might ask questions such as these: What do you know? What are you trying to find out? How can you organize the information you've found so far? About how large do you think the answer will be? Suppose there were only 3 people, how many different ways could they line up in single file? Or, on the other hand, suppose there were 5 people?

Discuss the different counting strategies that emerge during the lesson. How do the children feel about each strategy? Are there advantages or disadvantages to each? Remember there is no one best way to organize a solution to this problem.

Continue the investigation with problems such as these: How many different ways can 5 people line up single file? 6 people? 12 people? Invite the children to pose some "what if's" to explore.

Activity 2

Blackline 189 (see following page) contains additional counting problems that can be explored in class or as homework.

Activity 3

The counting problems on Blackline 190 (see following page) are set in a different context and are fun to try. We suggest doing them in a separate lesson where children have a chance to develop and share strategies for counting the required paths.

Note the two answers in Problem 1; children may decide to count according to different rules. You might also ask the children to write their names (or those of a friend) in a similar triangular arrangement. How many different paths can they find to spell their name? Encourage them to predict an answer.

Activity 4

Invite your children to try the puzzle problems on Blackline 192 (see page 183) on another day. These problems offer additional experiences that draw upon children's general organizational and problem-solving skills.

Opening Eyes to Mathematics | **Arrangements** | **Blackline–189**

1. Debby has 3 colored cubes: 1 red, 1 green and 1 blue. She wishes to arrange these 3 cubes in a line. How many different arrangements can she make?

2. Suppose Debby also has a yellow cube to put with the red, blue, and green ones. How many different ways can she arrange all 4 cubes in a line?

3. In Problem 2, how many different ways can Debby select 2 of the 4 cubes and arrange them in a line?

4. Stone Soup Company makes different kinds of vegetable soups. Each kind always uses 2 vegetables that have been chosen from 5 different available vegetables.

a. How many different combinations of 2 vegetables can the company form?

b. Suppose each kind of soup were made from a combination of 3 of the 5 vegetables. Now how many combinations can be formed?

c. How many combinations are possible if 4 vegetables are used? How about if 5 vegetables are used?

Opening Eyes to Mathematics | **Paths** | **Blackline–191**

1. One day Amy wrote the letters of her name like this:

```
  A
 M M
Y Y Y
```

She said her name could be spelled in 4 ways by following different paths. Her friend, Sally, found 6 different paths for spelling Amy. Both girls used this rule: Each path starts with the letter in the top row, goes to a letter in the second row, then to one in the third row. Can you figure out how each girl arrived at her answer?

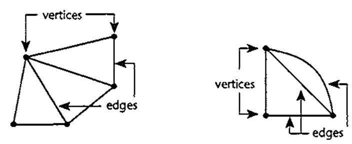
2. Debby wrote the letters of her name in a triangular arrangement, too.

```
    D
   E E
  B B B
 B B B B
Y Y Y Y Y
```

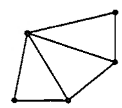
If Debby forms paths like Amy and Sally did, how many ways can she find to spell her name?

Opening Eyes to Mathematics **Tracing Designs** **Blackline–192**

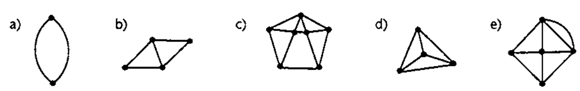
All of the designs on this sheet consist of edges joined by vertices.



1. Here's a puzzle that's fun to try. Karl claims he can trace this entire design without lifting his pencil and without going over any edge more than once. Can you do the same?



2. Some of the designs in this problem can be traced just as you did in Problem 1 and some can't be. See if you can identify the designs that can be traced. Remember no edge can be traced more than once and your pencil can't be lifted from the paper.



3. Create two designs—one that can be traced under the rules of Problems 1 and 2 and one that can't be. However, do not label which is which! Instead, offer your designs to some friends and see if they can predict which design can be traced.

Teacher Tips

Some groups may reach a point where they can make predictions about other related problems. There may also be children whose organization may remain somewhat random. All of this is to be expected as a natural part of problem solving. The children will have more opportunities for developing organizational strategies since counting problems are examined in later grades.

Published Piece

(Optional) Group displays of solutions to the problems in this lesson make nice published pieces. As an alternate, we have asked each child to write a personal account of their solution for one of the problems in a bby book.

Assessment

This would be a good time to use a group assessment such as that found on Blackline 190.

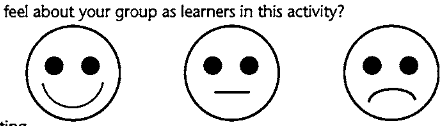
Opening Eyes to Mathematics | Group Assessment | Blackline–190

GROUP ASSESSMENT

1. How do you feel about your product?

Explain your rating.

2. How do you feel about your group as learners in this activity?



Explain your rating.

WE ARE PROUD OF:

WE WISH:

IF WE GET TO REVISE THIS PIECE, WE WILL:

WE LEARNED: (use back of page)

Pattern Reminder

Invite your children to explore one of the remaining Pattern Generalizations found on Blacklines 169–176.

36 Standard Units of Liquid Measure

You Will Need

- Chapter 7, Measurement
- containers which hold standard capacities: 1 or 2 liter bottles, 1 or 5 gallons, 1 or 2 quarts, pints, cups, etc.
- filler material such as rice, pea gravel, water, sand, available
- miscellaneous containers

Your Lesson

This activities of this lesson parallel those of Lessons 8, Standard Units of Length, and 25, Standard Units of Weight, in this volume. They also are related to the problems explored in Contact Lesson 184, Volume 2. In fact, if your children haven't done this lesson previously, you might do it first.

Discuss the containers you have gathered. What type of items are sold in these containers? Is something like sugar ever sold or measured in them? What type of items aren't sold in them? Why? What are the units of measure? How do they compare? Which units seem appropriate to use when finding the capacity of a peanut butter jar? of a swimming pool? etc.

This discussion can highlight distinctions between liquid and solid measure. These distinctions are not always clear-cut, however. For example, sugar is typically sold by weight, yet recipes call for cups of sugar.

Invite suggestions for ordering the containers by capacity. About how many cups in a quart? in a liter? How does a quart compare with a liter? etc.

Examine other containers in the room. Search for some that would hold about a quart (or some other amount). Estimate first and then do some measuring. Try different units. Which units seem appropriate for particular containers?

Here is an experiment you might conduct as a continuation of the lesson: How much water would be displaced if an object (such as an orange) is gently dropped into a container of water?

The children can decide how to conduct such an experiment: What equipment is needed? How shall data be gathered and displayed? What happens if the conditions of the experiment are varied: different objects? same object, shaped differently? different amounts of water in the container? etc.

The children might do further research on the topics of liquid displacement and density of objects. Children generally find the story of Archimedes and his experiments interesting.

Homework

Consider problems such as these for homework.

- Make a list of containers in your home that hold these amounts: a gallon, a liter, a pint, about a quart.
- If you filled your bathtub up to the drain opener, about how many gallons of water would it hold? How many quarts? How many liters?
- As you walk through the grocery, make a list of products that are sold by weight and of ones sold by liquid measure. What different sizes or capacities are displayed? What do you think accounts for the different containers that are used?

Journal Entry

What size of liquid serving is appropriate for you during each meal? What container would you choose to hold all the liquids you drink in a normal day?

Daily Computation

Ask your children to discuss liquid measure relationships in terms of "shifting the unit". Consider questions such as these:

Suppose a quart is the unit of liquid measure. Then what is the measure of a cup (in terms of quarts)? What is the measure of a gallon? of 5½ gallons? of a liter? etc.

Suppose now a cup is the unit of liquid measure. Then what is the measure of a quart (in terms of cups)? of a gallon? of 5½ gallons? of a liter? etc.

37 *Reflective and Rotational Symmetry*

You Will Need

- Chapter 11, Geometry
- Volume 3, Appendix D: Reflective and Rotational Symmetry
- a large rectangular sheet of plain paper
- overhead pattern blocks
- a transparency of Blackline 193 (Symmetry)

for each child

- copies of Blacklines 193 (Symmetry) and 194 (Pattern Block Symmetry Display)
- pattern blocks
- scissors and tracing paper available

Your Lesson

Children are generally quite interested in objects or designs that are symmetrical. Even the drawings of very young children, for example, often illustrate an awareness of symmetry. It seems that symmetrical objects offer a sense of balance that is appealing.

Several earlier lessons in Opening Eyes contain activities involving line (or reflective) symmetry (e.g., Volume 1, Contact Lessons 26 and 27). The activities of this lesson can help your children broaden their understanding of symmetry to include rotational symmetry. Please refer to Appendix D in this volume for related background information.

Activity 1

Begin the lesson by inviting your children to share examples related to their current understanding of symmetry.

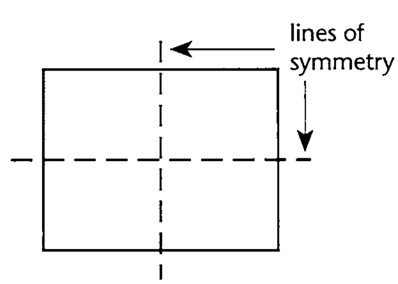
Ask the children to identify some objects that, in their opinion, are:

- symmetrical in some way. How would the children describe the symmetry of each object?
- not symmetrical. What makes these objects unsymmetrical?

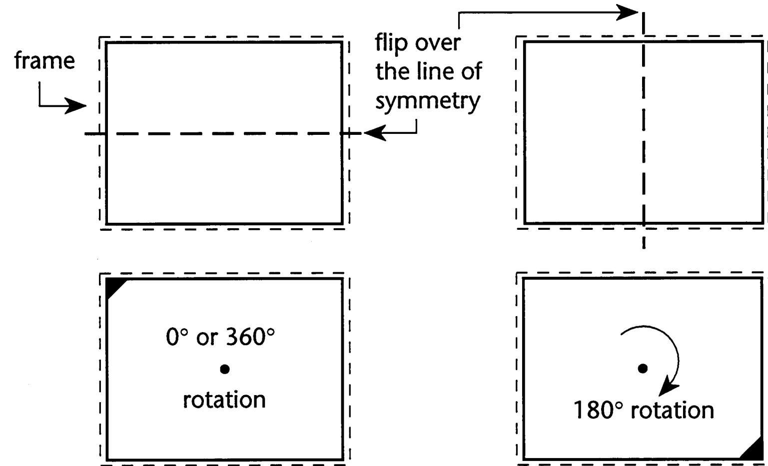
Hold up an object that displays rotational symmetry (e.g., a pinwheel, a picture of a starfish, a quilt block design). Do the children feel this object has any symmetry? Can the class picture turning the object about its center to a new position where it looks the same as it did to begin with? Discuss.

Activity 2

Hold up a rectangular sheet of plain paper and ask the children to describe its symmetry. They are likely to identify familiar lines of symmetry.

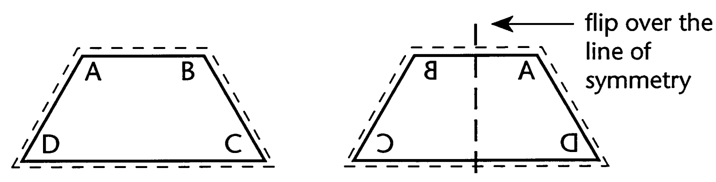


Introduce the frame tests for reflective and rotational symmetry. Do this by discussing the following question: How many different ways can we fit the paper into its frame? (Please see sample dialogue in Appendix D.)



The paper can fit in its frame by flipping over each of its two lines of symmetry. It can also fit by rotating about its center ½ of the way around (180°) and all the way around (360°). Rotating all the way around (360°) is equivalent to not rotating at all (0°). (Please see Teacher Tips)

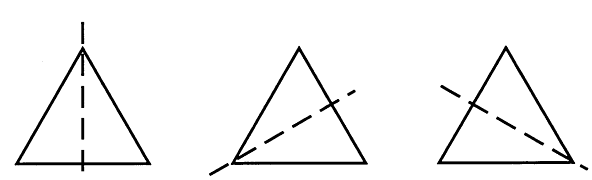
Display a red trapezoid from a set of pattern blocks at the overhead and draw a frame around it. Also have each child make a frame for a red trapezoid at their desk. Pose these problems for investigation: How many different ways can the trapezoid fit into its frame? How can one be sure the ways are different? Describe the symmetry of the trapezoid.



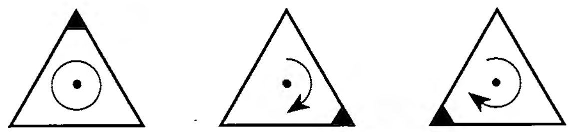
Here are two ways to fit the trapezoid into its frame. The trapezoid has reflective symmetry but no rotational symmetry.

Discuss the children's work, inviting volunteers to demonstrate their thinking at the overhead.

Have each child investigate the symmetry of the green triangle in their set of pattern blocks with the help of the frame test. Discuss.



The triangle can fit into its frame by flipping it over each of its three lines of symmetry.

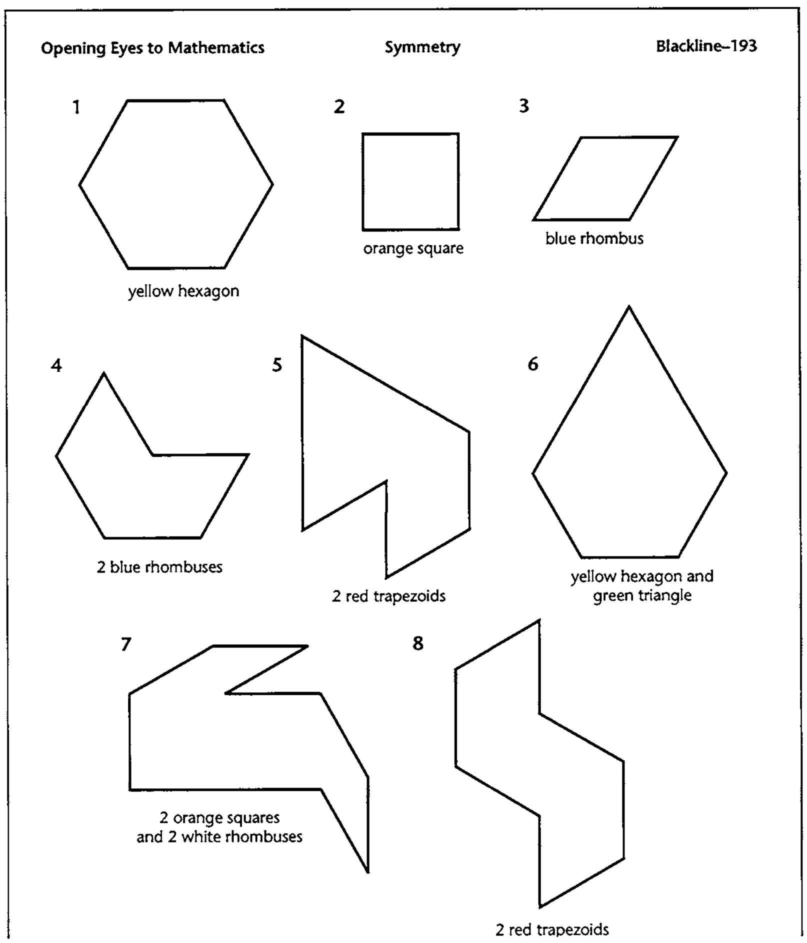


360° (or 0°) 120° 180°

The triangle can also fit by rotating about its center ⅓ of the way around (120°), ⅔ of the way around (240°), and all the way around (360°).

Activity 3

The shapes on Blackline 193, Symmetry, are frames for pattern block designs.



Have the children investigate the symmetry of each shape and then share their work with their neighbors or with the whole group. A transparency of Blackline 193 might be helpful during the sharing time.

This may be an appropriate time to discuss the order of rotational symmetry an object has. For example, the blue rhombus has rotational symmetry of order 2. (Also, please see Teacher Tips.)

Here is the answer key to Blackline 193:

1. Six lines of symmetry; rotational symmetry of order 6.
2. Four lines of symmetry; rotational symmetry of order 4.
3. Two lines of symmetry; rotational symmetry of order 2.
4. One line of symmetry; no rotational symmetry.
5. No symmetry.
6. One line of symmetry; no rotational symmetry.
7. No symmetry.
8. Rotational symmetry of order 2; no reflective symmetry.

Activity 4

Distribute copies of Blackline 194 (Pattern Block Symmetry Display) to each team of two children. Ask the teams to create a pattern block design that falls within each of the categories listed on the blackline. Have the teams identify their designs by category and display them at their tables. Invite teams to view and discuss the displays by taking a field trip to nearby tables.

Opening Eyes to Mathematics **Pattern Block Symmetry Display** **Blackline–194**

Create at least one pattern block design for each category. Make each design with at least 4 blocks. Note: The blocks don't have to be different.

Category 1: Reflective symmetry only.

Category 2: Rotational symmetry only.

Category 3: No symmetry at all.

Category 4: Both reflective and rotational symmetry.

Category 5: Rotational symmetry of order 3.

Teacher Tips

If appropriate, you may wish to use the children's work on Blackline 193 to discuss rotational symmetry in terms of degree measure. That is, a rotation of "a quarter way around" is a rotation of 90°. Similarly, one "a third of the way around" is a rotation of 120°, etc.

Tracing paper is sometimes helpful when using the frame test to investigate the symmetry of a figure. After drawing the frame for the figure, make a separate tracing of the figure and examine how to fit the tracing into the frame. In the same way, two copies of a figure, drawn on separate transparencies, may be used to demonstrate the frame test at the overhead.

Journal Entry

Using 2 green triangles and 1 orange square, make a pattern block design which has reflective symmetry only. Make a second design with these shapes that has rotational symmetry only. Show a sketch of each design and explain your thinking.

38 *Fractions—"Little House" Problem*

You Will Need

- Chapter 8, Fractions
- copies of Blacklines 195 (Little House Problem) and 196 (Nickel A Day) for each child

for each group of four children

- pattern blocks
- copies of Blacklines 62–67 (Pattern Block Shapes)

Your Lesson

Children seem to have strong feelings regarding the "fairness" of a situation. For example, they quickly point out when, in their opinion, the rules of a game are fair or the division of cookies is unfair. Probability lessons such as "Who Would You Rather Be?" (Lesson 6, Volume 3) focus on the fairness of chance situations. Similarly, Sample Lesson 2 in Chapter 8 of the Teaching Reference Manual (page 70) describes ways children might handle the fractions that emerge when dividing 3 cookies evenly among 4 people.

This lesson also examines the question of dividing objects (cookies) "fairly" among a group of children. We use an excerpt from Laura Ingalls Wilder's book, *Little House in the Big Woods*, as a context for investigating the particular problem of how to divide 2 cookies fairly among 3 children (see Blackline 195).

Opening Eyes to Mathematics **Little House Problem** **Blackline–195**

LITTLE HOUSE IN THE BIG WOODS
by
Laura Ingalls Wilder

Chapter 10: Summertime

Sometimes Ma let Laura and Mary go across the road and down the hill, to see Mrs. Peterson. The Petersons had just moved in. Their house was new, and always very neat, because Mrs. Peterson had no little girls to muss it up. She was a Swede, and she let Laura and Mary look at the pretty things she had brought from Sweden—laces, and colored embroideries, and china.

Mrs. Peterson talked Swedish to them, and they talked English to her, and they understood each other perfectly. She always gave them each a cookie when they left, and they nibbled the cookies very slowly while they walked home.

Laura nibbled away exactly half of hers, and Mary nibbled exactly half of hers, and the other halves they saved for Baby Carrie. Then when they got home, Carrie had two half-cookies, and that was a whole cookie.

This wasn't right. All they wanted to do was to divide the cookies fairly with Carrie. Still, if Mary saved half her cookie, while Laura ate the whole of hers, or if Laura saved half, and Mary ate her whole cookie, that wouldn't be fair, either.

They didn't know what to do. So each saved half, and gave it to Baby Carrie. But they always felt that somehow that wasn't quite fair.

As a group:

1. Restate the problem in your own words.
2. Use pattern blocks to design a Swedish cookie.
3. Solve the problem using your cookies as the model.
4. Write a letter to Mary and Laura describing your interpretation of the problem and your solution. You may need to justify your reasoning.
5. Use pattern block paper to illustrate your letter.
6. Create another problem that could be solved in a similar manner.
7. Be prepared to share your product with the class. Your presentation should include show-and-tell at the overhead projector.

Distribute Blackline 195 to each child and place pattern blocks at each table. Ask the children to read the story on the blackline and address the problems at the bottom. These problems call for the children to clarify their understanding of the situation described in the story and to find ways of resolving the situation.

Children often make some general suggestions for Laura and Mary:

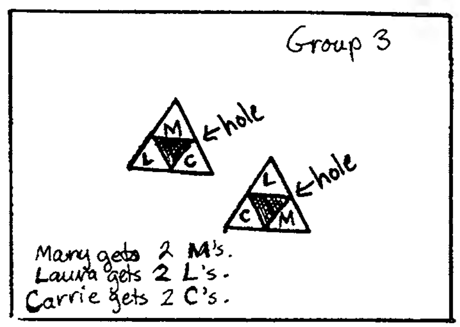
Ask for a third cookie!

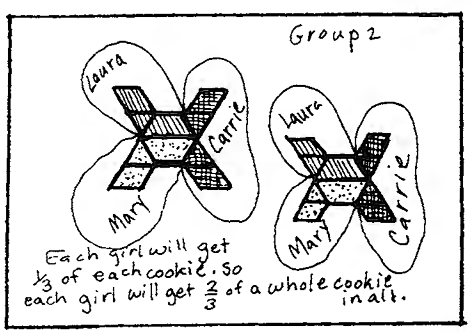
Over a three day period, there will be 6 cookies. Let Laura eat the 2 cookies on the first day. Give Mary 2 on the second day and give the third day's cookies to Carrie. That way, each child gets 2 cookies every third day.

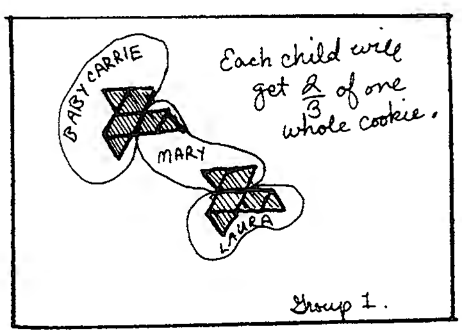
On the first day, Laura and Mary get half a cookie each, just as in the story. Carrie gets the other cookie. On the second day, Mary gets the whole cookie and the other two children get half a cookie. On the third day, Laura gets the whole cookie and half a cookie goes to Mary and to Carrie. Altogether, each child gets 2 cookies in 3 days.

Carrie is just a baby, so she gets smaller portions. Laura and Mary could give Carrie less. It doesn't have to be even.

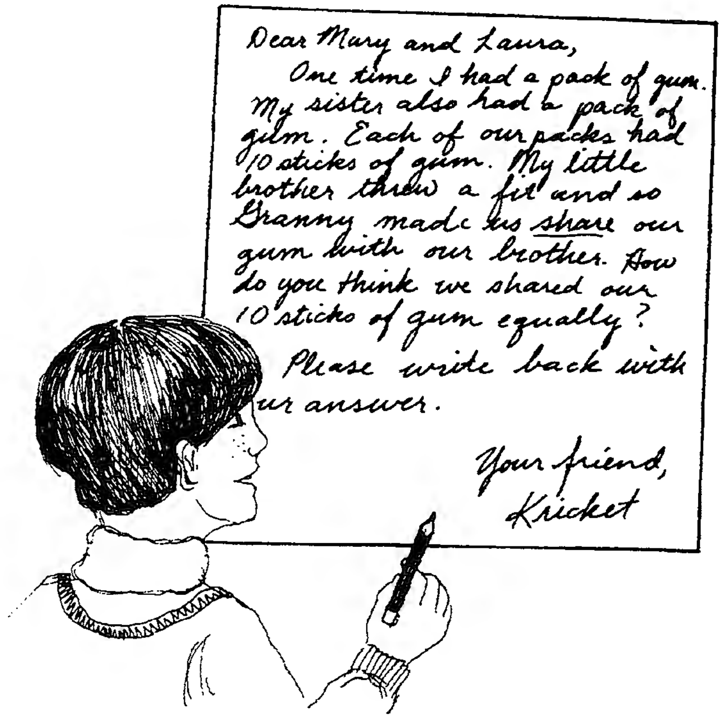
Some specific ways of dividing pattern block "Swedish cookies" also are shared.







We like to make this activity a language arts experience, as well as a mathematical one. We ask our children to compile their solutions and models into a letter of advice to Laura and Mary (see Journal Entry below). This letter also contains examples of similar problems suggested by the children.



Our children seem to enjoy learning about each other's solutions in the sharing time that follows the group work.

The problem on Blackline 196, Nickel A Day, is a nice follow-up to this lesson. Your children may enjoy pursuing it during another class session.

Opening Eyes to Mathematics **Nickel A Day** **Blackline–196**

Mary and Laura were offered the choice between receiving a nickel a day for a whole year or $25. What choice would you recommend to them? Why?

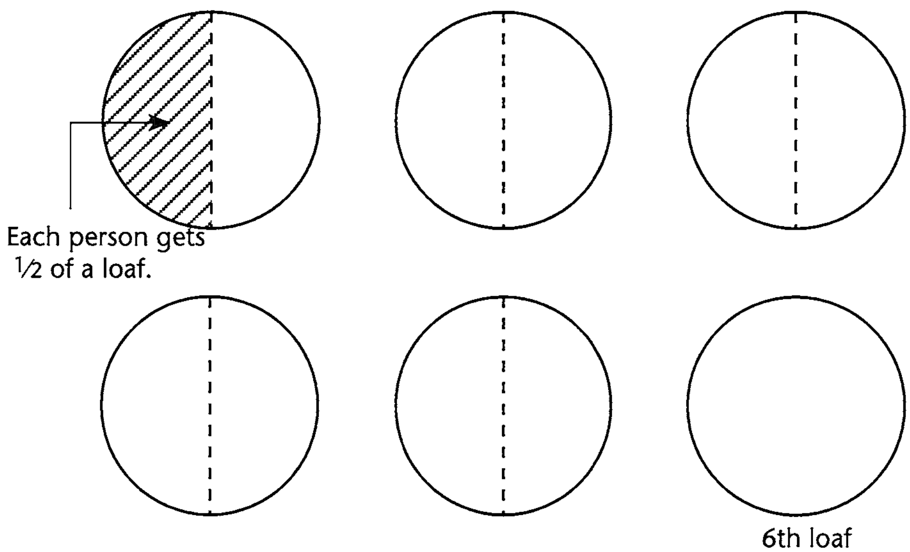
Support your answer with a model or picture. Write some sentences and equations to explain your thinking.

Teacher Tips

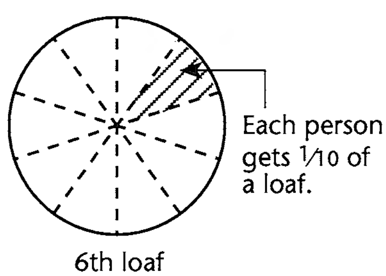
You may wish to link this lesson with others in this program. For example, ask the children to describe the symmetry of their pattern block cookies. Or, ask the children to build a "cookie" that has exactly one line of symmetry (or rotational symmetry, etc.). (See Volume 1, Contact Lessons 26 and 27, and Volume 3, Lesson 37.)

Consider connecting this lesson to a unit on ancient history. The ancient Egyptians solved the problem of dividing 6 loaves evenly among 10 people by giving each person 6⁄10 of a loaf. However, because they only understood unit fractions (i.e., fractions with a numerator of 1 like 1⁄2 or 1⁄5), they reported the answer of 6⁄10 as 1⁄2 + 1⁄10. (See *The Rhind Mathematical Papyrus* by A. B. Chace listed in the references of Appendix C for more information and examples.)

Here is a visual way of modeling the Egyptians' solution for dividing 6 loaves evenly among 10 people. Step 1: Divide each of the first 5 loaves into 2 equal parts. Give each of the 10 people 1⁄2 of a loaf.



Step 2: Divide the remaining loaf into 10 equal parts. Give each person 1⁄10 of a loaf.



Altogether, each person get 1⁄2 + 1⁄10 of a loaf.

Journal Entry

Send a letter of advice to Laura and Mary that describes your suggestions for splitting the 2 cookies fairly. Include any other thoughts or problems you wish to share with the two girls.

Published Piece

(optional) The children can compile their work on Blackline 195 into a published piece. Alternatively, you might ask each child to describe a time in their life when they faced a dilemma similar to that of Laura and Mary. This description can be put together in a "Tell It All" book.

Pattern Reminder

Are there any remaining Pattern Generalizations (Blacklines 169–176) left to explore? If so, this might be a good time to discuss one of them.

39 Cube Cover Project

You Will Need

- Chapter 7, Measurement
  for each child
- small wooden cubes
- Blackline 73 (2-cm Grid Paper)
- scissors, tape, materials for making published pieces
- Blackline 197 (Cube Cover Extensions)

Your Lesson

Here is a project our children enjoy and that we have used for portfolio assessment purposes. The project is based on the activities found in Volume 1, Contact Lessons 70–72. In those lessons, children created "cube covers" and explored ideas related to surface area. Some of the main activities are summarized here; you might use them to introduce this lesson.

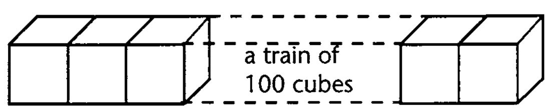
Activity 1

By way of review, give each child a collection of 6–8 cubes and discuss the following questions:

Suppose each square face of a cube is 1 unit of area and the cube itself is the unit of volume. What is the surface area of a cube? (6 square units, assuming all faces are counted.) What is the volume of the cube? (1 cubic unit.)

Suppose now 2 cubes are linked in a train? Using the same units of area and volume as above, what is the surface area of the train? (Answer: 10 square units.) What is the volume? (2 cubic units.) Discuss.

How about the surface area of a train of 3 cubes? a train of 8 cubes? a train of 100 cubes? etc. Encourage the children to use the visual clues found in smaller trains to help predict the surface area of larger trains. Invite volunteers to share their thinking.

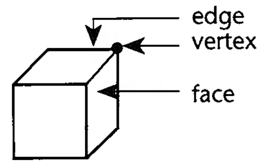


"I see 4 faces around each cube—that's 400—plus the 2 faces at the ends. The surface area is 402 square units."

"The end cubes have 5 faces showing. The other 98 cubes in the middle each have 4 faces. The surface area is 5 + 5 + 4(98) = 402 square units."

Activity 2

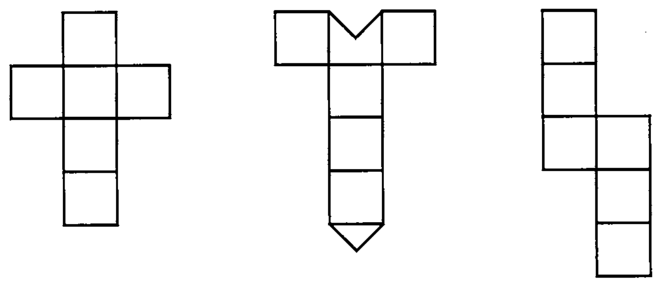
Ask each child to examine a wooden cube and discuss some of its attributes. This can be a time to identify the vertices, faces and edges of the cube.



A cube has 8 vertices, 6 faces and 12 edges.

Now ask each child to create covers for their cubes from single pieces of paper. The covers are to have no gaps or overlaps and, if necessary, may be taped together from smaller pieces.

Children may work in groups or individually. When using the activity to generate published pieces, however, we encourage individual products.



Provide plenty of exploration time and then invite the children to share some of their creations. Select a few of the covers and compare them. How are they alike? How are they different?

Activity 3

Have each child create a published piece of their work that includes the following parts:

Part 1: Their restatement of the cube cover activity and the problem(s) they solved.

Part 2: Their cube covers from Activity 2. (We encourage our children to glue each of their covers to a page titled, Cover For One Cube. We also suggest gluing just one face of each cover to the page. This permits a reader to test each cover by wrapping it around a cube.)

Part 3: A written description of each cover. This description can include features of the cover (e.g., surface area) and comments about why the cover "works". (Help your children, as needed, with any related vocabulary.)

Part 4: A report of their work on the extensions found on Blackline 197.

Opening Eyes to Mathematics **Cube Cover Extensions** **Blackline–197**

1. Tape 2 cubes together. Create several covers for this solid from single pieces of paper. Each cover is to have no gaps and no overlaps.

2. Create a solid shape by taping 3 or more cubes together. What covers can you make which have no gaps and no overlaps?

3. Suppose you work in a cube-making factory and your job is to determine how to wrap 24 cubes with a cover that has no gaps and no overlaps. What cover would you make that uses the most paper possible? What cover would you make that uses the least amount of paper possible?

4. Write a letter which explains why you chose the two covers you made in Part 3 above.

Teacher Tips

The purpose of the letter in Part 4, Blackline 197, is to offer children a chance to demonstrate their creativity and also to promote a discussion of why a manufacturer might choose one type of cover over another.

Even if you are using other assessment instruments besides portfolios, this project is a way for your children to strengthen their understanding of several ideas (such as surface area) and their spatial visualization skills. Providing written descriptions and justifications of their solution processes are, in themselves, instrumental for "cementing learning" in the child's mind.

Many of the activities of this lesson are extended further in *Visual Mathematics.*

40 *Areas & Similarity, Part I*

You Will Need

- Chapter 11, Geometry
- a transparent square geoboard and rubber bands
- a transparency of Blackline 141 (Square Dot Paper)
- a square geoboard and rubber bands for each child

Additional materials for Activity 1

- a transparency of Blackline 198 (Areas and Similarity, Activity 1)
- several copies of Blackline 198 for each group of four

Additional materials for Activity 2

- a transparency of Blackline 199 (Areas and Similarity, Activity 2)
- a copy of Blackline 199 for each child
- several sheets of Blackline 141 (Square Dot Paper) for each group of four

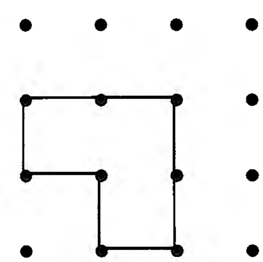
Your Lesson

Undoubtedly your children have had experiences with shapes that have been shrunk or blown-up. Perhaps some have used cameras with zoom lens or played computer games where shapes can be enlarged. These are situations involving similar objects—objects having the same shape but not necessarily the same size. In this set of lessons your children are invited to examine some of the properties of similar figures. This takes place within the context of several geoboard area activities.

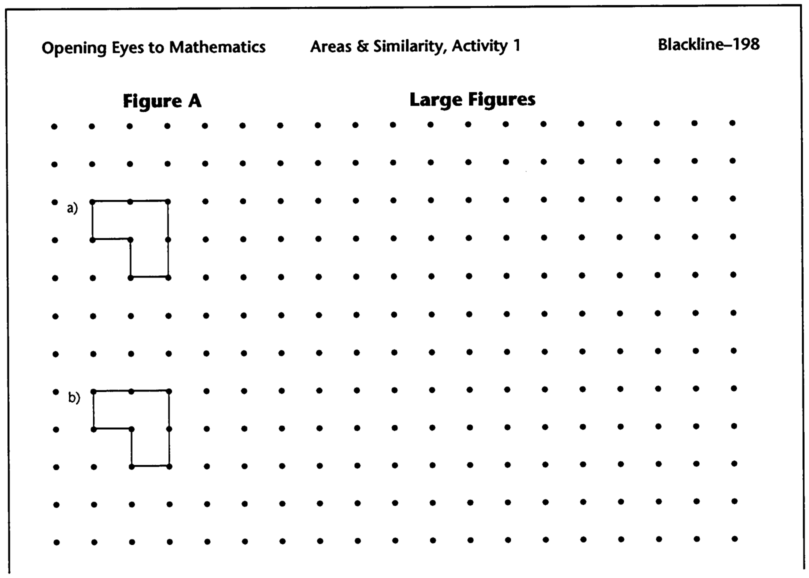
Activity 1

Display Figure A (shown below) at the overhead and ask your children to describe it. In particular, what is the area and perimeter of this figure? What symmetry does the figure have?

Figure A

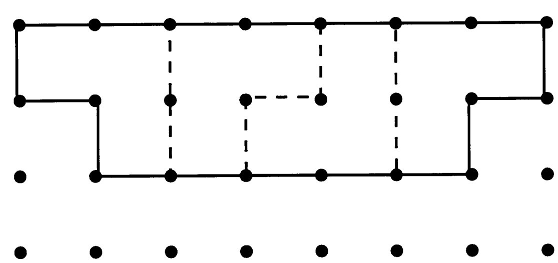


Divide the class into groups of four and have the children form Figure A on one of their boards. On the other boards, ask them to build figures whose area is four times the area of Figure A. Ask them to do this in several ways, recording each answer on Blackline 198.

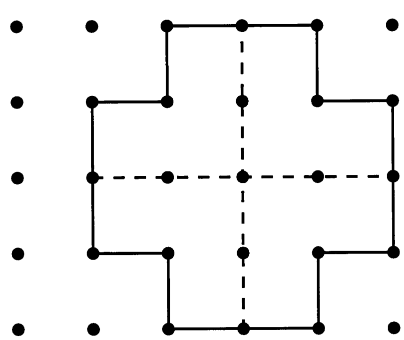


Conduct a show-and-tell discussion of the results: How did the groups proceed? What strategies did they use to arrive at answers? Have them demonstrate the required area relationship for some of their answers and describe any symmetry that is present.

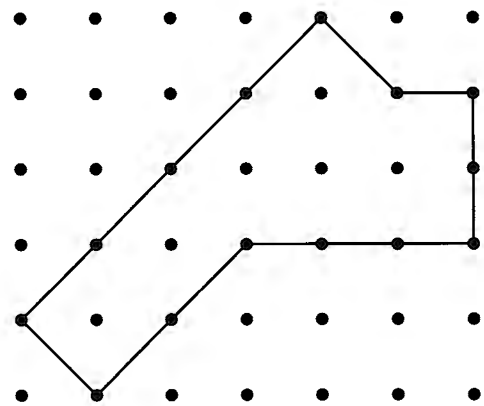
GROUP 1 *We put four figure A's together like this. Our new shape still has a line of symmetry.*



GROUP 2 *Ours has some lines of symmetry and rotational symmetry.*



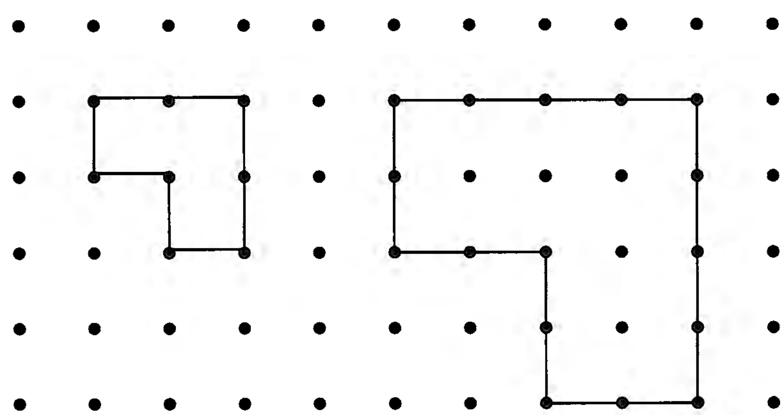
GROUP 3 *We just started making shapes that had an area of 12. Here's one that doesn't have any symmetry.*



Ask the groups to do address the following two problems:

- In your opinion, do any of your larger figures have the same shape as Figure A? Explain.
- The area of the larger figure is supposed to be 4 times that of Figure A. If you haven't already done so, make a larger figure that also has the same shape as Figure A. Record your answer on Blackline 198.

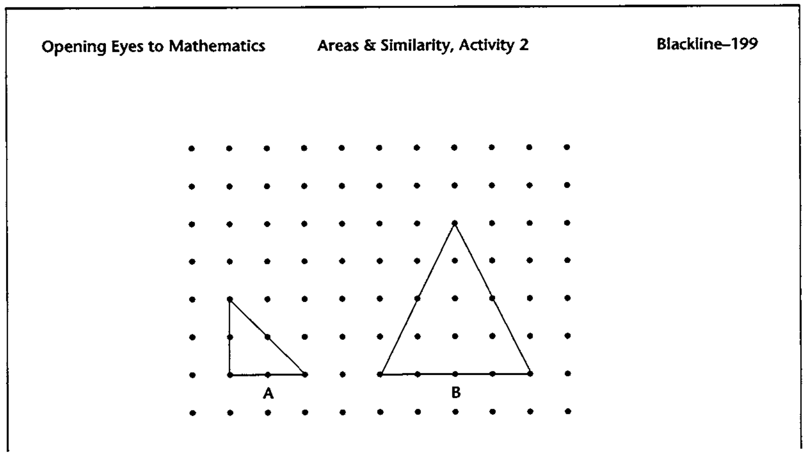
These problems are intended to motivate an informal discussion of similarity. Your children will likely have some interesting views about what figures have the same shape. How do their views agree with the mathematical meaning of similar shapes? The figures in the picture below are (mathematically) similar to one another (please see Teacher Tips). Display and discuss these figures, should they not be offered by your children.



The smaller figure has been blown up to the larger one. The sides of the smaller figure have doubled in length but there has been no change in the angles. Discuss these relationships and any other observations the children have.

Activity 2

Display a transparency of Blackline 199 at the overhead and distribute a copy of this blackline to each child.



Working in small groups, have the children answer these questions:

- How are shapes A and B alike?
- How are these shapes different?
- What is the area of each shape?

In the discussion that follows, invite each group to share (and demonstrate) one likeness and one difference. Here are some things that may be shared.

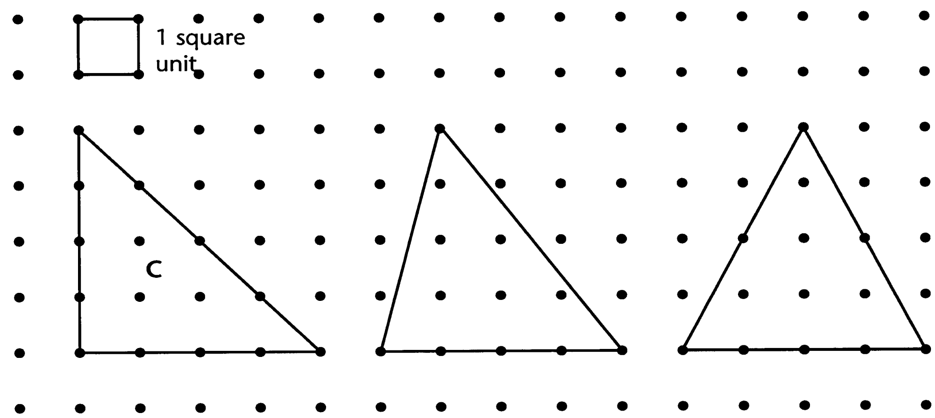
Likeness	Difference
Both shapes are triangles.	The area of A is 2. B has area 8.
Both have 3 sides.	A has a right angle, but B doesn't.
Both have line symmetry.	Perimeters are different
Both are isosceles triangles.	B has "inside" dots, but A doesn't.

You can add observations of your own to the list. Lists will vary, of course, but there will be plenty of opportunities to introduce (or review) geometric concepts and vocabulary. For example, a group might observe that both triangles are isosceles; however, shape B is also a *right triangle* since it contains a right angle. This can be contrasted with other kinds of triangles.

Notice that the area of triangle B is 4 times that of triangle A. Ask the groups to build some other triangles whose areas are 4 times that of triangle A. Have them make one of their answers be a triangle that is similar to triangle A. Answers may be recorded on square dot paper.

Provide sufficient time here as this task may prove challenging. Within the groups, it is important for your children to try out ideas, discuss strategies and verify their answers with one another. Encourage the children to record some observations or conclusions to share later with the class.

There are many triangles which have an area 4 times that of triangle A. Some of them are pictured here.



Any triangle with a base of 4 linear units and a height of 4 linear units will have an area of 8 square units.

Of all the possible triangles, only the one marked C is similar to triangle A.

Here is an extension your children might enjoy: Continue the search for triangles with area 4 times that of triangle A. How many different ones are there? Sort (or classify) the answers in some way. Notice that a classification of triangles has been started in the above illustration. If that illustration were continued, what other categories of triangles would there be?

Teacher Tips

It is not the intent to push formulas in Activity 2. In fact, we suggest you avoid them (unless responding to a formula suggested by a child). Your children will develop several visual strategies for determining the required areas and will likely discover some interesting relationships.

Mathematically, similar figures have the same shape, but not necessarily the same size. In the case of polygons, this means that corresponding angles are congruent and that corresponding sides are proportional. Therefore, if one polygon is blown up to a larger size, the angles will remain the same and each side will be enlarged by the same scale factor.

Homework

Have the children discuss the concept of similarity with a family member. What examples of (mathematically) similar objects can be found around the house or neighborhood?

41 *Areas & Similarity, Part II*

You Will Need

- Chapter 11, Geometry
- a square geoboard and rubber bands
- transparencies of Blacklines 141 (Square Dot Paper) and 154 (Isometric Dot Paper)
- rulers, tracing paper and Blacklines 141 (Square Dot Paper) and 154 (Isometric Dot Paper) available
- chart paper and general materials for making displays

Your Lesson

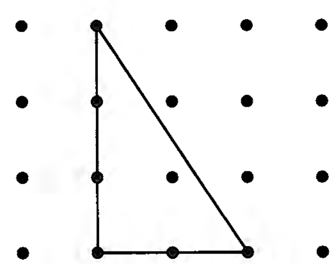
Listed below are some suggested activities for continuing the similarity activities done in Lesson 40 of this volume. Each activity is an opportunity for your children to broaden their understanding of similarity and to strengthen their problem-solving (and problem-posing) abilities.

Activity 1

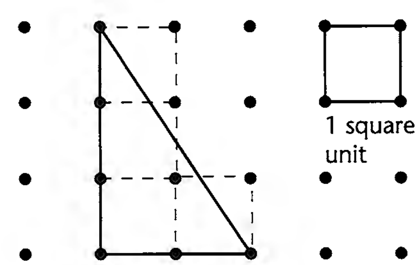
Ask each group of four to do the following:

- Build a right triangle of their choice on a square geoboard.
- Sketch their triangle on square dot paper and glue their sketch to a piece of chart paper.
- Determine the area and perimeter of their triangle and record it on their chart.

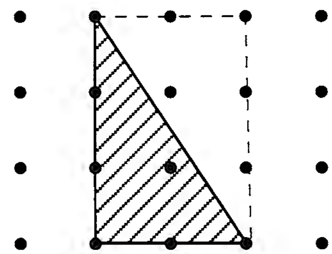
LUKE *Let's use this triangle.*



MELISSA *Okay. For sure the area is less than 4 square units. Look at this.*

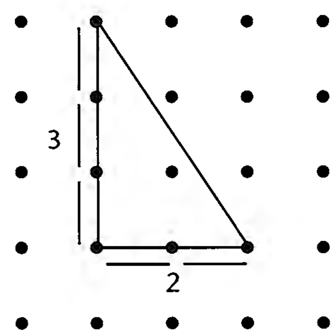


MARIA *I think the area is 3 because I can see half of a 2 by 3 rectangle.*



DOMINIC *You're right. The long side of the triangle splits the rectangle in half.*

MELISSA *What about perimeter? It's got to be at least 5 because of these 2 sides.*



MARIA *The long side looks to me to be about 3½ linear units. I think the perimeter is about 8½ linear units.*

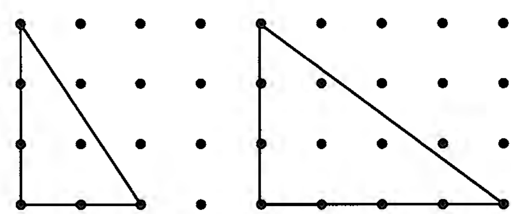
DOMINIC *We could find the length of each side by measuring with a ruler.*

LUKE *Or else we could use a string or some other marker.*

Have the groups now construct two different triangles that are similar to their right triangle. Ask the groups to draw these triangles on square dot paper and determine the corresponding areas and perimeters. Note the strategies groups use to do this. In particular, look for use of estimation or visual thinking techniques.

DOMINIC *We have to make another triangle that is the same shape as ours but larger. Well, it has to have a square corner (*right angle*).*

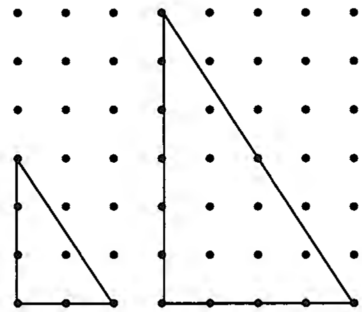
MARIA *Let's try this one.*



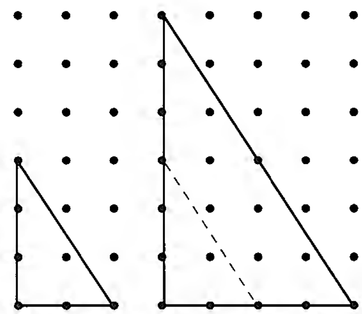
No, I don't think so. It doesn't look the same after all.

MELISSA *You've made the bottom of our triangle longer. Maybe we should make the other sides longer, too.*

LUKE *Try twice as long.*

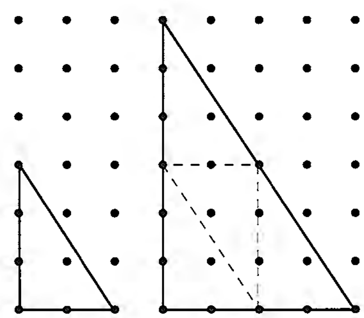


MARIA *That does look like the same shape. I can even see the triangle we started with! (She draws the line AB in the following illustration.)*



LUKE *Looks like the long sides are parallel.*

DOMINIC *I see our triangle four times. Look.*

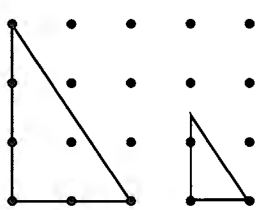


MELISSA *So the area is four times larger but the sides are only twice as large.*

MARIA *What if we made the sides three times as long?*

As you circulate among the groups, try to gain insight about the children's thinking and use of language. Be prepared to facilitate discussions of related geometry (see, for example, Luke's comment about parallel lines in the dialogue) and encourage the search for geometric relationships.

Have the groups add their drawings, results and any observations to their charts.

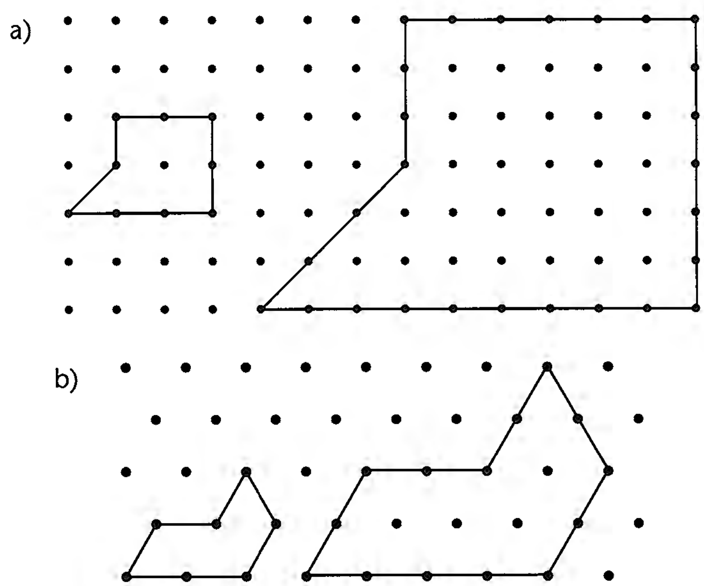


Ask for some predictions. For example, invite the groups to sketch and describe (on their charts) a triangle similar to their right one, yet smaller than it (example shown to the left). How about a triangle, similar to their right one, with sides 100 times as long?

Post the charts and invite groups to share their work. The sharing time offers another opportunity to discuss related geometry. For example, corresponding angles within similar triangles are congruent. This can be tested by tracing the angles of one triangle and superimposing the tracing on the angles of the other triangle. Angles themselves may be discussed in relation to a right angle: an angle smaller than a right angle is referred to as *acute*; an angle larger than a right angle is called *obtuse*.

Activity 2

You may wish to repeat Activity 1, only this time have the groups begin with different right triangles or with other shapes. Perhaps, too, some groups would like to work with Blackline 154, Isometric Dot Paper [see part b) of the following illustration].



Activity 3

Invite your children to pose additional problems, related to the activities of this and the previous lesson, that they would like to explore.

Encourage your children often to assume the role of problem-posers. You may have to offer some general help here. Fruitful mathematical problems often come from varying the parameters of a situation. For example, in Activity 2, Lesson 40, children were asked to create *triangles* with *areas four* times that of a given *right triangle*. How might the italicized parts be changed to create new problems?

42 Pattern Block Explorations

You Will Need

- Chapter 11, Geometry
- Volume 3, Appendix D: Reflective & Rotational Symmetry
- overhead pattern blocks

for each group of four children

- pattern blocks
- copies of Blacklines 62–67 (Pattern Block Shapes) available

Your Lesson

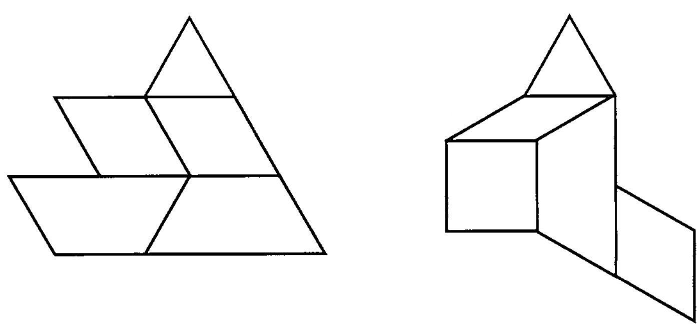
This lesson consists of some pattern block activities that focus on different measurement and geometric concepts. Each activity begins with a problem that children investigate in small groups. Results are then shared or displayed for discussion purposes. You can adjust the problems within each activity to suit your class.

Activity 1

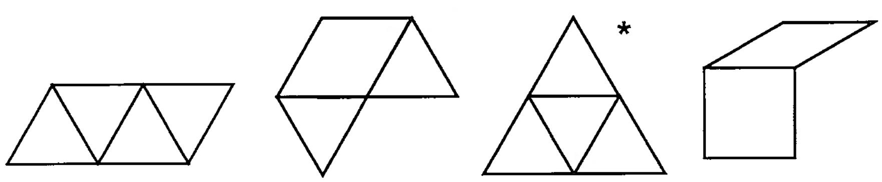
This activity is an extension of those presented in Contact Lessons 151–153, Volume 2. Have the children place a green triangle in front of themselves and tell them this block has a perimeter of 1 linear unit. Ask them to explore such problems as:

What is the perimeter of a yellow hexagon? (2 linear units) What is the perimeter of a red trapezoid? ($1\frac{2}{3}$ linear units)

What is the perimeter of each pattern block design shown here?

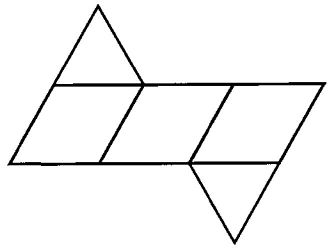


Create some pattern block designs that have a perimeter of 2. Try to find all the different possibilities, including one that is similar to the green triangle (please see Activity 2). How can you be sure all have been found?

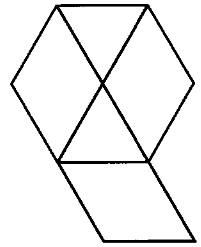


Here are examples of designs that have a perimeter of 2 linear units. The design marked with an asterisk is similar to the green triangle.

Form some designs consisting of 3 blue rhombuses and 2 green triangles. What perimeters can be made? What symmetry is exhibited by each design?



perimeter: $3\frac{1}{3}$
rotational symmetry of order 2
no reflective symmetry



perimeter: $3\frac{2}{3}$
no symmetry

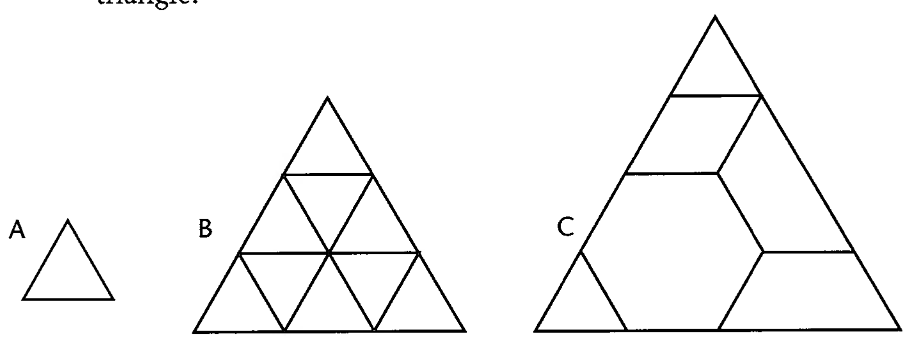
Activity 2

In Activity 1, your children created a design, having perimeter 2 linear units, similar to the green triangle. This assumed the perimeter of the green triangle was 1 linear unit. If you wish, this is an opportunity to explore additional problems related to similarity.

Ask your children to create other designs that are similar to the green triangle. Ask them to describe these designs and then explore these problems:

Examine the perimeters of your designs. How are they related to the perimeter of the green triangle?

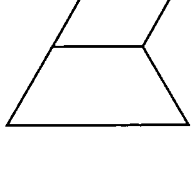
How are the areas of your designs related to that of the green triangle?



B's perimeter is 3 times A's.
B's area is 9 times A's.

C's perimeter is 4 times A's.
C's area is 16 times A's.

Describe an design that is similar to the green triangle but which has perimeter 10 times as large. How about one whose perimeter is 25 (or some other number) times as large?



This activity may be repeated using other blocks (or designs) in place of the green triangle. For example, your children might enjoy creating designs that are similar to the one to the left.

Activity 3

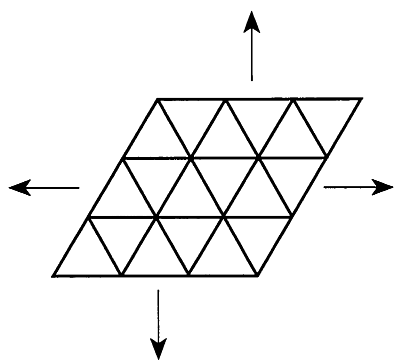
It's time for you and your children to be the problem-posers! Here are some starters for you.

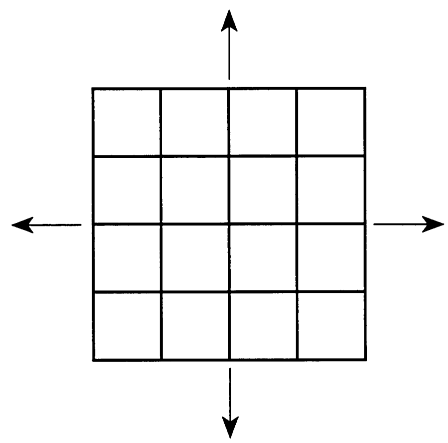
Ask the children to create some problems like those in Activities 1 and 2. These problems may be presented to a neighboring group, used as journal entries, or included in an assessment portfolio.

In Activity 1, the *green triangle* was assumed to have a *perimeter* of *1 linear unit.* Consider changing the italicized parts of this assumption. For example, what if the yellow hexagon (or some other block or design) has a perimeter of 1 (or some other number like ½ or ⅔) linear unit?

Activity 4

Look at the following pattern block designs.



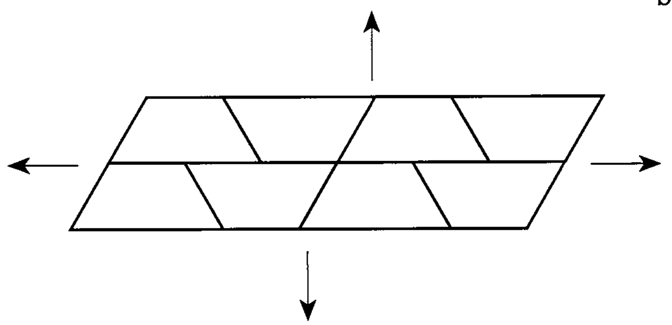


If it were extended infinitely in all directions, each design would cover a plane without any gaps or overlaps. The designs, therefore, will form tessellations of the plane (see Teaching Reference Manual, pages 91 and 92).

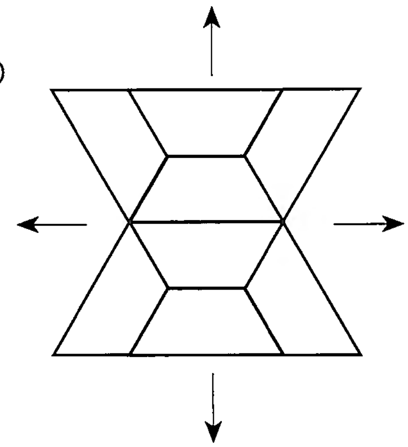
The above tessellations consist entirely of green triangles and orange squares, respectively. Ask your children to begin a tessellation which uses only red trapezoids. Have the children try to do this in a way that guarantees their design will, in fact, tessellate a plane. How can they be sure of this?

When ready, take the class on a "field trip" to observe the proposed tessellations displayed at different tables. Discuss some of the results. Is it clear that the designs, if extended, will complete a tessellation? Is there an identifiable set of trapezoids that repeats throughout the tessellation? Two possibilities are shown on the next page.

a)

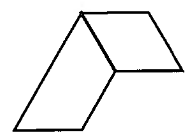


b)

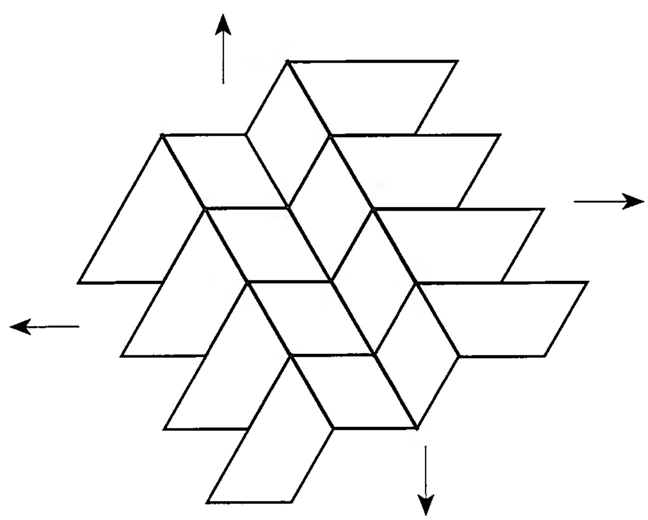


Ask the class to form other tessellations with pattern blocks. You may wish to do this in an open way at first and then have the children answer questions like these:

- Which pattern blocks will tessellate a plane by themselves?
- Imagine a new pattern "block" composed of a trapezoid and a blue rhombus, as if glued together like this:



Form a tessellation which uses only these "blocks". Here is one example:



- See if you can find another new "glued block" that can be used to form a tessellation.

Teacher Tips

Consider making these activities part of an art lesson. Children generally enjoy making colorful pattern block designs and tessellations using shapes cut from Blacklines 62–67.

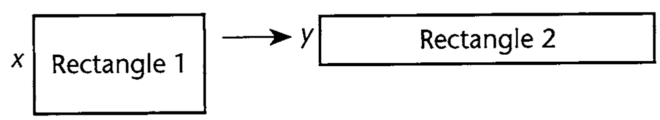
43 Paper Cutting Fractions

You Will Need

- Chapter 8, Fractions
- Blackline 73 (2-cm Grid Paper) and Blackline 200 (Equal Areas, Different Dimensions)
- transparencies of Blackline 25 (1-cm Grid Paper) and Blackline 73 (2-cm Grid Paper)
- scissors

Introduction

The activities of this lesson focus on several important mathematical concepts such as area, fractions, measurement, dimensions and similarity. The illustration below describes the main questions that are investigated. The investigations can promote greater spatial awareness on the part of your children and also anticipate later experiences (in *Visual Mathematics*) with division of fractions.



How can you cut Rectangle 1 and form Rectangle 2 so the areas of both rectangles are the same and dimension $y = \frac{1}{2}$ of dimension x?

If this is done, how are the other dimensions of the rectangles related?

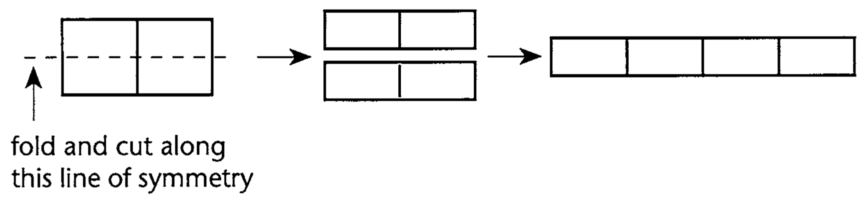
What happens if you try the same thing, only this time making dimension $y = \frac{3}{4}$ of dimension x?

Your Lesson

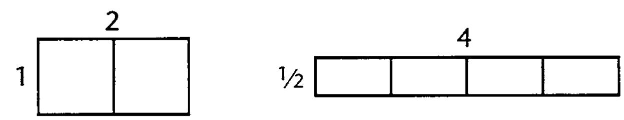
Activity 1

Present this problem to the class and ask the children to predict the answer: A rectangle is to be cut from grid paper. The area of the rectangle is 2 square units and one dimension is ½ linear unit. What is the other dimension?

Have the children cut the rectangle from 2-cm grid paper and note the required dimension. How close were their predictions? What strategies did the children use to obtain their rectangles? Here is one strategy that often occurs.



Have the children explore ways of describing the perimeter and area of the rectangle.



area = 2 square units
perimeter = 6 linear units

area = 2 square units
perimeter = 9 linear units

"The areas are the same—the pieces have just been arranged differently."

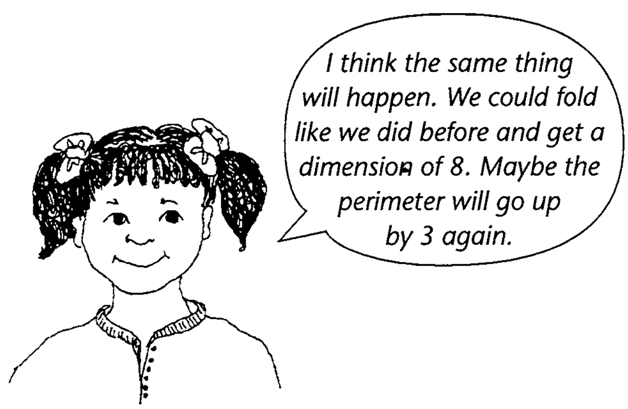
"One dimension is cut in half—the other is doubled."

"But the perimeter isn't doubled. It goes from 6 to 9."

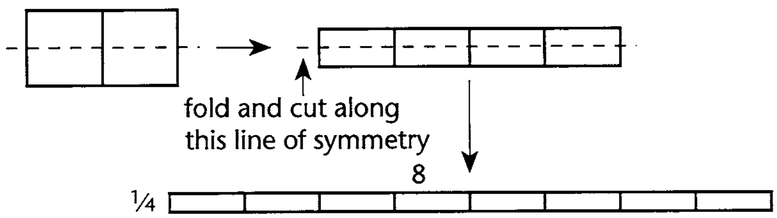
"The new perimeter has 3 more linear units. That's half way to doubling the old one."

Now pose this problem to the class: Your rectangle has an area of 2 square units. How can you cut it and form a new rectangle that has area 2 and one dimension of ¼ linear unit? What would be the other dimension of this rectangle?

Once again, have the class discuss the problem and make predictions before doing any cutting.



Then go ahead and cut! What is the other dimension? Have the children describe any changes in area or perimeter and explain why these changes occurred.



area = 2 sq. units
perimeter = 16½ linear units

"The area stayed the same, but look at the perimeter. It's now 16½. Why did it go up so much?"

Activity 2

The children may be ready to think generally about the above problems. The following problems can help with this.

Imagine a rectangle. Think of cutting this rectangle and rearranging the pieces to form a new rectangle where (a) the area stays the same and (b) one dimension of the new rectangle is ½ as long as a dimension of the old.

What can be said about the second dimension of the new rectangle?

Keep on doing this by cutting a dimension in half and forming new rectangles without changing the area. What happens to the other dimension?

Can the children visualize the second dimension doubling each time?

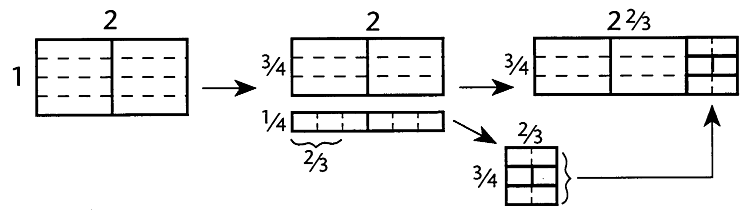
As an exercise, have them trace what happens to the second dimension when they start with a 44 by 44 square and apply the cutting procedure described above.

Activity 3

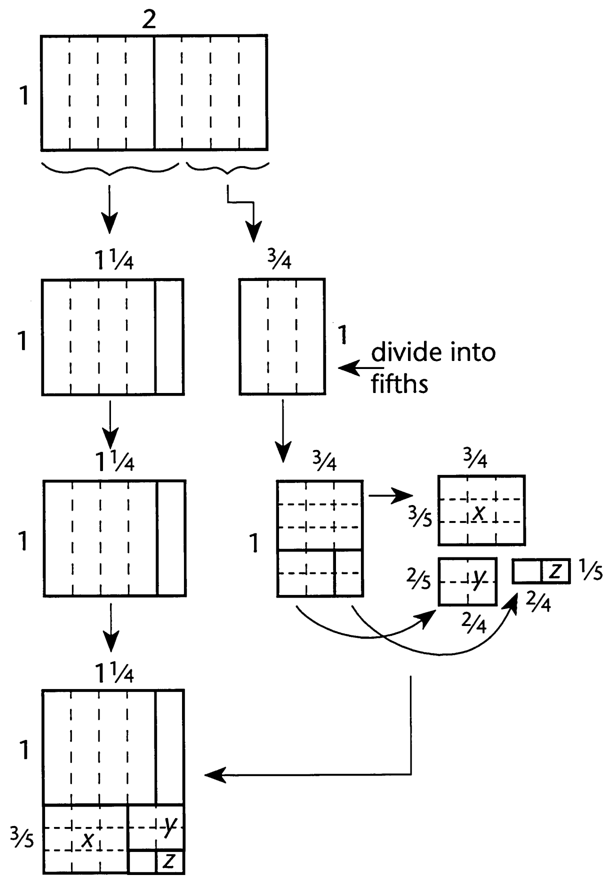
This activity explores questions similar to those in Activities 1 and 2. In each case, ask the children to do three things: Imagine the situation and predict answers to the corresponding question. Cut the required shape. Describe and justify their results.

Allow plenty of time for children to investigate, discuss and share. Encourage them to work together and watch for some creative solutions to come forth. Sample solutions for some of the questions are illustrated below.

(a) Cut a rectangle that has area 2 square units and one dimension of ¾ linear units. What is the other dimension?



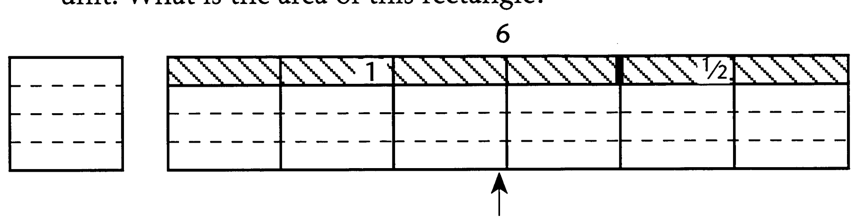
(b) Cut a rectangle with area 2 square units and one dimension of 1¼ linear units. What is the other dimension?



Area: 2
Dimensions: 1¼ by 1⅗

(c) Repeat Part (b) only this time make the given dimension 1½ linear units; 1¾ linear units; 3 linear units.

(d) Cut a rectangle that has dimensions of 6 linear units and ¼ linear unit. What is the area of this rectangle?



1 square unit

The 6 by ¼ rectangle has area 1½ square units

(e) Repeat Part (d) only this time make the given dimensions $\frac{2}{3}$ linear units and $\frac{1}{2}$ linear units; $\frac{1}{3}$ and $\frac{1}{3}$ linear units.

Activity 4

Have the class measure the perimeter of a rectangular room in your building using a standard measuring tool such as a yardstick or meterstick. With this information, have the children work the problems on Blackline 200.

Opening Eyes to Mathematics — Equal Areas, Different Dimensions — Blackline–200

1. You just determined the perimeter of a room that has the shape of a rectangle. Make a sketch of this rectangle and describe how you found its perimeter. How can you calculate the area of the room?

2. Make a scale model of the floor of your room on centimeter graph paper.

3. Imagine cutting one dimension of the room in half and making a new rectangular room with the same area. What would be the dimensions of the new room?

4. Suppose you keep making new rectangular rooms by cutting a dimension in half but keeping the area the same. What can be said about the dimensions of the new rooms?

Discuss the responses to Problems 3 and 4 of the blackline. Would the new rooms have dimensions that are reasonable in the eyes of the children? (Also, please see Teacher Tips.)

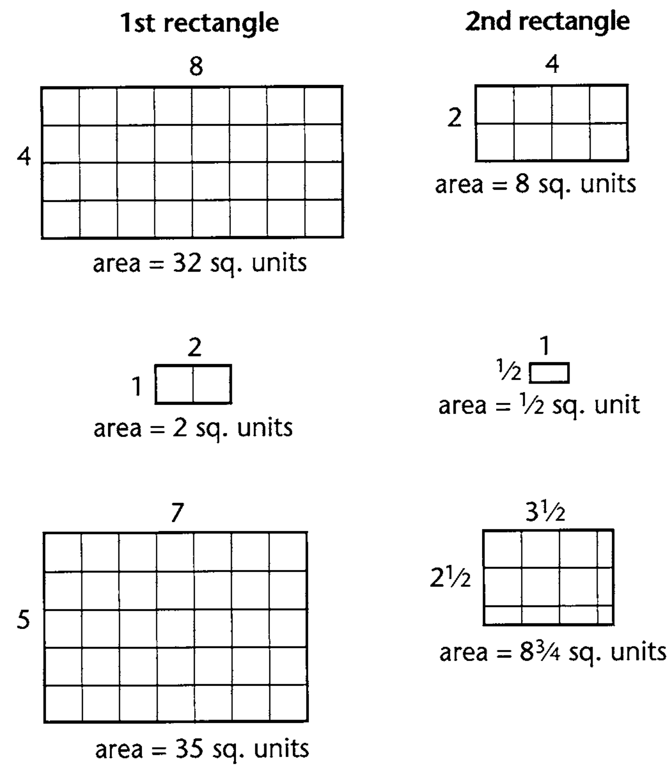
Activity 6

Use this activity to connect with Lessons 40 and 41, Areas and Similarity, of this volume.

Ask your children to make predictions about the following situation. Then have them explore the problem and share their strategies.

Start with a rectangle of your choice. Think about cutting each dimension of your rectangle in half. Use these new dimensions to make a second rectangle. How does the area of the second rectangle compare to the area of the first one?

Here are some examples:



"So far, the area of the second is ¼ the area of the first. Why is that? Will it always be true?"

Note: It is customary to use the following language to describe what is taking place in this situation:

- The second rectangle is similar to the first by a scale factor of ½. One could also reverse the statement: The first rectangle is similar to the second by a scale factor of 2.
- The two rectangles have the same shape and the dimensions of the second are ½ of the corresponding dimensions of the first.

Have the children investigate further by perhaps looking at other pairs of shapes where one is similar to the other by a scale factor of ½. Some children may also wish to examine what happens if a scale factor other than ½ is used.

Teacher Tips

The visualization children bring to the above activities can be useful when picturing room dimensions.

On a historical note, you may wish to conduct a library project about the Golden Rectangle. People as far back as the ancient Greeks were first conscious of this rectangle, which is said to have dimensions that are most pleasing to the eye. It is found in art and architecture, and is related to many interesting mathematical ideas. How do the "classroom" rectangles your children measured compare with the Golden Rectangle?

Homework

Consider these problems with a family member or friend.

What standard measuring tool(s) would you use to measure the dimensions of a regulation soccer field? What is the area of this field?

Suppose a new rectangular soccer field is formed with the same area as a regulation field, but with one dimension half as long. Describe the perimeter of this new field. How would the game itself be affected if played on the new field? Would you like to see the new field used for soccer?

44 *Spinners & Graphs, Part I*

You Will Need

- Chapters 9, Probability, and 10, Data Analysis and Graphing
- transparencies of Blacklines 201 (Spinner A) and 202 (Spinner Graphs)
- materials for graphing
- chart paper and markers for each group of four children

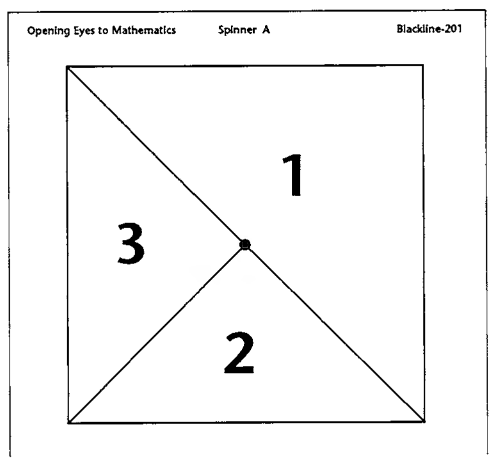
for each child

- Blackline 201 (Spinner A), bobby pins or paper clips can be used for pointers
- Blackline 202 (Spinner Graphs)
- Blackline 203 (Spinner Graphs Response Sheet A)

Your Lesson

The spirit of this lesson is similar to that of Lessons 33 and 34, Opinion Polls, in this volume. The lesson provides added experiences with interpreting graphs, making decisions about claims, and reflecting about what can reasonably occur in chance situations. The setting of the activities can also motivate discussion of congruence, area and angle.

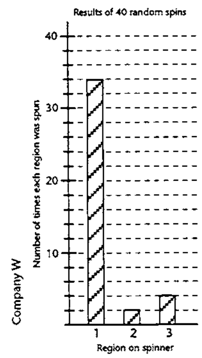
Divide the class into groups of four and then form teams of two children within each group. Blackline 201, Spinner A, shows a square that has been divided into 3 regions; the needle is at the center of the square. Distribute this spinner and copies of Blacklines 202 and 203 (see following page) to each team.

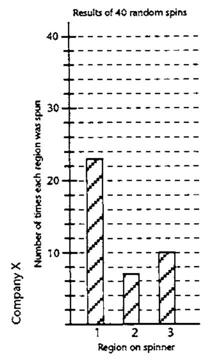


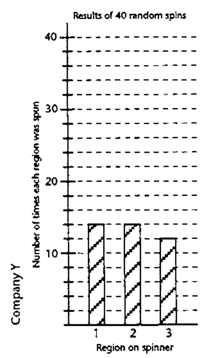
Demonstrate a few spins of spinner A at the overhead and, referring to Blackline 202, present this situation to the class: Representatives from four companies (W, X, Y and Z) each make the following claim, "I made 40 random spins of spinner A and the results of these spins are shown on my company's graph."

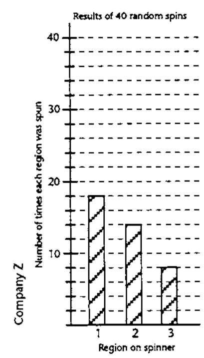
Upon examining spinner A and the four graphs, how do the children feel about each claim? Ask each child to think privately about this question for a few moments and to then, with their teammate, respond to the questions on Blackline 203. When ready, have the teams within each group compare their responses and prepare a report of their work for large group discussion.

Opening Eyes to Mathematics **Spinner Graphs** **Blackline–202**









Opening Eyes to Mathematics **Spinner Graphs Response Sheet A** **Blackline–203**

Representatives from four companies (W, X, Y, and Z) each make this claim: "I made 40 random spins of spinner A and the results are shown on my company's graph." Examine each graph and answer the following questions.

1. How do you feel about each claim? Explain your reasoning.

2. Is it possible that the results on each graph could have come from 40 random spins of the spinner? Explain.

3. Think about the graphs from each company. In your opinion, which category in the following chart best describes the results shown on each graph? Mark each choice with the company's letter.

Not likely to come from 40 random spins of spinner A	Somewhat likely to come from 40 random spins of spinner A	Highly likely to come from 40 random spins of spinner A

4. Explain the reasoning behind your choices in Question 3.

The questions on Blackline 203 seek to capture the children's reactions to these claims. Some children may base their initial responses on personal feelings, such as "anything's possible". Others may obtain bring experimental or theoretical information to their discussion.

GROUP 1 *We're not so sure about Company W's graph. There seems to be way too many 1's.*

TEACHER *Why does that bother you?*

GROUP 1 *Look at the spinner. Region 1 is half of it. So we think it should come up about half the time.*

GROUP 2 *We agree. Regions 2 and 3 also have the same amount of area. They should come up about the same number of times. (Note: Please see discussion about this remark in Lesson 45, Spinners and Graphs, Part II.)*

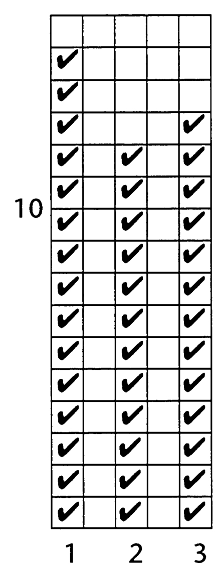
TEACHER *How did you decide that regions 2 and 3 have equal areas?*

GROUP 2 *Well, they looked the same. Then we traced region 2 and put it on region 3. It fit right on top.*

TEACHER *Yes, they are congruent regions.*

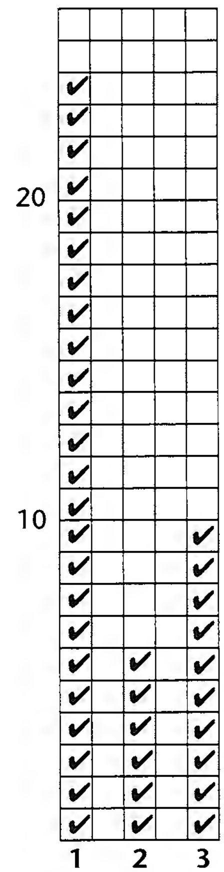
GROUP 1 *Region 2 should come up about ¼ of the time, since it is ¼ of the spinner. That's why we feel good about Company X's graph.*

GROUP 3 *We couldn't decide. So we did 40 spins of our own and made a graph. We got a graph like Company Y. All the regions came out about the same.*



GROUP 2 *Were your spins "all right"? Or maybe you didn't spin enough.*

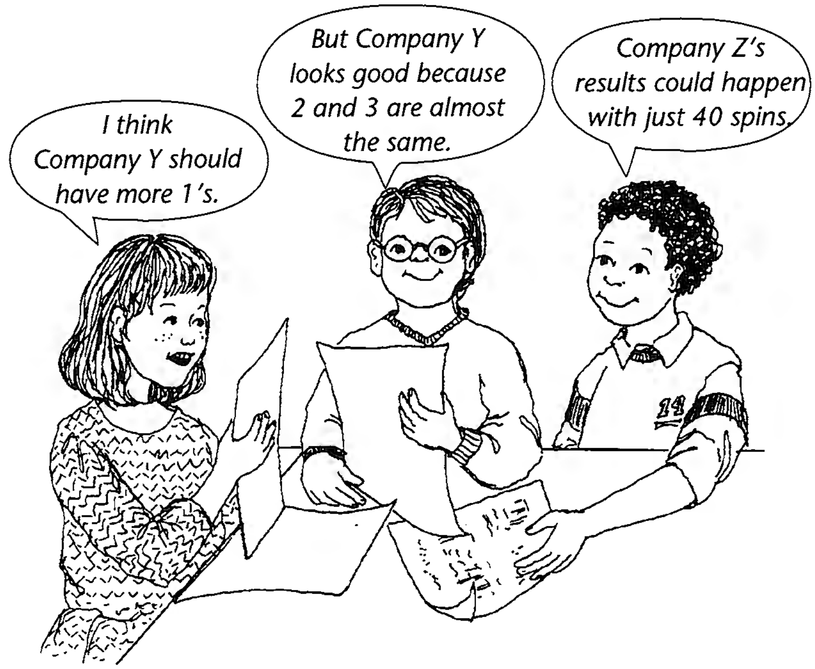
GROUP 4 *We did an experiment, too, but our needle seem to get stuck a lot. We had different people spin and got these results. There are a lot of 1's. Maybe that's what happened to Company W.*



The dialogue reflects different ways to make decisions about chance situations (i.e., personal judgement, experimental evidence or theory). The children are likely to initiate these strategies; if not, do so yourself. Here are some questions that might be helpful:

Look at spinner A. Which region do you think has the best chance of coming up? Why? How do you feel about the other regions?

- Suppose Company W's representative made 40 more spins? What do you predict will happen?
- Do you think you'll get results like Company W's if your team made 40 spins? Why don't you see what happens? Do you think 40 spins are enough?
- Take a poll of the teams by compiling the responses to Question 3 on Blackline 203. Do the compiled results suggest any overall class feelings? Or is there little consensus among the teams?



The questions on Blackline 203 are also intended to motivate a discussion of "possible" events versus "probable" ones. It is important to distinguish between these two notions and to provide time for children to reflect about them.

GROUP 5 *We think each claim is okay, since all the graphs are possible. Anything's possible!*

TEACHER *(addressing the class) Can you think of a situation in your life that is possible and likely to happen? Can you think of one that is possible but not likely to happen?*

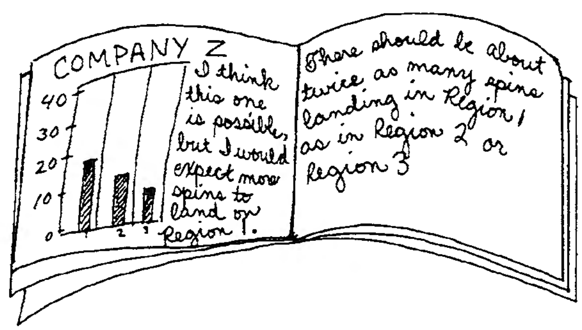
Continue the lesson by asking each group to create another graph (of 40 random spins of A) that they would confidently place in each of the categories shown in Question 3 of Blackline 203. Challenge them to make these graphs without actually conducting more experiments. That is, see if they can predict results they would place in each category. Ask them to explain the thinking behind their choices. How confident are they of their responses?

Allow time for the groups to share their work with the entire class. They could post their graphs on a class chart or create a book on their results (see Published Piece below). In any case, take this opportunity to note any changes of thinking that have occurred for your children as a result of the previous discussion.

Creating these graphs can also promote an awareness of the variability associated with chance situations. For example, in samples of 40 spins, several sets of results have a reasonable probability of occurring. There is no single answer, even though it is theoretically true that region 1 should occur twice as often as region 2.

Published Piece

(Optional) Your children might enjoy compiling their work on the above activities into a book. Some of the "chapters" of the book could focus on their beliefs about each claim, graphs for each category in Question 3 of Blackline 203, and suggestions for further exploration.



Pattern Reminder

There may be some Pattern Generalizations (Blacklines 169–176) left to explore. If so, consider looking at one of them; or, let your children create new ones for each other.

45 Spinners & Graphs, Part II

You Will Need

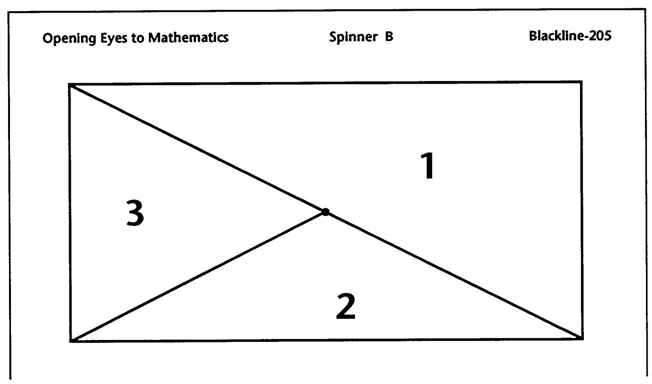
- Chapters 9, Probability, and 10, Data Analysis and Graphing
- transparencies of Blacklines 202 (Spinner Graphs) and 205 (Spinner B)
- materials for graphing
- chart paper and markers for each group of four children

for each child

- Blackline 202 (Spinner Graphs)
- Blacklines 204 (Spinner Graphs Response Sheet B)
- Blackline 205 (Spinner B), bobby pins or paper clips can be used for needles
- Blackline 206 (Spinner Graphs Homework)

Your Lesson

This lesson repeats the activities of Lesson 44, using Spinner B instead of Spinner A. Spinner A shows a square divided into three regions, whereas the regions of Spinner B form a rectangle that is not square. In both cases, the needle is at the center of the spinner.



Divide the class into groups of four and then form teams of two children within each group. Distribute spinner B and copies of Blacklines 202 and 204 to each team. Demonstrate a few spins of spinner B at the overhead and, referring to Blackline 202, present this situation to the class:

Representatives from four companies (W, X, Y and Z) each make the following claim: "I made 40 random spins of spinner B and the results of these spins are shown on my company's graph."

Upon examining both spinner B and the four graphs, how do your children feel about each claim?

As in Lesson 44, the following activities can elicit discussion about this question.

Have each child think privately about these claims and then, with their teammate, respond to the questions on Blackline 204.

Opening Eyes to Mathematics **Spinner Graphs Response Sheet B** **Blackline–204**

Examine the spinner and each of the graphs you've been given and then discuss these questions with your partner. Record your observations in the space provided.

1. How do you feel about each claim? Explain your reasoning.

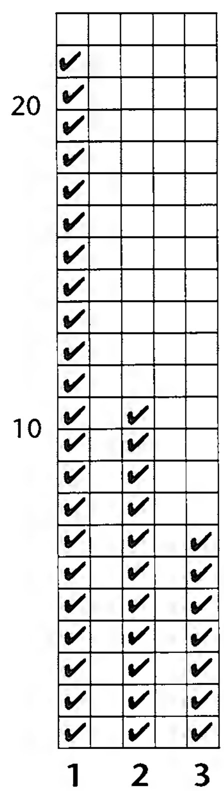
2. Is it possible that the results on each graph could have come from 40 random spins of the spinner? Explain.

3. Think about the graphs from each company. In your opinion, which category in the following chart best describes the results shown on each graph? Mark each choice with the company's letter.

Not likely to come from 40 random spins of spinner B	Somewhat likely to come from 40 random spins of spinner B	Highly likely to come from 40 random spins of spinner B

4. Explain the reasoning behind your choices in Question 3.

Ask the teams within each group to compare their responses and then report to the class. What strategies or reasoning did groups use to decide about each claim? Did any gather experimental evidence? Or attempt to think theoretically?



Compile the team responses to Question 3 on Blackline 204. Do the compiled results suggest any overall class feelings? Or is there little consensus among the teams?

After reports have been shared, ask the groups to create another graph (of 40 random spins of B) that they would confidently place in each category shown in Question 3, Blackline 204. Encourage them to attempt this without actually conducting more experiments. Have them explain the thinking behind their choices and comment about how confident they are of their responses.

The discussions related to these activities will likely be similar to those of Lesson 44. Look for opportunities to talk about some of the geometry exhibited by spinner B.

GROUP 1 *We like Company Z's graph* (shown to the left). *Our experiment gave us a graph like theirs.*

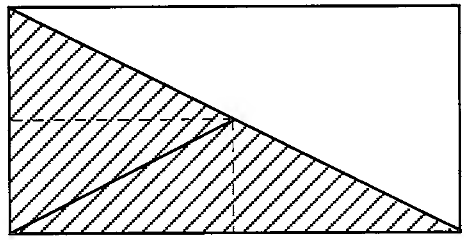
GROUP 2 *We think region 1 should come up about half the time. It takes up half the spinner—just like with spinner A.*

TEACHER *What about regions 2 and 3?*

GROUP 1 *We got region 2 more often. Last time, 2 and 3 had the same area. This time we're not so sure.*

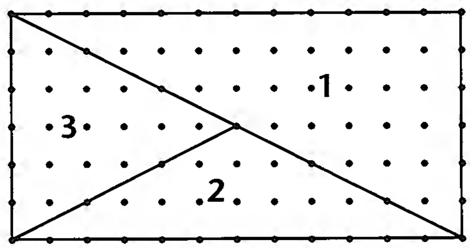
The teacher asks the groups if the areas of regions 2 and 3 are the same.

GROUP 3 *We can show they are the same. We made 4 congruent triangles.*



"The shaded triangles are congruent."

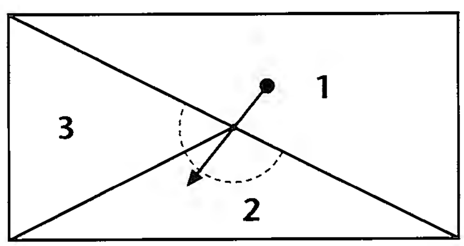
GROUP 4 *They are the same. We traced the regions on dot paper and counted squares—just like on the geoboard.*



"There are 72 little squares like this ☐ in the entire spinner. 36 of them are in Region 1. Region 2 has 18 of these squares and so does Region 3."

TEACHER *So the areas of regions 2 and 3 are the same just like with spinner A. Does that mean each of these regions should come up about ¼ of the time?*

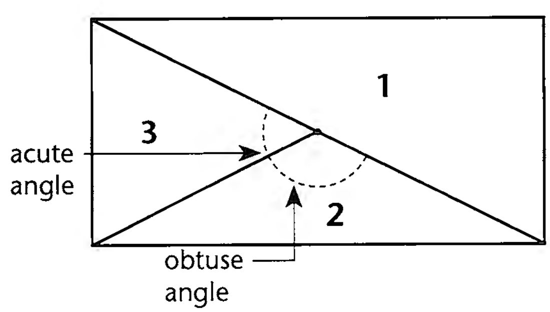
GROUP 1 *We don't think so—it seems there are more opportunities for the needle to land in region 2.*



"There are more places for the needle to land in region 2 than in region 3."

So region 2 should occur more than ¼ of the time, but not as much as ½ the time.

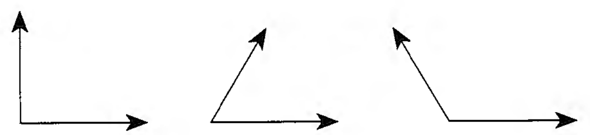
The teacher takes the time to identify the kinds of angles formed at the center of spinner B.



Teacher Tips

The regions of spinners A and B exhibit the same kind of area relationships. In both cases, region 1 contains half the area of the spinner, and regions 2 and 3 each contain ¼ of the entire area. However, because of the obtuse angle shown in the preceding illustration, the needle of spinner B should theoretically land in region 2 more often than in region 3.

The following diagram summarizes some of the vocabulary related to angles. The activities of this lesson offer a good context for introducing/reviewing these angle concepts. You may wish to discuss these concepts further.



right angle measures 90°	acute angle measures less than 90°	obtuse angle measures more than 90°

Homework

Ask your children to work with friends or family members on the Blackline 206 problem.

Some children could post their graphs from this problem and invite the others in the class to predict what the corresponding spinners look like.

Opening Eyes to Mathematics | **Spinner Graphs Homework** | **Blackline–206**

Create a spinner of your own and make three graphs (of 40 random spins) labeled as shown here:

1st graph: These results are not likely to come from my spinner.

2nd graph: These results are somewhat likely to come from my spinner.

3rd graph: These results are very likely to come from my spinnner.

Assessment

The homework above would make a good portfolio entry.

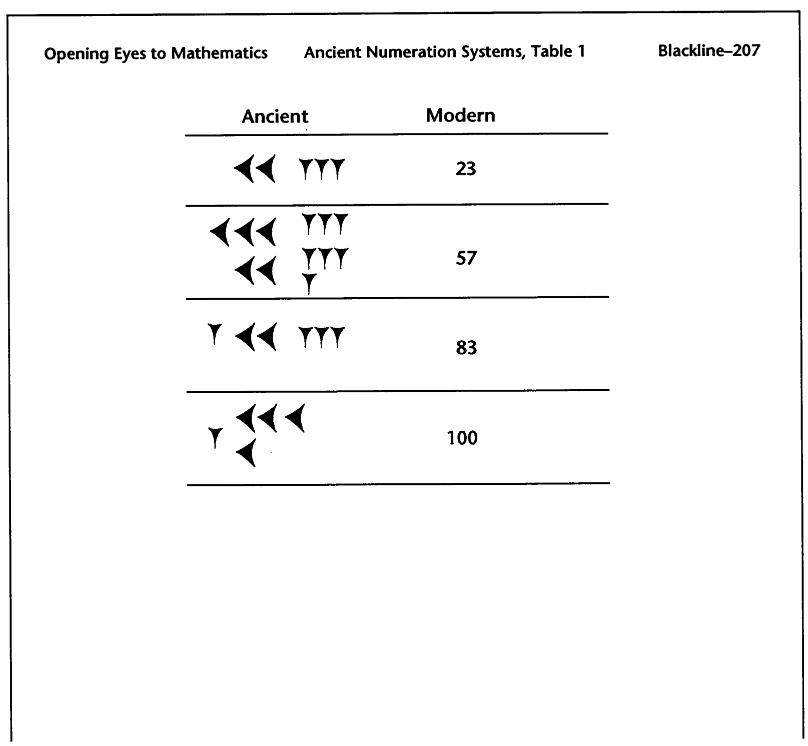
You Will Need

- Chapter 4, Place Value, and Chapter 6, Multiplication
- transparencies of Blackline 207 (Ancient Systems, Table 1) and Blackline 208 (Ancient Numeration Systems, Table 2, 2 pages)
- calculators, chart paper and general materials available
- a copy of Blacklines 207 (Ancient Systems, Part II) and Blackline 208 (Ancient Numeration Systems, Table 2, 2 pages) for each child

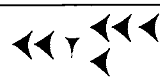
Your Lesson

In Lesson 24 of this volume, children assumed the role of archeologists attempting to decipher the ancient Egyptian numeration system. They are asked to assume a similar role in this lesson, whereby they use available clues to decipher a numerical table. As the "code is cracked", the features of the ancient Babylonian numeration system are uncovered and discussed.

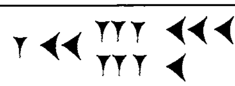
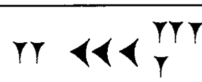
There is value in examining the Babylonian system, principally because it was a place value system that used a base of 60. We hope you will join the children in examining the tables on Blacklines 207 and 208 for clues to understanding this system. You may find it helpful to do this before reading the background information given throughout the lesson. To help you do this, we have included copies of all three blacklines here.



Opening Eyes to Mathematics | Ancient Numeration Systems, Table 2 page 1 | Blackline–208

	Babylonian	Modern
a)		143
b)		342
c)		248
d)		
e)		
f)		2423

Opening Eyes to Mathematics | Ancient Numeration Systems, Table 2 page 2 | Blackline–208

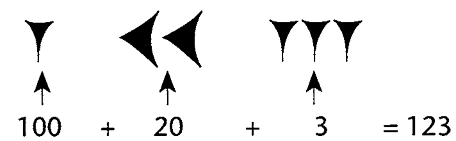
	Babylonian	Modern
g)		3661
h)		5200
i)		7234
j)		7472
k)		3627
l)		750
m)		12½
n)		a number of your choice

Activity 1

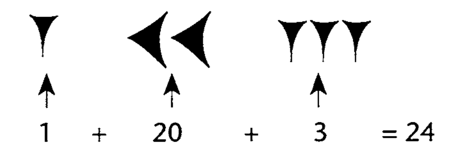
Display the entries in the left-hand column of Blackline 207 at the overhead, asking the children to imagine these are numerals from an ancient society. Tell them that some information is known about these numerals and reveal the first two entries in the right-hand column.

Ask them to predict the numbers that correspond to the third and fourth numerals in the left column. Children are likely to offer various predictions here based on the limited information available to them.

JEAN *The first two are just like our number system. The 𒁹 stands for 1 and the 𒌋 is 10. I think it continues that way, so the third number must be 123.*

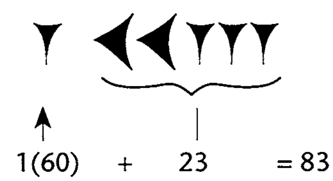


MARCUS *I agree with Jean; but the third number could also be 24. Maybe these numbers are like the Egyptian ones.*

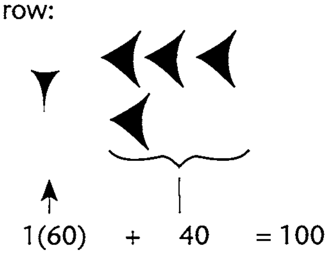


Reveal the entire third row of the transparency and give the children an opportunity to propose hypotheses that would explain the 83. These hypotheses can be tested against the fourth row. The children may suggest that the 𒁹 can stand for 1 or 60 depending on its position in the numeral. This interpretation also explains the fourth row of the transparency. If this suggestion doesn't occur, offer it yourself, since it illustrates a major feature of the ancient system (see following paragraph).

3rd row:



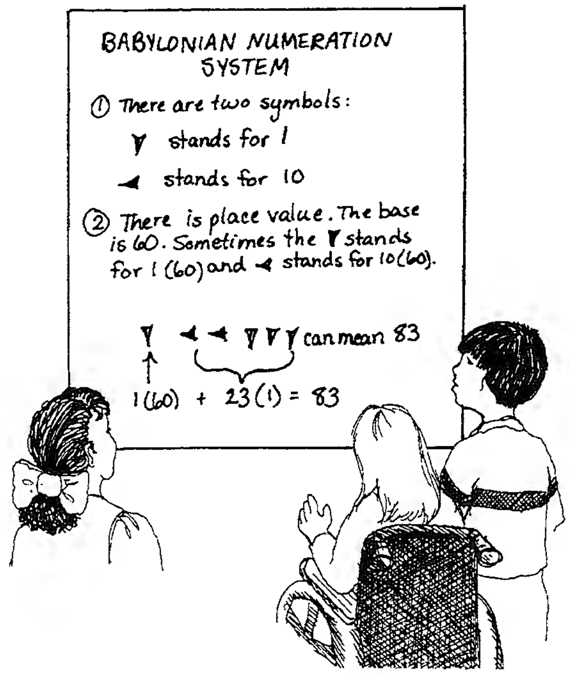
4th row:



Tell the class that numerals like those in the left column were used in ancient Babylonia. Such numerals were imprinted in clay tablets using two wedged-shape symbols: a narrow upright wedge 𒁹 and a broad sideways wedge 𒌋 with face values of 1 and 10, respectively. The Babylonians also used positional notation and a base of 60.

Activity 2

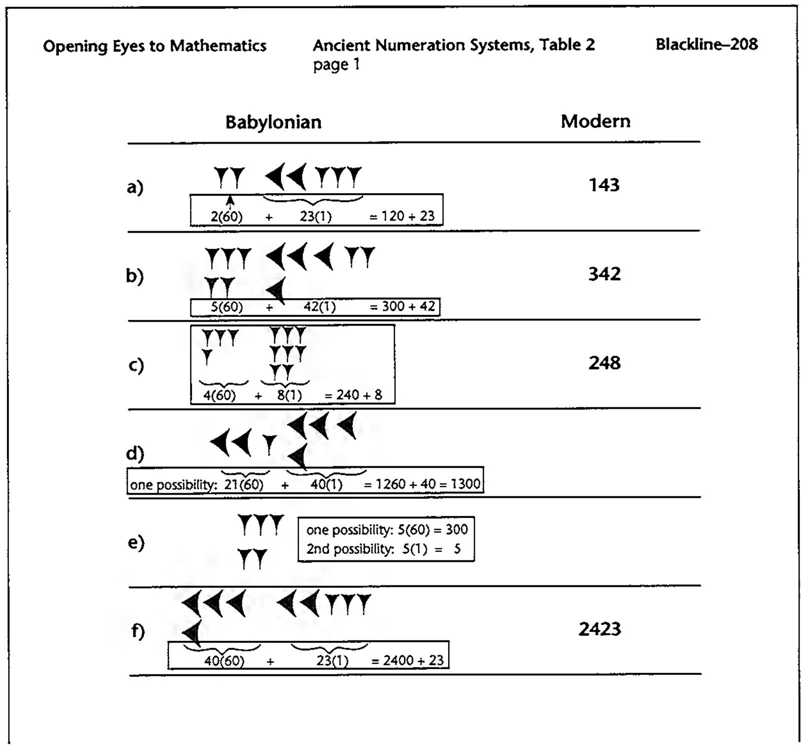
Begin a chart showing the information gathered thus far about the Babylonian system.



With this information, divide the class into groups and distribute page 1 of Blackline 208 (displayed on opposite page). Tell the children this blackline contains further clues about the Babylonian numeration system. Ask the groups to uncover the remaining mysteries of this system by completing the table. Allow enough time for the groups to attempt this task and to summarize their findings. It is likely that children will experience both progress and difficulty in deciphering the system. This experience can motivate the discussion that follows.

Display a transparency of Blackline 208, page 1, and ask the groups to share their thoughts about this page. Continue charting clues as they emerge. This is an opportunity for a very rich mathematical discussion, since the entire blackline (pages 1 and 2) contains several entries that can't be determined with certainty. Please don't expect groups to always be "right" or to agree. In fact, you can anticipate having to explain current understandings about this system as the discussion unfolds. Here are some things to expect about page 1.

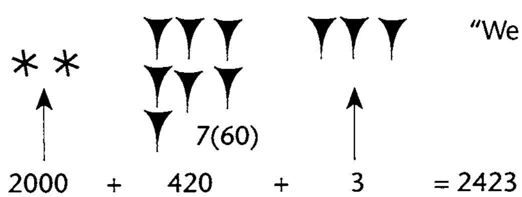
- Groups will likely use the information from Blackline 207 to obtain answers for parts a)–f) of Blackline 208. These answers are shown on page 234.



Opening Eyes to Mathematics — Ancient Numeration Systems, Table 2 page 1 — Blackline–208

	Babylonian	Modern
a)	2(60) + 23(1) = 120 + 23	143
b)	5(60) + 42(1) = 300 + 42	342
c)	4(60) + 8(1) = 240 + 8	248
d)	one possibility: 21(60) + 40(1) = 1260 + 40 = 1300	
e)	one possibility: 5(60) = 300 2nd possibility: 5(1) = 5	
f)	40(60) + 23(1) = 2400 + 23	2423

- Children may invent other symbols.



"We made a ✱ for 1000."

This can be positively acknowledged, if it occurs. It is believed, however, that the Babylonians consistently used just the two wedge-shaped symbols and a base of 60.

- Some groups may be unsure about the answer for part e):

Does stand for 5(60) + 0(1) + 300 or 5(1) = 5?

This ambiguity is directly related to the absence of a symbol for zero in the Babylonian system. As the children will see when they do the second page of Blackline 208, the Babylonians generally left a gap to indicate a missing place value. According to Burton, however, this practice wasn't always observed and sometimes the gap was overlooked. Distinctions between numbers like 300 and 5 were made on the basis of the context such as a business or astronomical problem) in which the numbers were used.

When ready, have the groups continue to explore the Babylonian system by examining and completing parts g)–n) of Blackline 208, page 2. Continue to chart any new features about the system that are mentioned during the accompanying discussion. Some of these features are discussed in the following paragraphs and in the Teacher Tips.

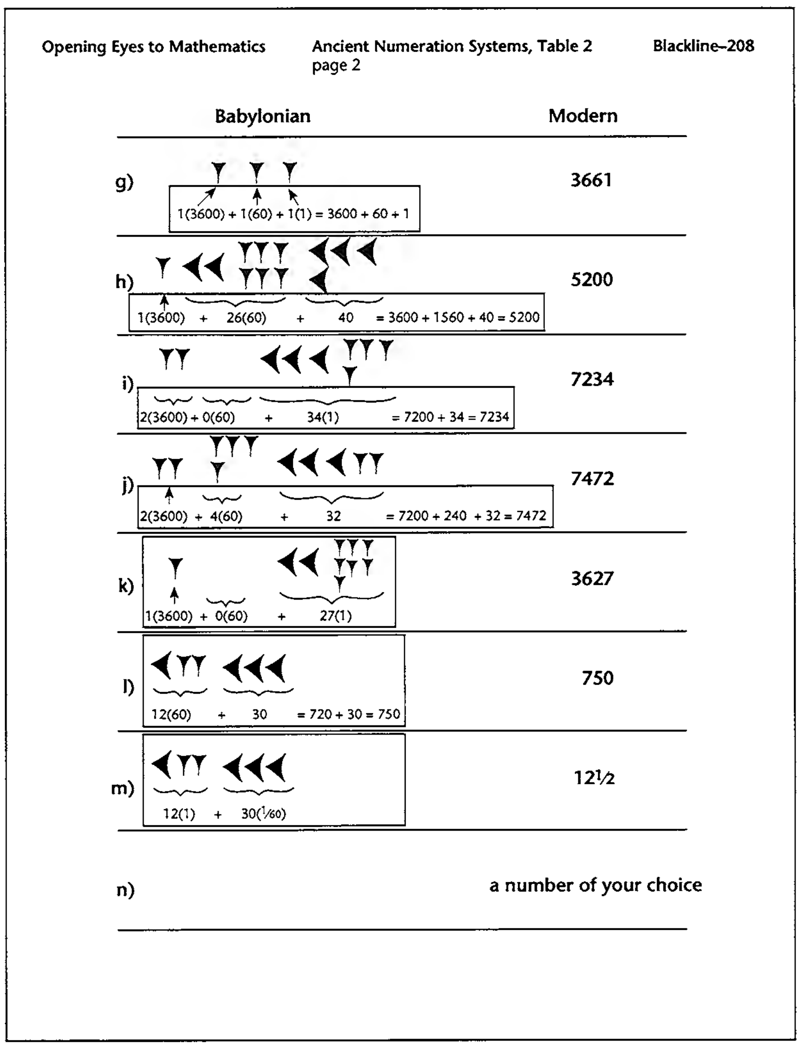
The second page of Blackline 208 may actually generate more questions than answers. Again, this is because there was no symbol for zero nor was there a way (other than context) of distinguishing the place values within a numeral.

Your children may, therefore, wonder about the answer 3661 to part g). This answer is explained by thinking about place value in base 60 (please see Teacher Tips).

 can represent 1(3600) + 1(60) + 1(1) = 3661.

Of course, it can also represent 3(1) = 3, and this ambiguity should be discussed.

The rest of the table can be interpreted as shown here. Be sure to allow time for children to report and discuss their thinking about each row.



Opening Eyes to Mathematics | Ancient Numeration Systems, Table 2 page 2 | Blackline–208

	Babylonian	Modern
g)	1(3600) + 1(60) + 1(1) = 3600 + 60 + 1	3661
h)	1(3600) + 26(60) + 40 = 3600 + 1560 + 40 = 5200	5200
i)	2(3600) + 0(60) + 34(1) = 7200 + 34 = 7234	7234
j)	2(3600) + 4(60) + 32 = 7200 + 240 + 32 = 7472	7472
k)	1(3600) + 0(60) + 27(1)	3627
l)	12(60) + 30 = 720 + 30 = 750	750
m)	12(1) + 30(1/60)	12½
n)		a number of your choice

Note the gap in the Babylonian numeral in part i). This gap signifies the absence any groups of 60. A similar gap is needed to report the Babylonian numeral for 3627 in part k).

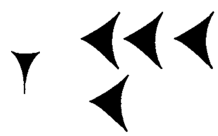
Notice, too, the answers for parts l) and m) look the same. Historical evidence suggests that Babylonians used "sexigesimal" (i.e., base 60) expansions for fractions in the same way we use decimals. However, their numeration system did not have the analogue of a decimal point, so there was no way (apart from context) to identify the ones column!

Upon completing the discussion of Blackline 208, ask the children to reflect about the Babylonian system. What features of this system do they like? What parts do they find difficult? How is this system like our numeration system? How is it different?

Activity 3

Here are some ways to continue the lesson.

Ask the children to consider a particular Babylonian numeral, but do not specify the units place. What possible base ten numbers could be represented by this numeral? Here is an example.



could mean
$$\begin{cases} 1(60) + 4(10) = 100 \\ 1(3600) + 4(60) + 0(1) = 3840 \\ 1(1) + 4(\tfrac{1}{60}) = 1\tfrac{4}{60} = 1\tfrac{1}{15} \\ \text{etc.} \end{cases}$$

Have the children conduct a library project focusing on the history of Babylon, or on other ancient numeration systems (Mayan, Chinese, African, Native-American, etc.) The references listed in Appendix C are possible sources of information about these topics.

Teacher Tips

You may find it helpful to link this lesson with previous place value experiences. Note the pattern of place values in the following chart.

	strip-mat	mat	strip	unit				
...	$5^3 =$ 125	$5^2 =$ 25	5	1	$\frac{1}{5}$	$\frac{1}{25}$	...	Base five
...	$4^3 =$ 64	$4^2 =$ 16	4	1	$\frac{1}{4}$	$\frac{1}{16}$	...	Base four
...	$10^3 =$ 1000	$10^2 =$ 100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	...	Base ten
...	$60^3 =$ 216,000	$60^2 =$ 3600	60	1	$\frac{1}{60}$	$\frac{1}{3600}$	...	Base sixty

One can see from the chart that base 60 is rather like base 10 in many ways. Our system, however, uses 0 and a decimal point to avoid the ambiguities of the Babylonian system.

To summarize, the distinguishing features of the Babylonian numeration system are:

- There were two symbols, 𒁹 and 𒌋, having face values of 1 and 10, respectively.
- Positional notation and a base of 60 were used.
- The first three (whole number) place values were 1, 60, and 3600 or 60 x 60).
- Sometimes a gap was used to signify the absence of a particular place value.
- There was no symbol for 0 and nothing (analogous to a decimal point) to indicate the units' place. The place values of the symbols in a numeral had to be determined from the surrounding context.

There have been various explanations offered as to why the Babylonians chose a base of 60. One is that 60 has a larger number of factors than 10, thereby making it possible to expand useful fractions such as ½ and ⅓ more conveniently. Another explanation is that astronomical and calendar considerations prompted the choice of base. We still see some of these considerations in our use of hours, minutes and seconds.

Journal Entry

Write the number 3247 as a Babylonian numeral. Describe your understanding of the Babylonian system. In your opinion, what are advantages does this system have? What are its disadvantages? Why do you feel the Babylonian system is no longer in use?

Published Piece

(optional) Your children might enjoy creating a numeration system of their own!

Reference

The History of Mathematics—An Introduction by David M. Burton Wm. C. Brown Publisher, Dubuque, IA 1991).

47 *Operations on Numbers, Part II*

You Will Need

- materials available for performing calculations, solving problems and displaying results
- for Activity 2, each child will need a copy of Blackline 209 (Sums and Differences)
- for Activity 3, each child will need several small slips of paper and a copy of pages 1–3, Blackline 210 (Operations With Numbers)

Your Lesson

In Lesson 17 of this volume, your children performed various arithmetical operations on two numbers. For example, what is the result of adding (multiplying, subtracting, etc.) 75 and 34? Reversing the question in different ways can lead to some interesting problems and further insights about number and operations.

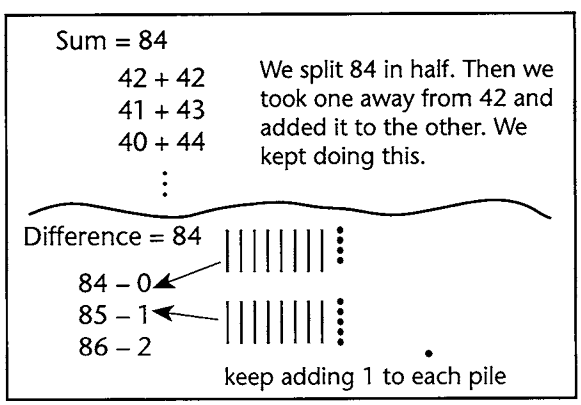
Activity 1

Here is one way to reverse the question: I'm going to add (subtract, multiply, divide) two numbers and tell you the result. The sum (product, difference, quotient) of two numbers is 84. What are the two numbers?

Pose such a question to your class and let the investigation begin:

- What strategies do the children use to determine an answer to the question?
- How many answers are there? Are the answers related to one another?
- What mathematical observations or conclusions do the children have about their answers?
- What happens if the 84 is replaced by 123 (or some other number).
- What happens if...? (Invite the children to make suggestions.)

You may make this an individual or group investigation. Children can explain their thinking and strategies in small groups or with the whole class. Results can be presented in books, at the overhead or on chart paper.



Conduct Activities 2 and 3 in the same way.

Activity 2

This activity is an example of a "puzzle problem" similar to those found in *Math and the Mind's Eye, Unit 1, Activity 4,* and in *Visual Mathematics*. Each child will need a copy of Blackline 209 (Sums and Differences).

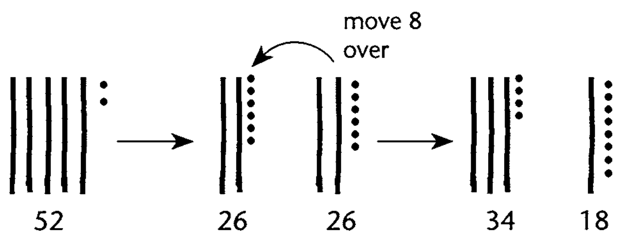
Display a transparency of Blackline 209 and ask the children to investigate the problem in small groups.

Opening Eyes to Mathematics | Sums and Differences | Blackline 209

When two numbers are added, the result is 52. When the same two numbers are subtracted, the difference is 16.

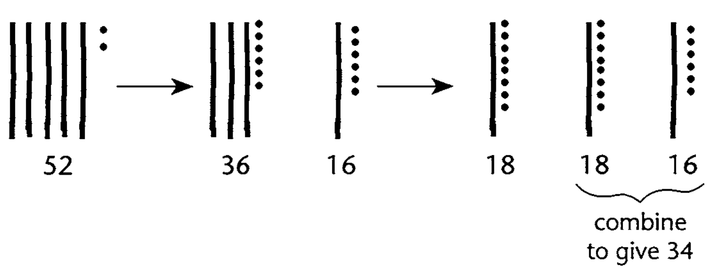
Develop some strategies for determining the two numbers.

GROUP 1 *We started with 52 and split it into two piles. Then we tried to fix the piles until we got 16 more in one than the other. Every time we moved one unit over, one pile was two larger than the other. We saw we could just move 8 over.*



GROUP 2 *32 + 20 = 52. That was easy, but the difference was 12. We knew we were close. So we tried 33 + 19—difference of 14. Then 34 and 18 worked.*

GROUP 3 *We put out 52 and removed 16. That leaves 36. Split the 36 in half and give the 16 to one of the halves.*



Discuss the strategies and have the groups continue to investigate. What happens if other numbers are used in place of the 52 and 16? What happens if ...?

Groups can also create puzzles of this sort for others in the class or at home to try.

Activity 3

Give several small slips of paper and a copy of page 1, Blackline 210, to each child.

Opening Eyes to Mathematics **Operations With Numbers** page 1 **Blackline 210**

Problem 1. Write these numbers on the front and back sides of two slips of paper. You can show four sets of numbers by placing the papers on the table in different ways. Write the sum of each set of numbers in the space provided.

	1st paper	2nd paper	
Front	4	7	
			Sums:
Back	12	20	

Do the same in Problems 2 and 3.

Problem 2.	1st paper	2nd paper	
Front	8	5	
			Sums:
Back	15	12	

Problem 3.	1st paper	2nd paper	
Front	14	25	
			Sums:
Back	68	37	

Problem 4. Here are the sums. What are the numbers?

	1st paper	2nd paper	
Front			
			Sums: 9, 12, 11, 14
Back			

Write Problem 1 of this blackline at the overhead. Ask each child to write the numbers indicated on two slips of paper.

	1st paper	2nd paper	
Front	4	7	
			Sums:
Back	12	20	

Now have the children place their papers so the two front numbers are showing. Enter the sum of these numbers, 11, on the overhead (children do the same on their blacklines).

	1st paper	2nd paper	
Front	4	7	
			Sums: 11
Back	12	20	

Ask the children to flip the second paper to show the 20 on the back. Write the sum of the exposed numbers, 24, on the overhead (children do the same).

	1st paper	2nd paper	
Front	4	7	
			Sums: 11, 24
Back	12	20	

Two more combinations can be exposed by flipping the papers and their sums can be recorded.

	1st paper	2nd paper	
Front	4	7	Sums: 11, 24, 19, 32
Back	12	20	

Ask the children to find the four possible sums in Problems 2 and 3. Discuss. (The possible sums for Problem 2 are 13, 20, 20 and 27. For Problem 3, the sums are 39, 51, 93 and 105.)

Continue by having the children work on Problem 4. Discuss in a large group. There are several solutions, two of which are presented here.

Solution 1:

	1st paper	2nd paper	
Front	2	7	Sums: 9, 12, 11, 14
Back	4	10	

Solution 2:

	1st paper	2nd paper	
Front	5	4	Sums: 9, 12, 11, 14
Back	7	7	

Now the investigation can really begin! How many solutions are there? Is it possible to find a solution where one of the front numbers is 8? where one of the front numbers is 3½? where one of the front numbers is 20? (You can phrase these questions to generate the specific computational practice or to work with different kinds of numbers.)

- How are the solutions related to one another? What mathematical observations or conclusions can one make?
- Try the problems on pages 2 and 3 of Blackline 210. What strategies are there?

Opening Eyes to Mathematics | **Operations With Numbers** page 2 | **Blackline 210**

Problem 5. Here are some more sums. What are numbers for each?

	1st paper	2nd paper	
Front			
			Sums: 22, 32, 20, 30
Back			

	1st paper	2nd paper	
Front			
			Sums: 56, 63, 51, 58
Back			

	1st paper	2nd paper	
Front			
			Sums: 140, 196, 183, 239
Back			

Problem 6. Make up some puzzles for your friends or family.

	1st paper	2nd paper	
Front			
			Sums:
Back			

	1st paper	2nd paper	
Front			
			Sums:
Back			

Can you make one a puzzle that uses three slips of paper?

Opening Eyes to Mathematics | **Operations With Numbers** page 3 | **Blackline 210**

Here are some more puzzles, only this time the numbers on the papers are multiplied. The first one is done for you.

1.	1st paper	2nd paper	
Front	3	4	
			Products: 12, 36, 36, 48
Back	9	12	

2. What are the products?

	1st paper	2nd paper	
Front	12	5	
			Products:
Back	6	15	

3. What are the numbers? How many answers can you find?

	1st paper	2nd paper	
Front			
			Products: 60, 180, 30, 90
Back			

4. Find a solution to Problem 3, where one of the front numbers is $^1/_2$.

	1st paper	2nd paper	
Front			
			Products: 60, 180, 30, 90
Back			

5. Try this one. What are the numbers?

	1st paper	2nd paper	
Front			
			Products: 112, 336, 224, 672
Back			

6. Make up a puzzle for a friend or family member.

Teacher Tips

"Reversing the question" is often a good strategy for problem posing. This strategy was also used in Lesson 7, Exploring Perimeter, of this volume. Generating What if? questions by varying the parameters of a problem (such as changing the numbers or operations in the above activities) is also a good problem-posing technique. Problem posing is an important part of problem solving. There are times when we, as teachers, will be the problem-posers. However, we should also provide plenty of opportunities for our children to play this role.

Of course, the problems we pose should be set at an appropriately challenging level. There are no general rules for doing this. Sometimes, a helpful guideline is to pose problems that involve math from the next one or two grade levels. Our own curiosity about mathematics, and that of the children, is also a good source of questions.

In any case, don't be reluctant to ask a question that seems hard (or that you may not have an answer for at the moment). After all, the nature of a problem is that it doesn't have a ready answer or solution. We encourage you to join the children in the search for solutions!

Activity 3 is an adaptation of one found in *Points of Departure—1*, a collection of open-ended activities published by the Association of Teachers of Mathematics, Kings Chambers, Queen Street, Derby, England DE1 3DA (see Starting Point 12, "Discs").

Homework

We're confident that family members or friends will enjoy the challenge of helping your children continue with these activities.

Assessment

A checklist or clipboard notes can be used to record observations related to your children's problem solving abilities and to their understanding of number operations. You might look for evidence of such things as: understanding the problem, strategies attempted, cooperation and communication with others, extending and drawing conclusions.

Perhaps, too, your children can compile their work on an investigation of interest into a published piece or an item for their portfolios.

Pattern Reminder

You may wish to have your children examine one of the remaining Pattern Generalizations on Blacklines 169–176.

48 Standard Measures, Recipes & Fractions

You Will Need

- Day 1: Blackline 211 (Ice Cream Recipe)
- Day 2: copies of Blackline 212 (Play Dough Recipe) and Blackline 213 (Finger Paint Recipe) for each child.

Your Lesson

Thanks to Jane Bailey (of *Honey Milk Ball* fame—Contact Lesson 136, Volume 2) and Mary Lou Horn, we have some wonderful recipes to share. Of course, we've accompanied them with problems designed to strengthen your children's measuring skills.

Day 1. Consider the recipe on Blackline 211 from Jane for the first day. It is titled, *Roll 'em, roll 'em, roll 'em—ICE CREAM!*

Opening Eyes to Mathematics | Ice Cream Recipe | Blackline 211

Roll 'em, Roll 'em, Roll 'em—ICE CREAM!
from Jane Bailey

2 cups of Half-and-Half
½ cup sugar
1 tsp vanilla

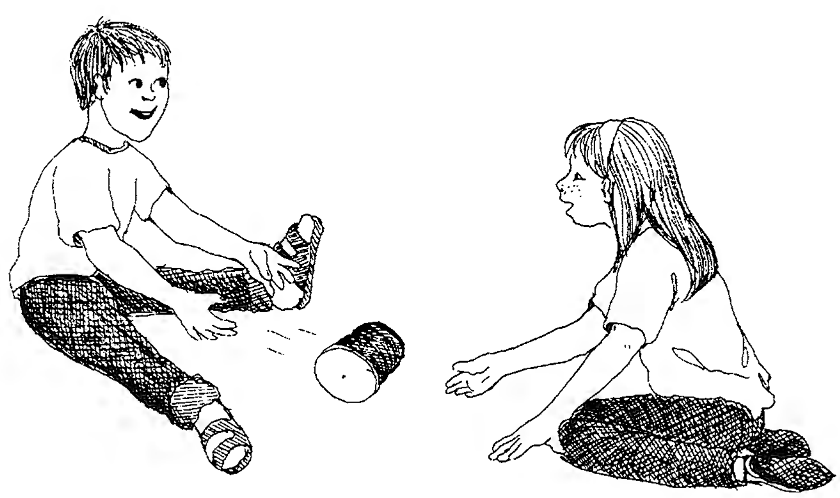
Put all 3 ingredients into a 1-pound coffee can with a plastic lid. Put this can inside a 3-pound coffee can. Layer around it with ice and rock salt. Secure a lid on the 3-pound can.

Roll the 3-pound can back and forth on the floor to a partner for 10 minues.

Remove the lid and drain off the ice water. Remove the inside can and its lid. The outer edges will be firm and the inside still soupy. Stir the ice cream with a rubber spatula and put the lid on again.

Place the smaller can inside the 3-pound can once more, layer it with ice and rock salt, and seal with its lid. Roll 5–10 minute more.

The ice cream is now ready to eat. This recipe serves 4 people.



Questions:

- Suppose you wish to double the recipe so it serves eight. How can this be done? Would it suffice to simply double the ingredients and use larger cans?
- Note how the can rolls. How far can you roll it before it bears to the left or right? What effect does condensation have on the roll? How is this effect explained?
- How far did your can roll altogether?

For added fun, have your children sit in a circle and roll the can among them.

Day 2. Ask the children to examine Mary Lou Horn's recipes and answer the questions on Blackline 212 and 213.

Opening Eyes to Mathematics | **Play Dough Recipe** | **Blackline 212**

PLAY DOUGH RECIPE
from Mary Lou Horn

This is a play dough recipe similar to the commercial type and more durable. Keep in a plastic bag or closed container when not being used.

1 cup flour
1 cup water
1 Tbl oil
1 Tbl alum
½ cup salt
2 Tbl vanilla
food coloring

Mix all dry ingredients. Add oil and water. Cook over medium heat, stirring constantly until it reaches the consistency of mashed potatoes. Remove from heat and add vanilla. Divide into balls and work in food coloring by kneading.

Question: How could you make playdough if you only had ½ Tbl alum? Rewrite the recipe to show correct proportions.

Opening Eyes to Mathematics | **Finger Paint Recipe** | **Blackline 213**

FINGER PAINTS
from Mary Lou Horn

1½ Tbl sugar
¼ cup cornstarch
1 cup cold water
food coloring

Mix the first two ingredients and then add the water. Cook over a low heat, stirring constantly until well blended. Divide the mixture into 4 or 5 portions and add a different food coloring and a pinch of detergent to each. The detergent facilitates cleaning up.

Question: Suppose you were mixing fingerpaint for a scout troop project. You need to triple the recipe above to have enough finger paint for your troop. Rewrite the recipe ingredients to reflect this change.

Daily Computation You might ask your children to examine other problems related to recipes.

49 Geoboard Shapes

You Will Need

- Chapter 11, Geometry
- a transparency of Blackline 214 (Geoboard Shapes)
- a transparent geoboard and rubber bands
- transparencies of geoboard and dot paper: Blackline 68 (Geoboard Paper), Blackline 110 (Geoboard Recording Paper) (adjoining boards); Blackline 141 (Square Dot Paper) and Blackline 154 (Isometric Dot Paper)

for each child

- a geoboard and rubber bands
- a copy of Blackline 214 (Geoboard Shapes)
- materials available: geoboard paper, square and isometric dot paper, tracing paper (or tissue paper), scissors, glue, chart paper, markers

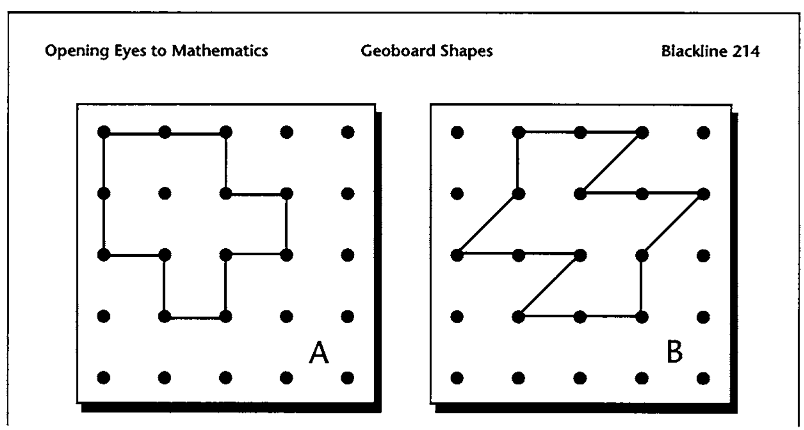
Your Lesson

This lesson consists of a geoboard exploration that is based on activities found in Contact Lessons 130–134, Volume 2. Explorations such as this can help children strengthen their problem-solving abilities and, at the same time, promote connections among number, shape, measurement and geometry.

The lesson also links naturally with Lesson 7, Exploring Perimeter, of this volume. Both lessons start out in similar ways, with the ensuing activities emerging from initial observations from the class about a geometric situation.

Activity 1

Divide the class into teams of two and distribute Blackline 214. Display a transparency of this blackline at the overhead and ask one member of each team to make shape A on their geoboard. The other team member makes shape B.

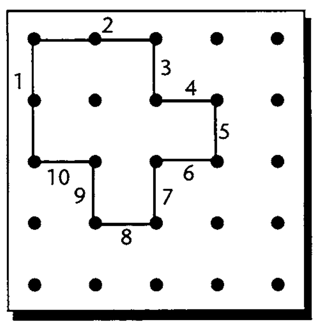


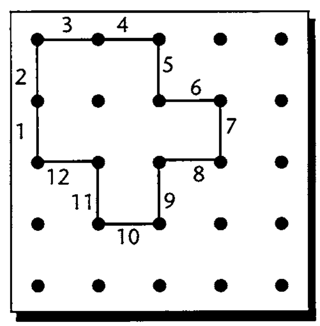
Ask each team to make a list of several similarities and differences between the two shapes. In the discussion that follows, you might have some teams share (and demonstrate) one likeness and one difference. Here are some things that are typically shared.

Likenesses	Differences
Same area	Different perimeter
Both have the same number of sides.	Shape B has acute angles; Shape A doesn't.
Both have some right angles.	A has a line of symmetry; B doesn't.
Both can be split into two congruent parts.	A touches 12 pegs; B touches 14.
Both have symmetry.	A has an inside peg.

The discussion offers an opportunity to review (or introduce) related geometric concepts and vocabulary.

- What is meant by an acute angle? What kind of angles does A have? Are all of B's angles acute? Find other examples of acute (right, obtuse) angles in the room.
- You say both shapes have the same number of sides. How many sides are there? Show us what you mean by "side". Does everyone agree? (Note: Some children may think about "side" in different ways. For example, some may say A has 10 sides; others may report 12 sides.)



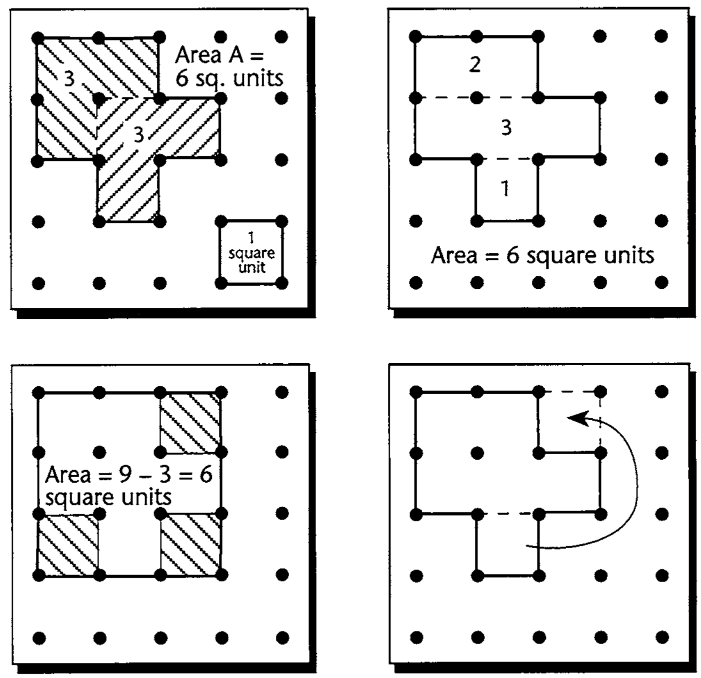


Lists will vary, of course, and so what happens next is somewhat open and up to you. (Be sure to add your own observations to the list.) Based on the above list, should an area problem come next? Or one on congruence? Or maybe one on symmetry? In any case, there is a decision to be made. (The following activities are merely samples.)

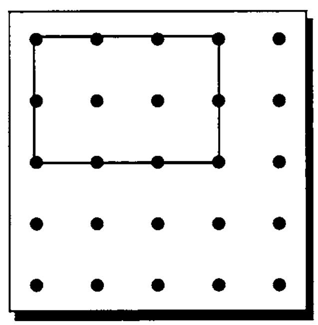
Activity 2

Suppose a team reports that both shapes have the same area. This observation can motivate problems such as these:

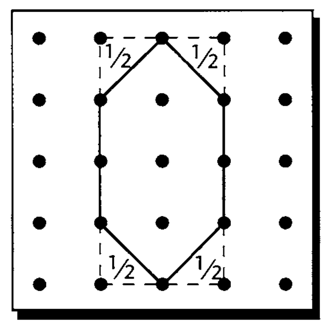
- What is the common area? Have the teams demonstrate different methods of determining this area.



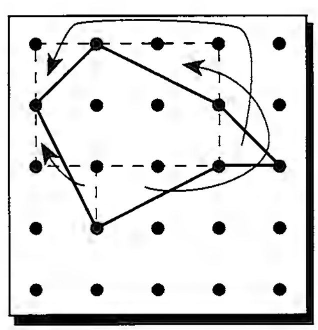
- Create other geoboard shapes that have an area of 6 square units (assuming the common area is reported as 6).



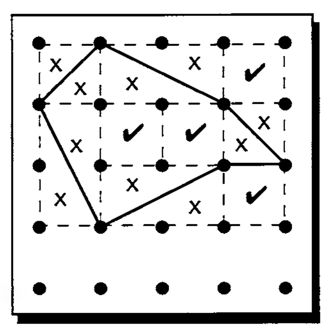
"I remembered 2 x 3 = 6."



"I made a 4 by 2 rectangle and took ½ away from each corner."

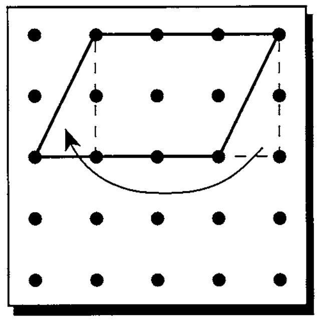


"I just tried things, but this has area 6 because you can fit the pieces into a 2 by 3 rectangle."

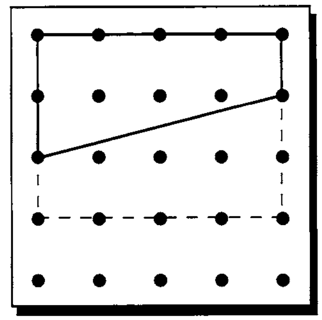


"The area is ½ of 12. The pieces inside match the ones outside."

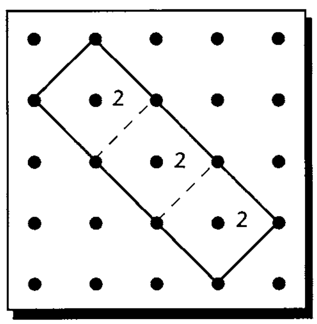
- Create some quadrilaterals that have an area of 6. (This can be repeated for triangles, parallelograms, etc., if you wish.)

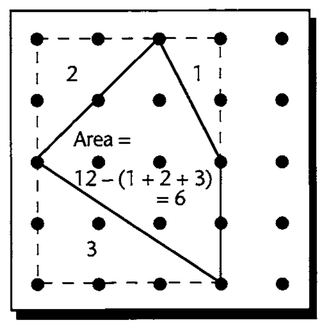


"Start with a 2 x 3 and move the triangle."



"Start with a 4 by 3 and cut it in half."





It is likely that much discussion will occur within and between teams. As this takes place, encourage the children to reflect about their work. How do they feel about their results? Does their work make sense? Can the results be explained to others? How can these results be shared?

Try to gain some insight into how the children proceed with this problem. As indicated in the illustrations, perhaps some will use trial-and-error. Others might draw upon related knowledge or "fix-up" a known shape. Still others might see some missing areas. Feel free to share your favorite methods as well.

We find it helpful to interrupt the working time on these problems for some sharing, particularly if some children are struggling unduly with them. Children often adopt some of the methods that are shared when they return to their own exploration.

What other questions would the children like to explore? Invite them to pose and investigate some related problems. Perhaps the teams can create a display or some other publication of their work.

Activity 3

(This activity may be conducted in the same way as Activity 2.)

Suppose one of the teams reports that shape A has one line of symmetry but no rotational symmetry, and vice-versa for shape B. Here are some problems that build upon this observation.

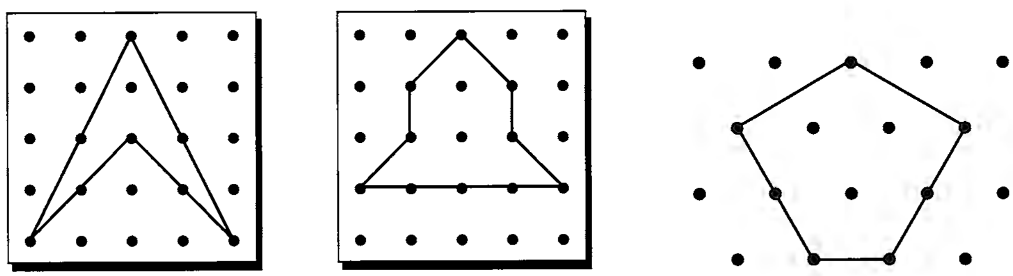
49 Geoboard Shapes (continued)

Distribute chart paper, markers and geoboard paper to each group of four children. Have the groups put these headings on their charts:

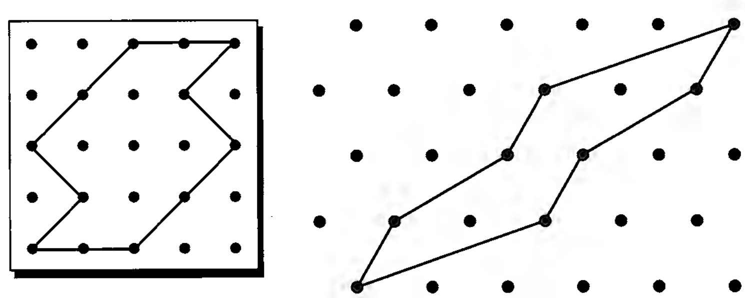
a) Reflective symmetry. No rotational symmetry.	b) Rotational symmetry. No reflective symmetry.	c) No symmetry.	d) Reflective and rotational symmetry.

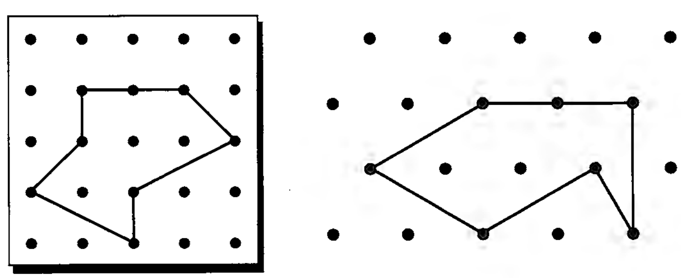
Now have the groups create examples of geoboard shapes which fall into each category and add them to their charts.

a) One line of symmetry. No rotational symmetry.

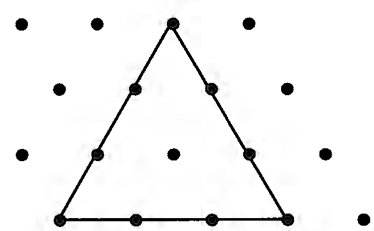


b) Rotational symmetry (order 2). No lines of symmetry.





c) No symmetry.

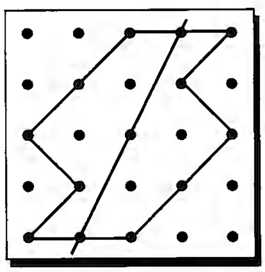


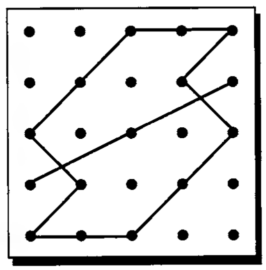
d) Rotational symmetry (order 3).
3 lines of symmetry.

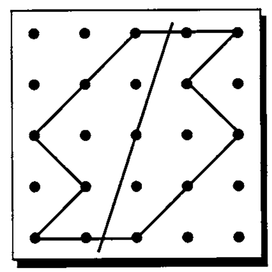
Have the groups post their charts and invite the class to examine the charts. Invite some of the groups to share the thinking they used to generate their examples. As in Activity 2, note the methods that are shared.

Ask the groups to take their charts back to their desks and investigate this question: Look at the shapes in each category. How many different ways can each shape be split into two congruent parts by a straight line? This question connects this lesson with Lesson 16, Congruence & Area, and Lesson 37, Reflective & Rotational Symmetry, in this volume.

Invite the children to share their work and make general observations about the problem. Children are often surprised by the number of different ways that shapes having rotational symmetry of order 2 can be divided into two congruent parts by a straight line. In fact, there is an infinite number of ways for these shapes and this can be discussed.







Care must be taken when thinking about the shapes that are not symmetrical. Some of these shapes can be split into two congruent parts by a straight line, while other can't be.

What other problems can the children pose about this situation?

Teacher Tips

As you can tell, this lesson is quite open in structure. We encourage you to remember that this openness is an integral part of problem solving and of a constructivist teaching philosophy. There may well be times when you or your children may feel uncomfortable. We have experienced that feeling on many occasions, too. When those times occur, take a deep breath, learn along with the children, and know it's okay if you don't have all the answers at your fingertips or if lessons don't come to an obvious point of closure. Be assured that you will be doing a great job by giving your children a chance to become better mathematical problem-solvers.

Homework/Journal

There will be various choices for you, depending on how the lesson goes.

Assessment

This is essentially a problem-solving lesson in geometry. Perhaps you can make some clipboard notes that describe your children's growth in problem solving and geometry.

Published displays or other products can also provide helpful information for assessment purposes.

You Will Need

- a copy of Blackline 215 (Discussion Card Area) for each child
- base ten area pieces and linear pieces available

Version 1

- a large, classroom set of Multiplication/Division Discussion Cards
- transparency of Blackline 183 (Base Ten Area Pieces)
- a large cube with faces labeled H, H, F, F, F and J, respectively. These labels correspond to selected base ten area pieces shown on Blackline 183. We call this cube the *unit's cube.*

Version 2 (for every two children)

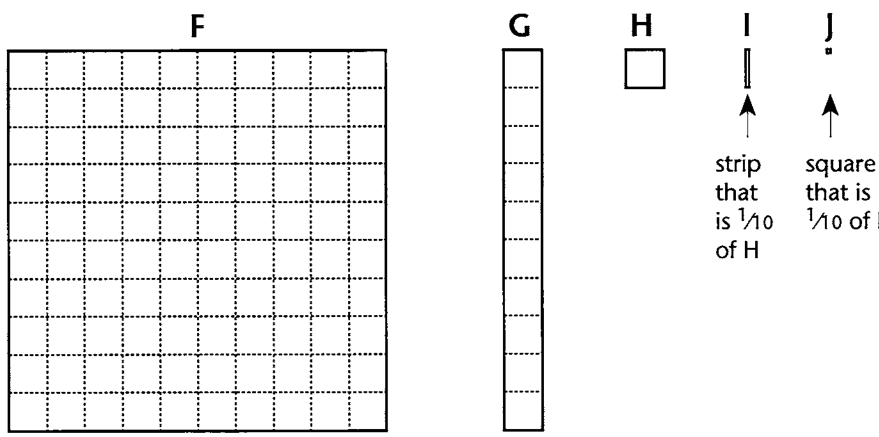
- a set of individual Multiplication/Division Discussion Cards
- a copy of Blackline 183 (Base Ten Area Pieces)
- a small, wooden cube with faces labeled H, H, F, F, F and J, respectively. These are called *small unit's cubes*.

Your Lesson

Discussion Card War was first introduced in Volume 2, Insight Lesson 82. This game is extended in this lesson to include the possibility of having different base ten pieces represent units of area.

Version 1 (large group)

Display a transparency of Blackline 183 and review the structure of the base ten area pieces. Ask the class to imagine piece J, the next smaller piece to the right of I. Invite a volunteers to describe piece J and sketch it on the transparency. Have the children also sketch J on Blackline 183.

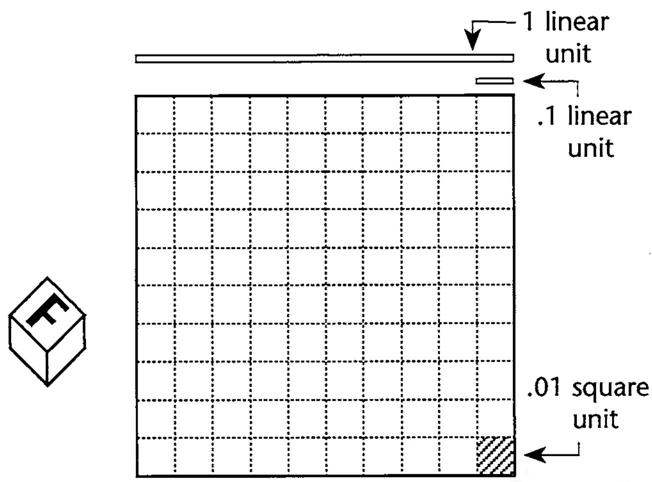


Divide the class into two teams and place the Discussion Cards face down in two piles.

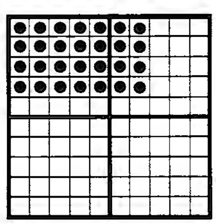
Before play begins, the children decide if the winner of each round is the team whose card shows the larger or smaller area. They also decide if the overall winner is the team with the most or least cards at the end of the game.

Toss the unit's cube. The letter that turns up determines which base ten piece is the unit of area for the round. Each team claims a stack of Discussion Cards and draws the top card in its pile. Team members discuss how to calculate the area of the array shown on the card. This area is to be reported in terms of the unit of area for the round.

When ready, each team shows its card to the other, naming the dimensions (factors) and area of its array. The winner of the round claims the cards being compared. (On any round where the selected cards have the same area, the teams draw 3 more cards each. They then compare the areas of the arrays on the third card, with the winner claiming all the cards. Pictured below is an example using piece F as the unit of area and the edge of F as a unit of length.

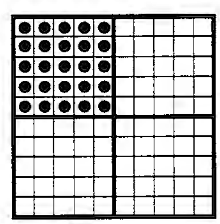


Team A's Card



Dimensions: .4 by .6 (linear units)
Area: .24 (square unit)

Team B's Card



Dimensions: .5 by .5 (linear units)
Area: .25 or ¼ (square unit)

Play continues until one team loses all its cards or until time is called.

Version 2 (for teams of two)

This version is played like Version 1, except two children play against each other. Opponents roll a small unit's cube to determine the unit of area for each round.

Version 3 (for teams of two)

This is played like Version 2, except no talking is allowed as children turn over their cards. On separate paper, each child writes a multiplication statement for the array pictured on their card. Children can then talk and can use their statements to compare the area of their arrays.

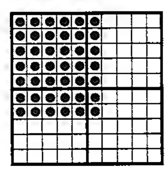
Teacher Tips

Discussion Card War is another game children enjoy playing at home. This is also a game your class will enjoy revisiting during the year. You can vary the letters on the cubes to include other base ten pieces. For example, label the faces of the cube H, H, F, F, D and J, respectively, where D is the next larger piece to the left of E on Blackline 183.

Following play of this game, you may wish to ask each child to attempt the problems on Blackline 215 (Discussion Card Area).

Opening Eyes to Mathematics | Discussion Card Area | Blackline 215

Examine this picture of a Discussion Card.



Problems:
1. Choose a unit of area and determine the dimensions and area of the shaded array. Explain your thinking.

2. Do the same as in Question 1 for other units of area. Identify the unit of area in each case.

Pattern Reminder

There may still be a Pattern Generalization (Blacklines 169–176) that hasn't been explored. If so, this might be a good time to examine it with your class.

51 Discussion Card Sorting & Probability

You Will Need

- Chapter 2, Sorting, and Chapter 9, Probability
- Volume 3, Lesson 6 (Who Would You Rather Be?)
- yarn necklaces, chart paper and materials for graphing

for every team of two children

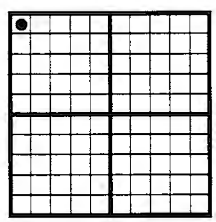
- a set of individual Multiplication/Division Discussion Cards
- 2 large loops of differently colored yarn
- Blackline 140 (Who Would You Rather Be?)

Your Lesson

Throughout this program, Multiplication/Division Discussion Cards have been used to visually portray many mathematical concepts and relationships, notably area, dimensions, factors, and counting patterns. The structure and visual nature of these cards also make them useful for sorting and probability activities.

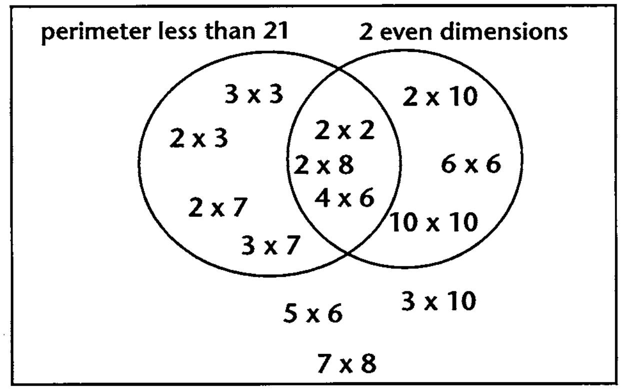
Activity 1

Give each team of two children two loops of yarn and an individual set of Multiplication/Division Discussion Cards. Tell them the smallest square (shaded below) on each card has area 1 square unit and perimeter 4 linear units.



Ask the teams to place all the cards showing shaded rectangular arrays with two even dimensions inside one of the loops. Then place all the cards showing shaded arrays with a perimeter less than 21 linear units inside the other loop.

Give the teams an opportunity to address the problem of where to place cards that belong in both loops. This problem can be solved by overlapping the loops, as if in a Venn Diagram.



This illustration shows some—not all—of the Discussion Cards for each region in the Venn Diagram. The cards are identified by the dimensions of the shaded arrays.

Some questions to discuss: How can the cards within each region of the loops be described? How about the cards that are outside the loops—how can they be described? How many cards show arrays with two even dimensions and perimeters less than 21? How many cards show arrays with an odd dimension?

The discussion can motivate questions related to probability: Suppose all the cards are placed in a sack and one is randomly drawn out. What are the chances that the array on the card will have a perimeter that is less than 21?

This question calls for an estimate of the chances a particular event ("the array will have a perimeter less than 21") will occur. As is true for any probability question, this estimate may be made on the basis of personal feelings, experimental results or theoretical considerations. How would your children respond at this point? Observing the distribution of the cards on their tables might prompt comments which are theoretical in nature. Some purely personal remarks might also emerge.

MAUREEN *I think there's a pretty good chance.*

TEACHER *Why do you feel that way?*

MAUREEN *Because 21 has always been my lucky number!*

LI *I'd say it's not quite even.*

TEACHER *What do you mean?*

LI *There aren't as many cards with perimeter less than 21 as those with bigger perimeters.*

ALICIA *Right. There are 30 cards with bigger perimeters, but only 25 with perimeters less than 20. It's almost even but not quite.*

LUIS *So the chances are not quite 50-50—a little less than 50%.*

TEACHER *Do you think the chances are as small as 25%?*

Twenty-five of the 55 Discussion Cards show arrays with perimeters less than 21. Theoretically, then, the chances of randomly drawing such a card are $^{25}/_{55}$ (or a bit more than 45%). It is not necessary to insist on this exact fraction, though you may wish to discuss it. It is appropriate to speak in terms of estimates as suggested in the dialogue, for these estimates can establish a sense of "reasonableness" about a situation. In this case, it would be reasonable to think the (theoretical) chances are close to 50%, but unreasonable to think they are higher than 50% or are very low.

Activity 2

Would the theoretical estimate discussed in Activity 1 be supported by experimental results? That is, if one were to repeatedly draw a card randomly from a full set of Discussion Cards, would one actually get shaded arrays with perimeters less than 21 about 45% of the time? Test this out by conducting a version of "Who Would You Rather Be?" (See Lesson 6 for materials and suggested procedures.)

Two children (or teams), A and B, play the following game:

A card is randomly drawn from a *complete* set of Discussion Cards and then *put back*. If the perimeter of the shaded array is less than 21, A gets 8 points. If the shaded array has a perimeter that is 21 or bigger, B gets 2 points.

If this game is played over and over, which child is likely to be ahead?

Activity 3

Here are examples of other probability questions that can be investigated during class or at home (possible answers are in parentheses). These questions may also emerge from sorting activities as illustrated above.

Suppose all the Discussion Cards are placed in a sack and one is randomly drawn out. What are the chances that the array on the card will have:

- a dimension that is a factor of 3? (A little less than 20%. Theoretically, the chances are $^{10}/_{55}$.)
- a perimeter less than 21 and two even dimensions? (Theoretically, the chances are small—only 7 cards out of 55 are in this category.)
- an even area? (Theoretically, quite high. There are 40 cards out of 55 in this category.)
- an odd perimeter? (For sure, zero!)
- an area of 48? (Not much chance.)

Teacher Tips

Estimates can be discussed in terms of language that is familiar to children (e.g., 50-50, 1 in 4, familiar percents like 50% or 25%). Of course, this is an opportunity to expand upon these terms.

- What do you mean by 1 in 4?
- What do you think is meant by 1 in 2? 2 in 5?
- What is an example of a event that has a 50-50 chance of occurring?
- How would you express 50% as a fraction?
- The chances (of getting a perimeter less than 21) are 25 out of 55. Try 25 divided by 55 on a calculator. Explain the decimal that results.

Journal Entry

Give an example of an event in your life that you feel has a 50% of occurring. Give a reason for your answer.

What's an example of an event in your life that you feel has less than 50% chance of occurring? Explain.

Assessment

How do your children make decisions in chance situations? In these activities, are their decisions influenced by experimental or theoretical information? Are the children aware of the three usual ways for estimating probabilities?

Appendix A *Area Models for Multiplication & Division*

The area model for multiplication is used in several *Opening Eyes* lessons. In this model, products are represented by areas of rectangles and factors by the corresponding dimensions of the rectangles. It is important, therefore, to distinguish the area of a rectangle from its dimensions.

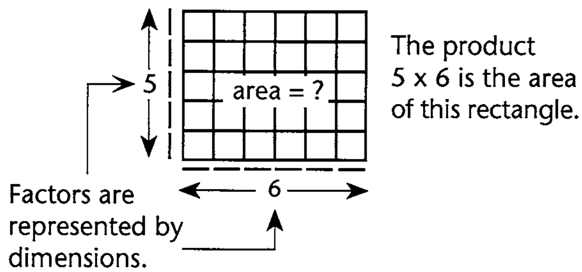


Diagram 1

One of the major advantages of the area model is its universality. In the world around us, people often use this model to estimate and measure carpet, wallpaper, window panes, materials for patios, etc. In mathematics, children encounter area models for both multiplication and division at several grade levels. Children soon learn that the same kind of pictures can be used to describe problems involving large numbers, fractions, or even algebra.

a)

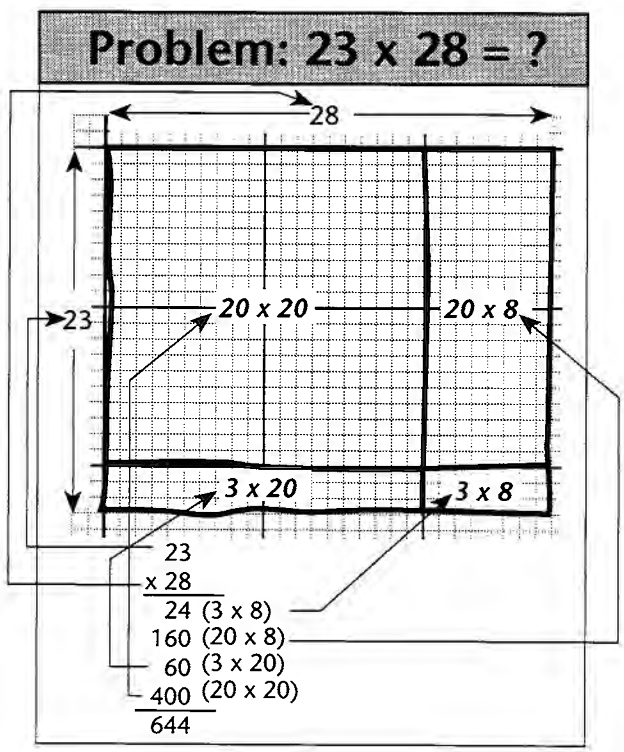
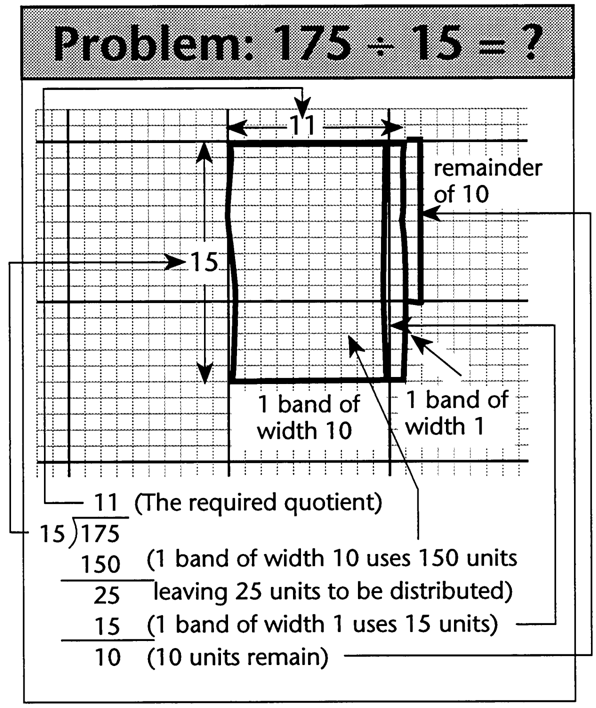


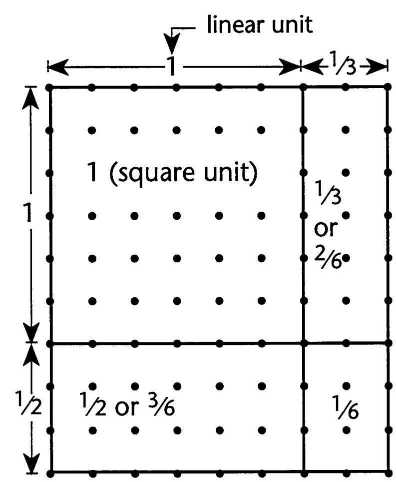
Diagram 2

(continued on next page)

b)

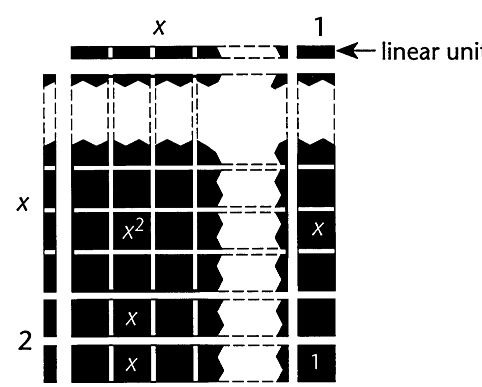


c) Problem: $1\frac{1}{3} \times 1\frac{1}{2} = ?$



$1\frac{1}{3} \times 1\frac{1}{2} = 1 + \frac{2}{6} + \frac{3}{6} + \frac{1}{6} = 1 + \frac{6}{6} = 2$

d) Problem: $(x + 2)(x + 1) = ?$

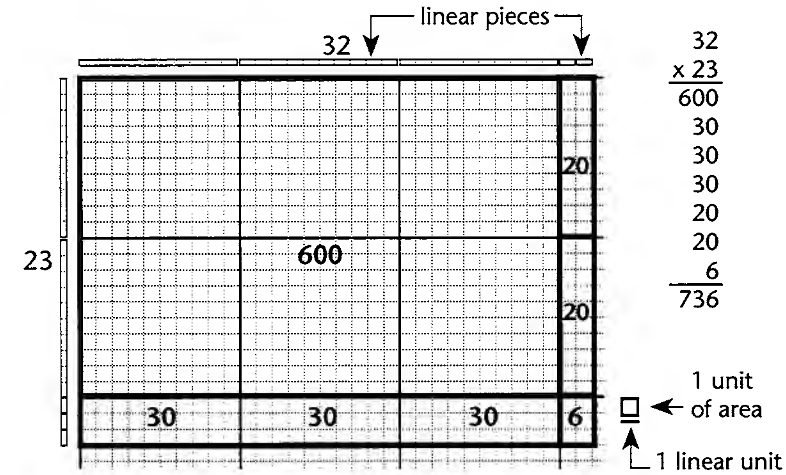


$(x + 2)(x + 1) = x^2 + 3x + 2$

Diagram 2 (continued)

In fact, a given array can model several products depending on the units of area and length chosen.

a)



b)

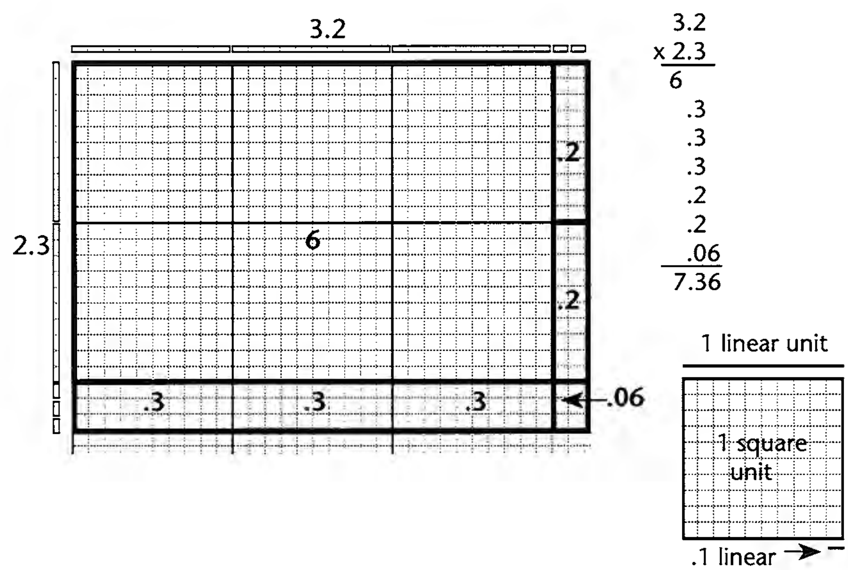


Diagram 3

Note how the dimensions of the rectangles in Diagrams 1 and 3 have been depicted. In previous *Opening Eyes* lessons, individual linear units were used to show dimensions of (reasonably sized) rectangles (Diagram 1). For other rectangles, such as those in Diagram 3, we use "base ten linear pieces" to depict dimensions. These pieces have lengths that correspond to the sides of each different base ten area piece. Note: Area pieces were called counting pieces in Volumes 1 and 2.

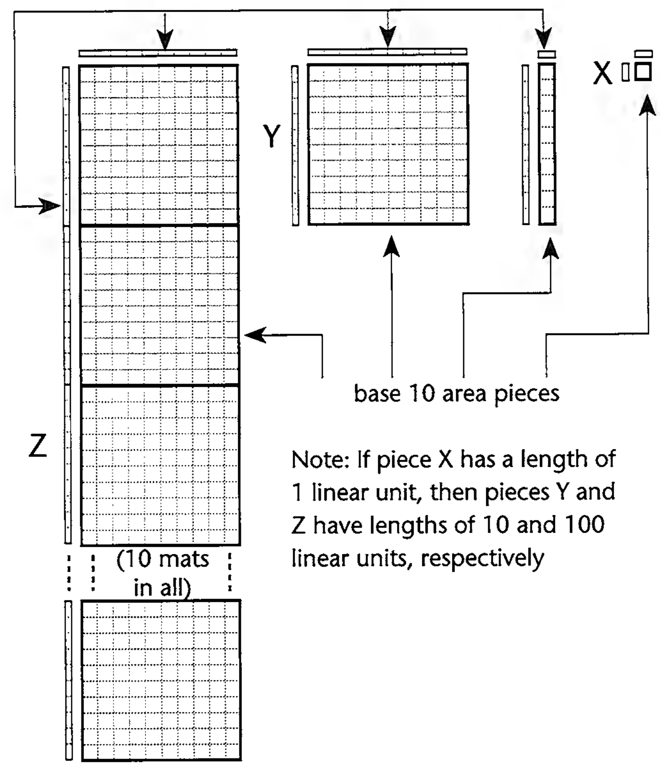
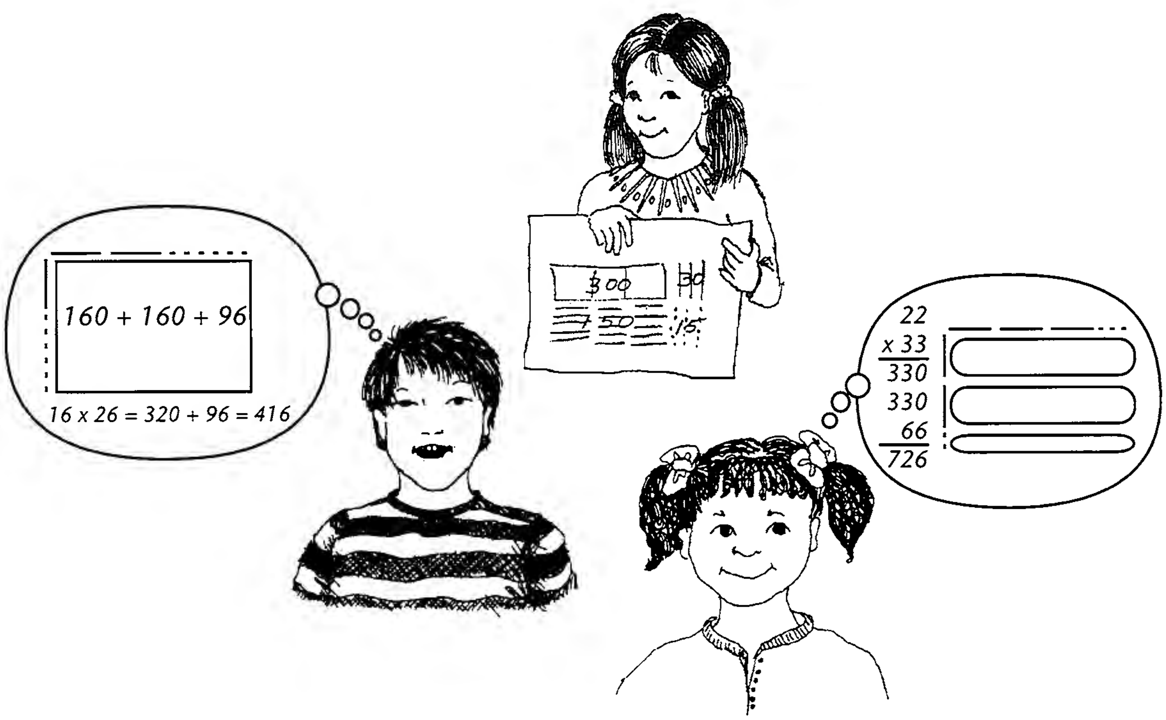


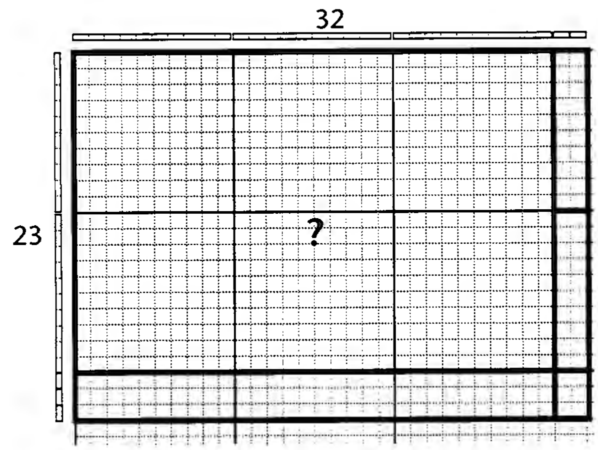
Diagram 4

Children can deepen their understanding of multiplication, and develop their ability to calculate products, by building, sketching or mentally visualizing rectangles. In fact, some children even imagine linear piece outlines of rectangles to help them.



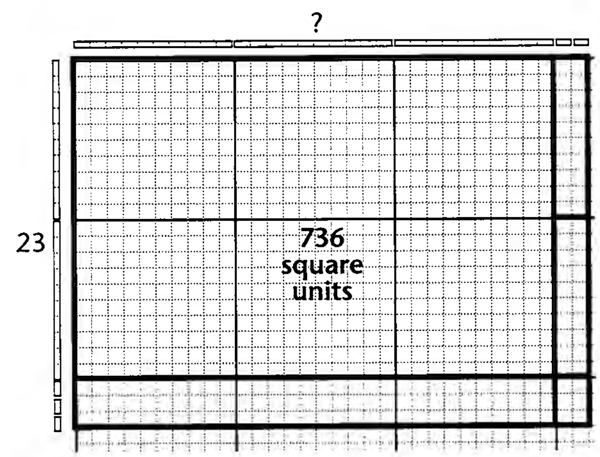
Because of the inverse relationship between multiplication and division, a given rectangle can model both operations. In fact, we generally discuss the two operations at the same time.

a)



32 x 23 is the total number of square units

b)



736 ÷ 23 is the missing dimension

Area models provide children with visual options for thinking about general multiplication and division concepts, and for calculating products and quotients. As discussed above, these are powerful options; Lessons 14 and 15 in this volume focus on these models. These lessons are based on activities found in *Math and the Mind's Eye* materials and are extended for older children in *Visual Mathematics*.

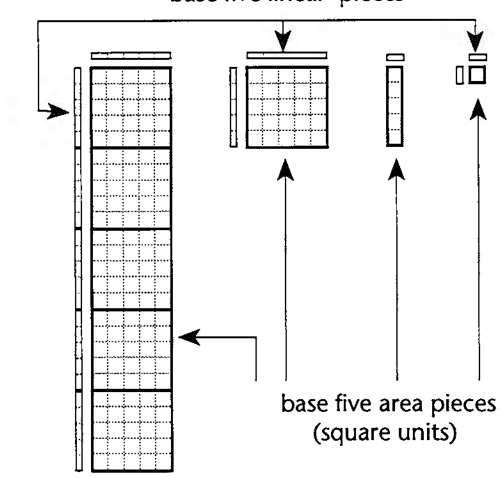
Please note the word "options" in the above paragraph. There are, of course, several models and strategies for calculating products and quotients. We recognize it may not be natural or practical to build (or even visualize) rectangles for every problem. In some cases, children may prefer to use repeated addition or subtraction models. At other times, they may find it appropriate to use mental arithmetic techniques or calculators.

We feel these different options should be acknowledged and discussed. We also encourage any attempt to apply general number or operation sense to a situation, and believe that any mathematically valid thinking about a problem should be accepted and shared.

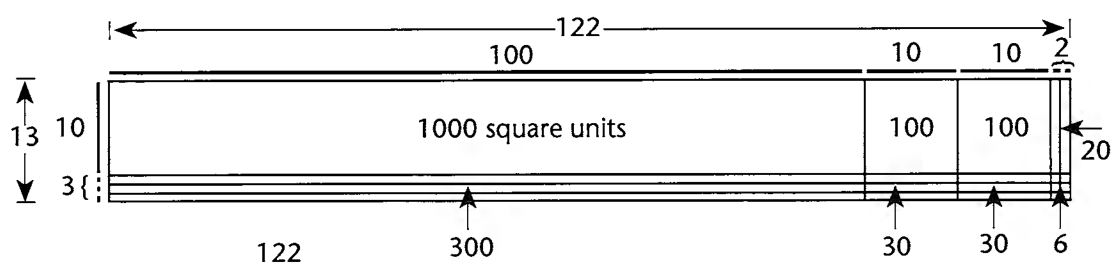
It is important that children have opportunities to construct their own solutions to particular problems. Consequently, while children gain specific experience with area models in Lessons 14 and 15, they are invited (and encouraged) to analyze multiplication and division situations in other ways elsewhere in the book (e.g., Lesson B, Daily Computation, and Lesson 17, Operations on Numbers, Part 1).

Note: Two types of base ten pieces in Diagrams 3 and 4. *Area pieces* are used to form rectangles; *linear pieces* are used to represent dimensions. Your children have been using area pieces and linear units throughout *Opening Eyes*. The additional linear pieces, however, are new to this volume and are introduced in Lesson 13.

Linear pieces can be created to correspond with other sets of area pieces.



We end this appendix by showing one more base ten multiplication.

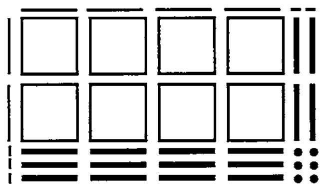



```
  122
 x 13
 1000
  200
   20
  300
   60
    6
 1586
```

Appendix B Operating on Numbers

Choose two numbers. What can be done with them? Well, one can operate on them in various ways or use them as a part of a problem:

$$\begin{array}{r} 42 \\ -\ 23 \\ \hline 19 \end{array}$$

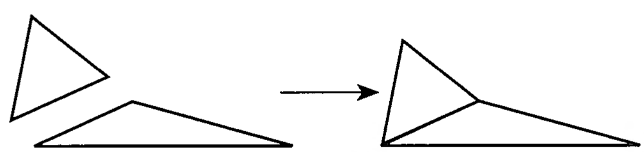


$$\begin{array}{r} 42 \\ \times\ 23 \\ \hline 800 \\ 40 \\ 120 \\ 6 \\ \hline 966 \end{array}$$

$$\begin{array}{r} 1 \text{ r } 19 \\ 23\overline{)42} \\ \underline{23} \\ 19 \end{array}$$

Sue has $42 and Jane has $23. How many more dollars does Sue have than Jane?

Operations (the act of "doing something" to two mathematical objects) are used throughout mathematics and comprise one of the big ideas of the subject. For example, we are familiar with the four fundamental operations of arithmetic. Two numbers are added, subtracted, multiplied or divided to generate another number. In geometry, two shapes can be "put together" to form another shape. The picture below depicts a geometric operation in which two triangles have been joined to form a quadrilateral.



In later grades, students learn to add and multiply two functions or two matrices.

Lesson 17, Operations on Numbers, Part I, of this volume focuses on arithmetical operations. In this lesson, children operate on numbers of their choice in different ways. The children also decide which operations to examine and which calculating options to use when performing those operations.

It's important to note two things about Lesson 17:

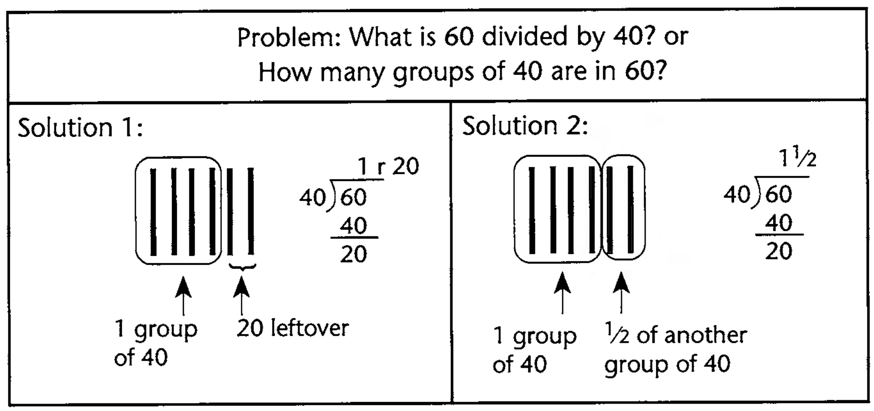
1. The activities of the lesson are intended to be very open-ended with lots of child-choice. Remember, there are no best methods or hard-and-fast rules for performing operations. Factors such as the following ordinarily influence the calculating option chosen by a child:

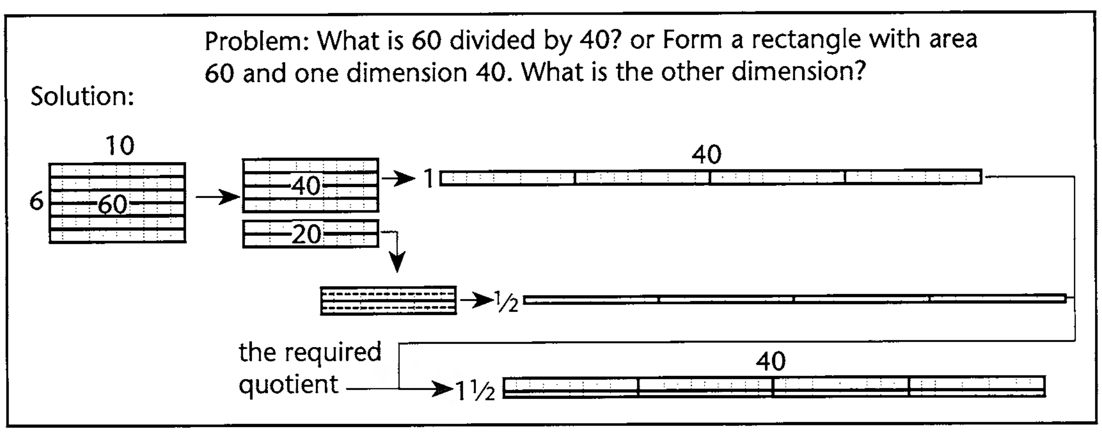
- The child's natural learning style and stage of development. Some may prefer to work with pictures; others may feel the need for pieces; other may work principally "in their head" or want to use a calculator.

- The complexity of the calculation itself. Finding the quotient of 234 divided by 17 should prompt a different strategy than finding the sum of 5 and 3.

2. In the lesson children are invited to evaluate their choices and explore various "What if's". This open exploration can generate some rich mathematical discussion and can easily take the lesson in an unforeseen (and potentially complex) direction! One scenario is presented in the body of the lesson. Two more twists are described in the following examples. Note the role of the calculator in each example.

Example 1. Suppose some children have chosen 40 and 60 for their numbers and proceed to divide. The children might begin by using base ten area pieces to help them divide 60 by 40. Several solutions are possible.

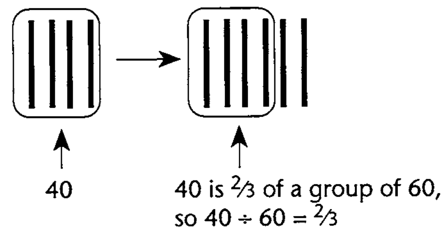




But now, what happens if the children

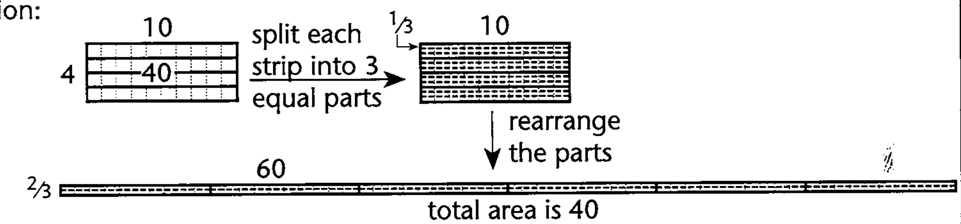
- do the division on a calculator? What explains the 1.5 that is displayed? What does this quotient represent?
- reverse the numbers and divide 40 by 60 with the help of base ten pieces?

Problem: What is 40 divided by 60? or How many groups of 60 are in 40?
Solution:



Problem: What is 40 divided by 60? or Form a rectangle with area 40 and a dimension of 60. What is the other dimension?

Solution:



The required dimension is ⅔. So 40 ÷ 60 = ⅔.

- divide 40 by 60 on a calculator? What explains the sequence of 6's following a decimal point? Does the calculator's answer seem the same as ⅔?

LUIS *Well, ⅔ is more than ½ and so is .6.*

MARLA *Think of money. Two-thirds of a dollar is between $.60 and $.70. Is the calculator's answer about that much, too?*

JIM *I'm not sure what all those 6's on the calculator mean. Do they keep on going?*

Note the role of estimation in the responses. See Teacher Tips for a way of visualizing ⅔ as .66666....

Example 2. Suppose some children have selected 5 and 0 for their numbers and begin by dividing 0 by 5.

IAN *Look, I divided 0 by 5 on my calculator—and the answer is 0.*

LESLEY *Well if you split 0 dollars evenly among 5 people, each gets 0 dollars!*

JANET *Think of an array. If the area is 0 and one dimension is 5, the other dimension has to be 0. That's weird!*

But, what if the children now divide 5 by 0? A calculator will report an error message when a division *by* 0 is entered. This message should be discussed, particularly since children are likely to be surprised by it.

JENNIFER *Let's see what happens if we put 5 in the calculator and then divide by 0. Look it says ERROR!*

ELI *I don't get it.*

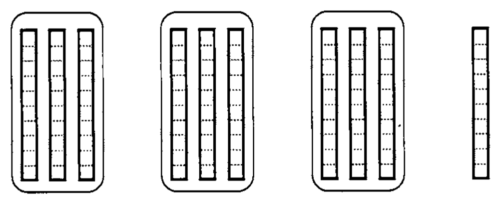
RENEE *Well, that's weird! Take 5 cookies, make groups of 0 cookies each. How many groups can you make? An awful lot!*

AMANDA *How can you have an array with 5 for an area and one dimension 0?*

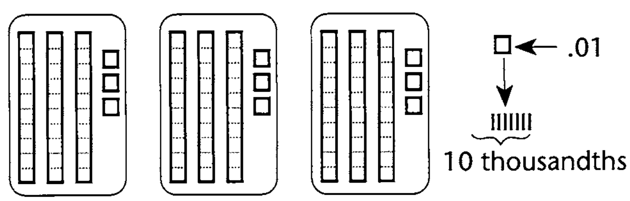
The children have raised some important issues in the above dialogue. In fact, It's because of questions like these that division by zero is left undefined in mathematics.

Teacher Tips: Here is a visual way of showing that ⅔ = .66666.....

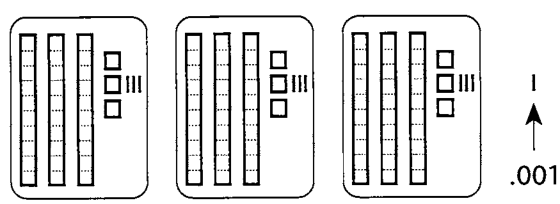
Divide a unit square into three equal piles of striplets (i.e., tenths). Each pile would get 3 tenths. There will be a tenth left over.



Divide the extra tenth into "squarelets" (i.e., hundredths) and give 3 hundredths to each pile. There will be a hundredth left over.



The extra hundredth can be divided into "tiny striplets" (i.e., thousandths), with 3 going to each pile and 1 thousandth left over.



Continue the process. Then ⅓ = .333... and ⅔ = .666...

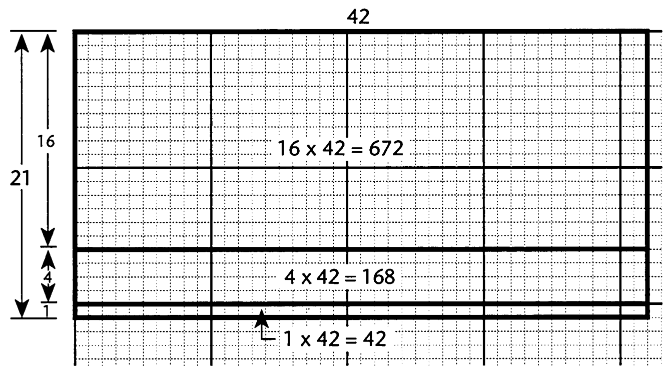
Appendix C *The Duplation Method of Multiplication*

The ancient Egyptians used the Duplation Method described in Lesson 24 of this volume to multiply whole numbers. This method of multiplication works because any whole number can be expressed as a sum of powers of 2. The Distributive Law can then be applied as shown in the following illustration.

Problem: What is 21 x 42

✔	1	42
	~~2~~	~~84~~
✔	4	168 or 4 x 42
	~~8~~	~~336~~
✔	16	672 or 16 x 42
	~~32~~	~~1344~~
		882

So 21 x 42 = 882



Because division is the inverse of multiplication, the Duplation Method may be used to divide. Here are examples showing how the Egyptians divided.

Example 1:

Problem: 156 ÷ 13

~~1~~	~~13~~	
~~2~~	~~26~~	
4	52	✔
8	104	✔
~~16~~	~~208~~	
	156	

156 ÷ 13 = 12, the desired quotient

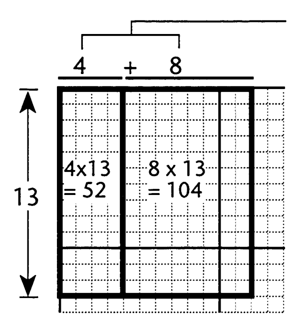
Example 2:

Problem: 129 ÷ 12

~~1~~	~~12~~	
2	24	✔
~~4~~	~~48~~	
8	96	✔
~~16~~	~~192~~	
½	6	✔
¼	3	✔
	129	

129 ÷ 12 = 10 + ½ + ¼, the desired quotient

The solution depicted in Example 1 above is analogous to constructing a rectangle with area 156 square units and one dimension 13. The required quotient is the other dimension of this rectangle.



the desired quotient

Note how the Egyptians handled the remainder in Example 2. The people did use a special symbol for ⅔; other than that, they only used unit fractions (i.e., fractions with a numerator of 1).

The Duplation Method is just one of several different methods for calculating products used in the past. These procedures were often related to people's understanding of number at the time and to the availability of materials (such as paper). Please see the references listed below for added information.

How do the algorithms of the past compare with those of today? What calculating options and procedures might be important in years to come, especially in light of changing technology? By thinking about such questions, one can argue that our "traditional" algorithms haven't always been standard, nor are they likely to remain so. They therefore need not be mandated. Perhaps, in fact, one of the procedures constructed by your children may "the" algorithm of the future!

Selected References:

1. The National Council of Teachers of Mathematics has several publications that contain background reading and classroom ideas related to the history of mathematics, three of which are listed here.

Chace, A. B. *The Rhind Mathematical Papyrus*. Reston, VA: NCTM, 1969.

National Council of Teachers of Mathematics. 31st Yearbook. *Historical Topics for the Mathematics Classroom*. Reston, VA: 1969.

Articles found in the *Arithmetic Teacher* (now called *Teaching Children Mathematics*) and the *Mathematics Teacher*. In particular, "Using Historical Materials in the Mathematics Classroom" by Abraham Arcavi specifically discusses the ancient Egyptian numeration system. *Arithmetic Teacher,* December 1987: 13–16.

2. A comprehensive discussion of past numeration systems and algorithms can be found in *Number Words And Number Symbols* by Karl Menninger. The M.I.T. Press, Cambridge, MA, 1970.

3. A description and bibliography of children's books related to mathematics can be found in *Read Any Good Math Lately? Children's Books for Mathematical Learning, K-6,* by David J. Whitin and Sandra Wilde. Heinemann, Portsmouth, NH, 1992.

4. The following articles discuss the use of numbers in Native-American and African societies, respectively.

Dixon, Roland B., and Kroeber, A. L. "Numeral Systems of the Languages of California." American Anthropologist, 9 (October-December, 1907).

Zaslavsky, Claudia. *Africa Counts*. Prindle, Weber & Schmidt, Inc. Boston, 1973.

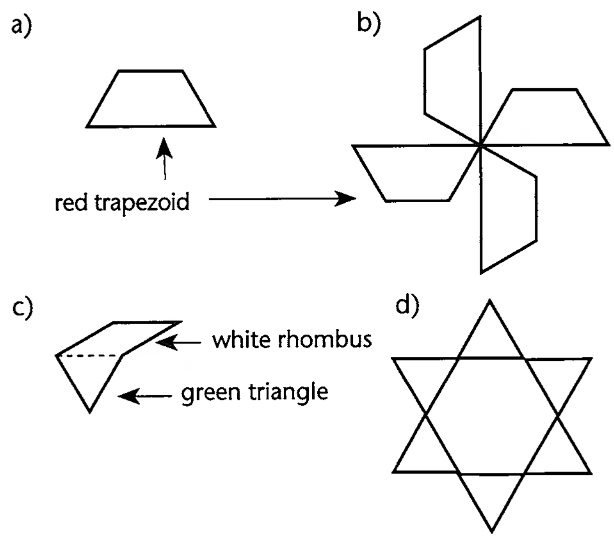
5. Burton, David M. *The History of Mathematics—An Introduction.* Wm. C. Brown Publisher, Dubuque, IA, 1991.

Appendix D Reflective & Rotational Symmetry

Lesson 37 of this volume contains several activities which can help your children construct some ideas about symmetry. These ideas are related to both reflective and rotational symmetry. What distinguishes these two types of symmetry? Given a shape, how can one test for its symmetry?

As in other lessons, these activities are introductory in nature and will be explored further in *Visual Mathematics*. Some of the basic ideas and classroom procedures are described more completely in *Math and the Mind's Eye* Unit X on symmetry.

First, here is some background information for you. Examine the following figures. How would you describe the symmetry, if any, of each figure?



One way to answer the question is to draw a "frame" around each figure, and see if the figure will fit in its frame in more than one way.

There is a line of symmetry in Figure (a) about which the figure can be flipped and still fit in its frame. The line of symmetry divides the figure into two congruent parts that are mirror images of one another. Because of this, Figure (a) is said to have reflective symmetry. Lines of symmetry have come up several times in *Opening Eyes* (e.g., Volume 1, Contact Lessons 26 and 27).

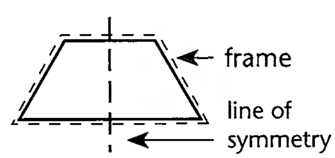
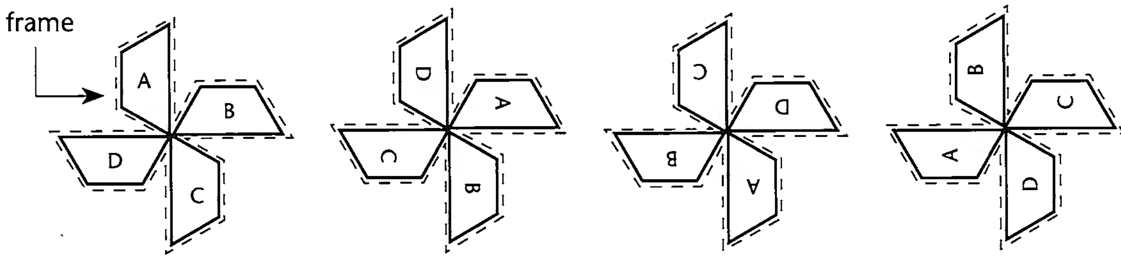


Figure (b) has rotational symmetry, but no reflective symmetry. It can be rotated about its center a quarter way around (90°) and still fit in its frame.



The figure can also be rotated half way around (180°), three-quarters way around (270°), and all the way around (360°—equivalent to not rotating it at all). Because of these 4 rotations, we say figure (b) has rotational symmetry of order 4.

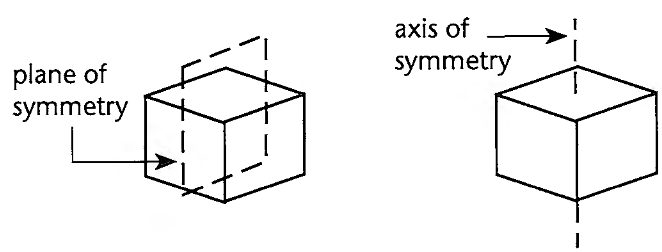
Figure (c) fits in its frame in just one way: "rotating" 0° (or 360°). It is standard to report that this figure has no symmetry. (Every figure can always be put back in its frame in this way.)

There are 6 ways to rotate Figure (d) back into its frame, so this figure has rotational symmetry of order 6. Figure (d) also has 6 lines of symmetry.

It is important to distinguish between reflective and rotational symmetry.

- A figure has *reflective symmetry* if it can be put back in its frame by flipping the (entire) figure about a line. The line is called a *line of symmetry.*
- A figure has *rotational symmetry* if it can be returned to its frame by rotating the (entire) figure about a point in more than one way. The point is called the *center of rotation.*

These ideas extend to three dimensions. An object in space has reflective symmetry if it can be divided by a plane into 2 congruent halves that are mirror images of one another. The plane is called a *plane of symmetry.*



The cube in the illustration also has rotational symmetry. Rotations of 90°, 180°, 270° and 360° about an *axis of symmetry* are some ways to return the cube to its "frame".

We suggest introducing the frame test in Activity 2 of Lesson 37. The following dialogue describes one way to do this.

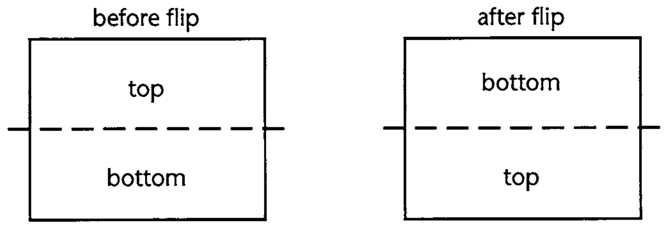
TEACHER *Here is one way to find out about the symmetry of a shape. (Holding a rectangular piece of paper against the chalkboard) I'm going to draw a "frame" around this paper. How many different ways can we fit the paper into its frame?*

JENNY *(demonstrating) Well, you can simply lift the paper up and put it back down.*

ALONZO *Look at this. You can flip it over and it will fit. (He demonstrates a flip over the horizontal line of symmetry.)*

TEACHER *Is that really a different way than Jenny's?*

ALONZO *Well, it looks the same when you're done. But really the top and bottom halves have been switched.*



TEACHER *How do the rest of you feel? Are the two ways different?*

ANDY *I think they're different. In Alonzo's way, we end up looking at the back side of the paper.*

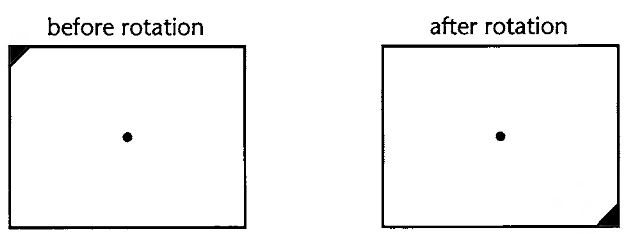
(The debate may continue, though, in this case, children are likely to feel the two ways are different. Even if they don't, the following ideas can be discussed.)

TEACHER *(referring to the horizontal line of symmetry) Alonzo has flipped the paper over this line of symmetry. Whenever a shape can fit into its frame by flipping over a line, we say the shape has reflective symmetry. The line of symmetry acts like a mirror, separating the paper into two halves that are mirror images.*

MARIA *That's like before. Remember when we showed ½ by folding a paper so the bottom half fit exactly on the top half?*

ALONZO *Only this time, the top and bottom parts have traded spots.*

NAOMI *I have another way that I think is different. You can turn the paper halfway around and put it back in the frame. (Naomi demonstrates a clockwise rotation of 180° about the center of the paper.)*



TEACHER *Why do you feel that's different, Naomi?*

NAOMI *Well, the paper hasn't been turned over like Alonzo's way. We still see the front side. But the corners have traded places.*

TEACHER *Naomi has demonstrated that the paper can still fit in its frame when it is rotated halfway around its center. (The teacher demonstrates by placing the paper in its frame, fixing the center with a pen, and rotating the paper 180°.) This shows the paper has rotational symmetry. This type of symmetry involves a different motion than line symmetry.*

JOE *Here's another example. Start with the paper in its frame like it was in the beginning. Now turn it like Naomi did, only go all the way around. See, it's back in the frame.*

JENNY *Hey, that's no different than mine!*

The children may suggest other ways to fit the rectangle in its frame. It is generally a good idea to discuss each suggestion, inviting the class to decide if a different way has been found. Perhaps someone will even suggest a flip about a vertical line of symmetry.

The dialogue contains some subtleties. Maria's fold creates a line of symmetry and demonstrates the congruence of the two regions created by that line. Alonzo's flip and Naomi's rotation involve movements of the *entire* paper (and not just one part moved onto the other) which return the paper to its frame. Moving the entire shape is an integral part of the frame test. The frame test, then, is a very general way to test for (and distinguish between) the two types of symmetry.

Materials Guide

The following general manipulative materials are needed to carry out the activities in Opening Eyes to Mathematics, Volume 3. It is possible that more than one teacher can share a classroom set. We have found that children will respect and carefully handle materials if we organize the materials and model their use. We have, therefore, included general suggestions for preparing and organizing and storing these manipulatives in the Materials Guide of the Teaching Reference Manual (pages 129–134).

✂ Materials marked with a ✂ can be prepared by you or your assistants. Some of these materials can also be purchased from The Math Learning Center (MLC) at the address given below.

$ Materials marked with a $ are available from MLC and will need little, if any, additional preparation on your part.

✳ Materials marked with ✳ can be stored in a central location and shared with a neighboring teacher.

Order Math Learning Center materials from

MLC Materials
PO Box 3226
Salem, OR 97302
(503) 370-8130
FAX: (503) 370-7961

Note: MLC Catalog symbols, preceded by a #, are included in this Guide.

Overhead Manipulatives

Each teacher will need access to an overhead projector and marking pens. If an overhead projector cannot be provided, enlarged replicas of overhead pieces can be used (see page 129 of the Teaching Reference Manual).

- $ Base Four Area Pieces, Overhead (#PG4O)—cut apart on heavy lines
- $ Tile, Overhead (#SQ75)—one set per teacher
- $ Pattern Blocks, Overhead (#PBO)—one set per teacher
- $ Large Base Ten Area Pieces, Overhead (#USMO)
- $ Geoboard, Clear (#GBC)
- $ Spinner, Overhead (#SPOH)

Student Manipulatives

- $ Large Base Ten Area Pieces (#USM)—one set per two children
- $ Large Base Ten Linear Pieces (#LEP)—one set per four children
- $ Linear Pieces, 2 cm Black (#LU)—one set per four children
- ✂$ Number Card Packet (#NCP)—(see pages 129 and 130 of the Teaching Reference Manual)
- ✂$ Discussion Cards, Individual (#IDIS)—(see page 130 of Teaching Reference Manual)
- ✳ Cubes, Wood ¾" (#CW75 or #CW755)—about 125 per four children
- $✳ Pattern Blocks (#PB or #PPB)—about 125 per team
- $✳ Geoboards (#GBP or #GBC)—one per child
- $ Calculator (#TI01 or #TI02)—one per child

✂$ Ten Strip Boards (#TSB)—one per child. Purchase these from MLC or create your own from Blackline 14.

✂ Group Fraction Bars—one set for every four children. See page 131 of the Teaching Reference Manual).

Large Group Manipulatives

✂$ Discussion Cards (#DIS)—one class set. Purchase from MLC and shade according to the directions. Laminate and store in resealable bags.

✂$ Large Number Cubes—one of each of those described below. Purchase these (Cubes, Foam—#CF300 or #CF400) or assemble five cubes by cutting the pattern shown on Blackline 3 from poster board, laminating and folding.

Using a permanent marker, label the faces on one cube with each of the following:

a. 0-1-2-3-1-2
b. 1–6, colored (or marked) red
c. 1–6, colored (or marked) blue
d. H, H, F, F, F, J

Keep the extra cube for occasional needs. This can be labeled with a dry, erasable marker when needed

✳ Scales—one for every four children. Have several kinds available.

✳ Measuring instruments such as rulers, yardsticks, meter sticks, etc.

$ (Optional) People Fraction Bars (see page 133 of Teaching Reference Manual)

General Materials

Chart paper

School Supplies: scissors, tracing paper or tissue paper, materials for graphing, blank transparencies, etc.

Blank overhead transparencies

Index

Index Key: TRM – Teaching Reference Manual; CL – Contact Lesson(s); IL – Insight Lesson(s); Les – Lesson(s)

Index (continued)