Unit IX / Math and the Mind's Eye

Picturing Algebra

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Picturing Algebra

1. **Toothpick Squares: An Introduction to Formulas**
Rows of squares are formed with toothpicks. The relationship between the number of squares in a row and the number of toothpicks needed to form them is investigated, leading to the introduction of algebraic notation and the use of formulas.

2. **Tile Patterns, Part I**
Tile patterns are used to generate equivalent expressions and formulate equations.

3. **Tile Patterns, Part II**
Algebraic expressions are represented as sequences of tile arrangements. Examining the properties of these arrangements leads to solving equations.

4. **Counting Piece Patterns, Part I**
The net values of arrangements in counting piece patterns are determined. Functional notation for net values is introduced.

5. **Counting Piece Patterns, Part II**
Counting piece patterns are used to introduce equations involving negative integers.

6. **Counting Piece Patterns, Part III**
Counting piece patterns are extended to include arrangements corresponding to non-positive, as well as positive, integers.

7. **Counting Piece Patterns, Part IV**
Counting piece patterns are used to introduce quadratic equations.

Math and the Mind's Eye materials are intended for use in grades 4-9. They are written so teachers can adapt them to fit student backgrounds and grade levels. A single activity can be extended over several days or used in part.

A catalog of Math and the Mind's Eye materials and teaching supplies is available from The Math Learning Center, PO Box 3226, Salem, OR 97302, 503-370-8130. Fax: 503-370-7961.

Math and the Mind's Eye

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Toothpick Squares: An Introduction to Formulas

O V E R V I E W

Rows of squares are formed with toothpicks. The relationship between the number of squares in a row and the number of toothpicks needed to form them is investigated, leading to the introduction of algebraic notation and the use of formulas.

Prerequisite Activity

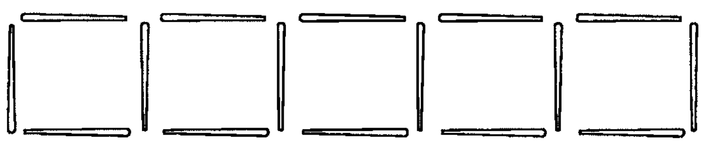
None, although experience in visual patterning (e.g., Unit 1, Activity 2, *Cube Patterns*, or Unit 1, Activity 3, *Pattern Block Trains and Perimeters*) is helpful.

Materials

Flat toothpicks.

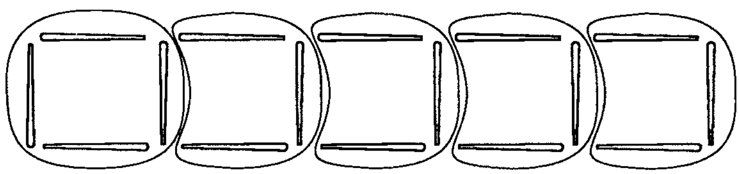
Actions

1. Distribute about 25 toothpicks to each student. Have the students form 5 squares in a row as shown here.



2. Ask the students if they can see ways, in addition to one-by-one counting, to determine the total number of toothpicks in the 5 squares. Discuss different ways of "seeing the total of 16."

(a)



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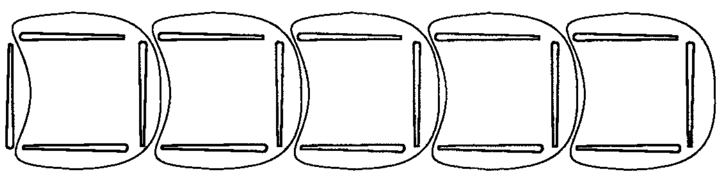

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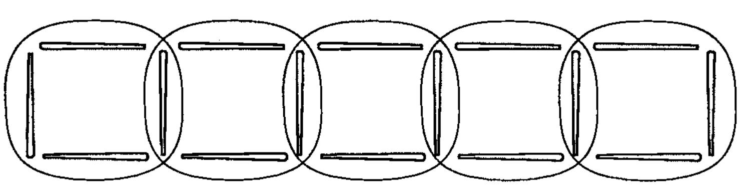

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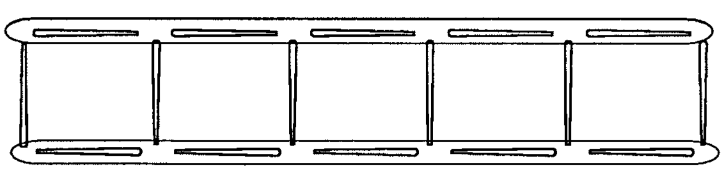


Comments

1. The pattern of squares can be drawn on the chalkboard or formed by placing toothpicks on an overhead projector.

2. Below are some ways of viewing the number of toothpicks. The students may find others. A master for a transparency which can be used to illustrate different methods of counting the toothpicks is attached (Master 1).

(a) One square of 4 toothpicks and 4 groups of 3: $4 + 4(3) = 16$.

(b) One toothpick at the left and 5 groups of 3: $1 + 5(3) = 16$.

(c) Five squares of 4 toothpicks with 4 toothpicks counted twice: $5(4) - 4 = 16$.

(d) Two rows of 5 toothpicks and 6 vertical toothpicks: $2(5) + 6 = 16$.

Actions

3. Ask the students to imagine extending the row of 5 squares to 12 squares and then predict the total number of toothpicks needed to build the 12 squares. Discuss the methods used to predict the total.

Comments

3. Twelve squares require 37 toothpicks. Here are ways of determining this, corresponding to the methods described in Action 2:

(a) $4 + 11(3) = 37$ (1 square of 4 toothpicks and 11 groups of 3),

(b) $1 + 12(3) = 37$ (1 toothpick on the left and 12 groups of 3),

(c) $12(4) - 11 = 37$ (12 squares of 4 toothpicks with 11 toothpicks counted twice),

(d) $2(12) + 13 = 37$ (2 rows of 12 toothpicks and 13 vertical toothpicks).

Actions

4. Have the students determine the number of toothpicks if the row of squares is extended to

(a) 20 squares,
(b) 43 squares,
(c) 100 squares.

Discuss.

Comments

4. In determining their answers, a student is likely to use one of the methods discussed in Action 3. You can ask them to verify their work by using one of the other methods suggested.

Actions

5. Tell the students to suppose you made a row of toothpick squares and to suppose you have told them how many squares are in your row. Working in groups of 3 or 4, have the students devise various ways to determine the number of toothpicks from this information.

Comments

5. Having students discuss with one another their ideas for determining the number of toothpicks may help them clarify their thoughts.

A student may suggest a method that works for a specific number of squares, say 45. If this happens, you can ask the student how their method would work no matter what the number of squares is.

Actions

6. For each method a group has devised, ask them to write verbal directions for using that method. Suggest they begin each set of directions with the phrase, "To determine the number of toothpicks, ...". Encourage the groups to review their written directions for clarity and correctness.

Comments

6. You may have to explain to the students that "verbal directions" means directions expressed in words, without using symbols.

Actions

7. Ask for a volunteer to read one set of directions from their group. Record them, as read, on the chalkboard or overhead. Discuss the directions with the students, revising as necessary, until agreement is reached that following them, as written, leads to a correct result. Repeat this action until directions for several different methods are displayed.

8. Have the students suggest symbols to stand for the phrases "the number of toothpicks" and "the number of squares". Discuss their suggestions.

Comments

7. A master for an overhead transparency that can be used in recording the directions is attached (Master 2).

If a set of directions is suspected to be incorrect, you can suggest to the students that they test the directions for specific instances. For example, if the number of squares is 20, following the directions for finding the number of toothpicks should result in 61 toothpicks, as determined in Action 4.

Possible directions corresponding to the methods described in Action 2 are:

(a) "To determine the number of toothpicks, multiply one less than the number of squares by three and add this amount to four."

(b) "To determine the number of toothpicks, add one to three times the number of squares."

(c) "To determine the number of toothpicks, multiply the number of squares by four and then decrease this amount by one less than the number of squares."

(d) "To determine the number of toothpicks, double the number of squares and then add to this amount one more than the number of squares."

8. While the choice of symbols is a matter of personal preference, it is helpful to choose symbols which are easily recorded, not readily confused with other symbols in use, and are suggestive of what they represent. For example, "the number of squares" might be represented by n (the first letter of the word "number"), or by S (the first letter of the word "square"). The latter choice may be preferable since it is not as likely to be taken to mean "the number of toothpicks".

Actions

9. From the suggestions made in Action 8, select symbols to represent the number of toothpicks and the number of squares. Have the students use these symbols and standard arithmetic symbols to write each set of directions in symbolic form. Point out to the students that a set of directions written in symbolic form is called an *algebraic formula.*

10. For each set of directions displayed in Action 7, ask for volunteers to show their formulas. Discuss.

11. Discuss symbols and their role in writing mathematics.

Comments

9. If the students work in groups, they can assist one another in writing appropriate formulas. If the issue is raised, you may want to suggest the use of "grouping" symbols, such as parentheses, to avoid ambiguities.

10. If the validity of a formula is in question, you can ask the students to test it to evaluate the number of toothpicks given a specified number of squares.

Following are formulas corresponding to the directions listed in Comment 7. In the formulas, T stands for the number of toothpicks and S stands for the number of squares.

(A symbol, such as S or T, that stands for a quantity that can have different values is called a *variable.*)

(a) $T = 4 + 3(S - 1)$,
(b) $T = 3S + 1$,
(c) $T = 4S - (S - 1)$,
(d) $T = 2S + (S + 1)$.

Some students may write "$3S - 1$" for "$3(S - 1)$" in formula (a). If this happens, you can comment on the need to distinguish between "subtracting 1 from 3 times the number of squares" and "subtracting 1 from the number of squares and then multiplying by 3". Parentheses are used to make this distinction.

Other ambiguities may arise. They can be discussed as they occur.

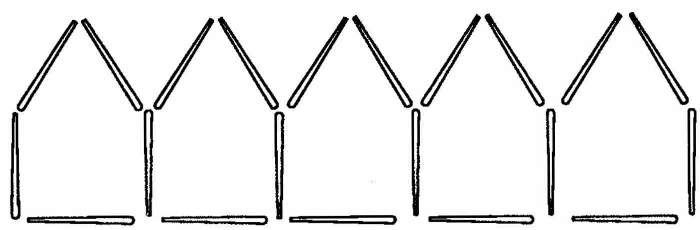
11. One way to begin the discussion is to ask the students what they perceive as advantages or disadvantages in using symbols rather than words.

The use of symbols enables one to write mathematical statements concisely and precisely. However, it can obscure meaning if the reader is unfamiliar with the symbols used or lacks practice in reading symbolic statements.

Actions

12. (Optional.) If the row of squares in Action 1 is extended until 142 toothpicks are used, how many squares will there in the row?

13. (Optional.) Form a row of pentagons with toothpicks as shown. Ask the students to write a formula relating the number of toothpicks used with the number of pentagons in the row.



Comments

12. There are 47 squares.

Some students may arrive at the answer by a "guess-and-check" method. Other students may use their knowledge of how squares are formed: "After 1 toothpick is placed, there are 141 left and it takes 3 more to form each square. So $141 \div 3$, or 47, squares are formed."

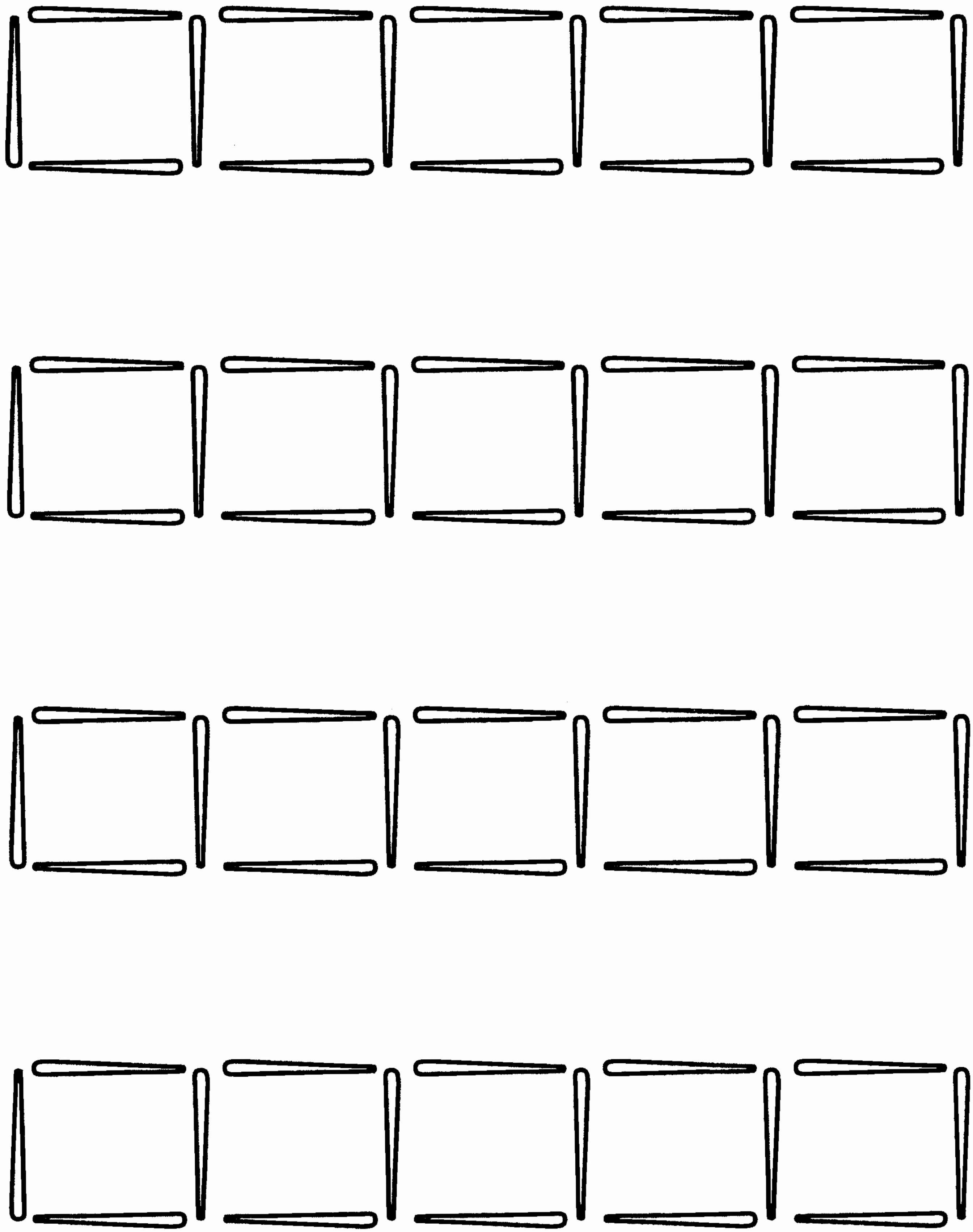
You may wish to point out to the students that an answer may also be arrived at by replacing T by 142 in the formula in Action 10 and determining what S must be to have equality. In arriving at an answer, they have determined the solution of the *equation*: $142 = 3S + 1$.

13. If T is the number of toothpicks used and P is the number of pentagons formed, then

$$T = 1 + 4P.$$

In giving a formula, it is necessary to give the meaning of symbols like T and P that do not have standard meanings.

This formula can be written in other forms. Also, students might choose symbols other than T and P to represent the number of toothpicks and the number of pentagons.



To determine the number of toothpicks,

To determine the number of toothpicks,

To determine the number of toothpicks,

To determine the number of toothpicks,

Tile Patterns, Part I

O V E R V I E W

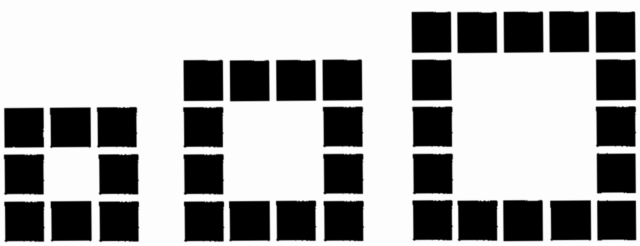
Tile patterns are used to generate equivalent expressions and formulate equations.

Prerequisite Activity
Unit IX, Activity 1, *Toothpick Squares.*

Materials
Tile or counting pieces.

Actions

1. Distribute tile to each student or group of students. Display the following sequence of tile arrangements on the overhead. Have the students form the next arrangement in the sequence.



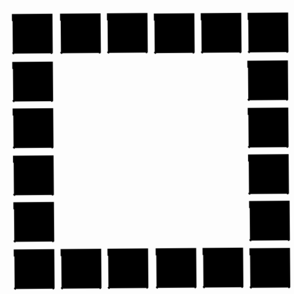
2. Ask the students to determine the number of tile in the 20th arrangement. Have a volunteer explain their method for arriving at an answer. Illustrate on the overhead.

Comments

1. Each student or group of students will need at least 30 tile. Counting pieces, introduced in Unit VI, Activity 1, *Counting Piece Collections*, can be used instead of tile.

Asking the students to form the next arrangement helps them focus on the structure of the arrangements.

Most students will form the fourth arrangement as shown below. If someone forms another arrangement, acknowledge it without judgement, indicating there are a number of ways in which a pattern can be extended. Tell the students you want them, for now, to consider the series of arrangements in which the pattern shown is the next arrangement.



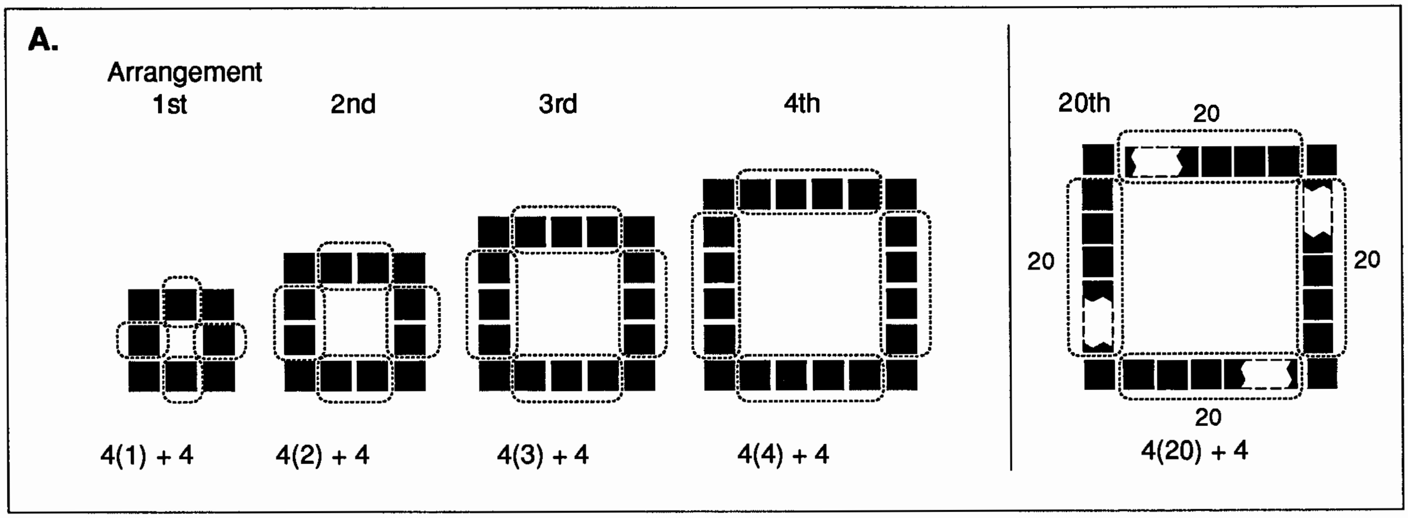
2. There are various ways to determine that 84 tile are required to build the 20th arrangement. One possible explanation: "On each side, the 20th arrangement will have 20 tile between corners. Since there are 4 sides, the number of tile required is 4 times 20 plus the 4 corner tiles."

Continued next page.

Actions

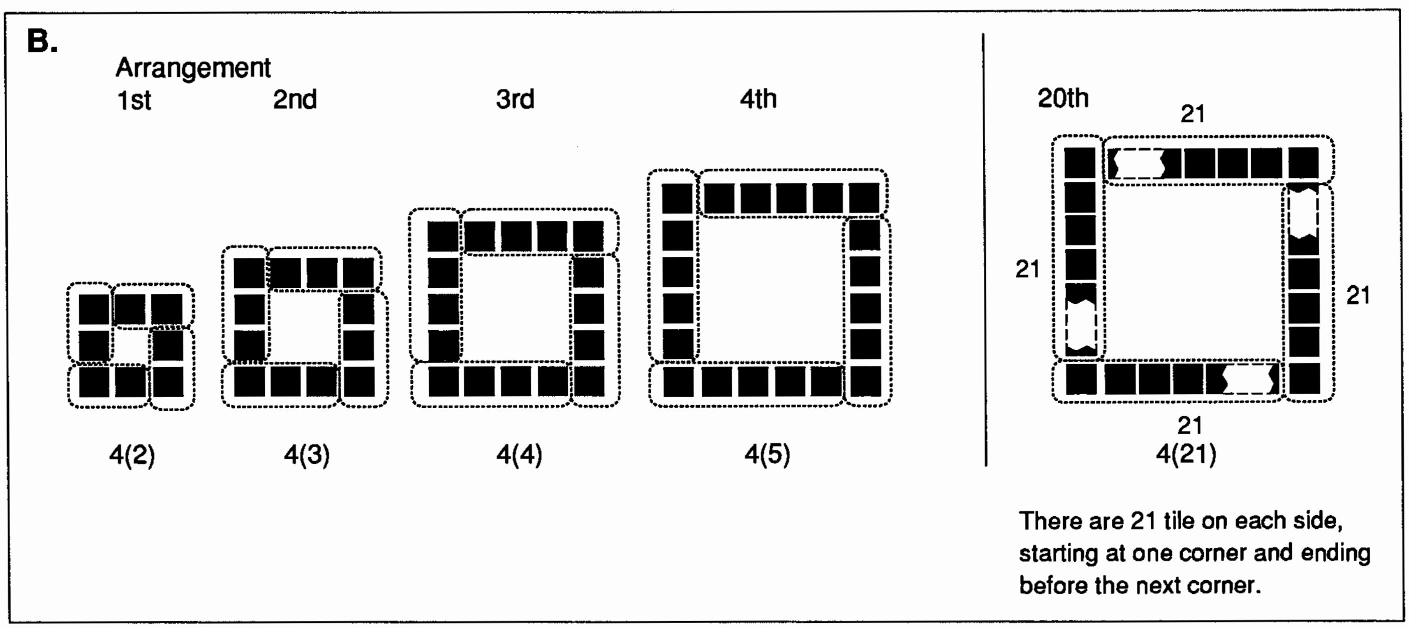
Comments

2. *Continued.* A master for a transparency on which to illustrate ways of viewing arrangements is attached (Master 1). The above method might be illustrated as follows. The illustration shows how the first 4 arrangements, and the 20th, are viewed.



3. Distribute paper copies of Transparency 1 to be used as recording sheets by the students. Ask them to record their method of viewing the number of tile in each arrangement. Have them work in groups to devise and record other methods. Ask for volunteers to present their methods.

3. Below are some other ways of viewing the 20th arrangement:



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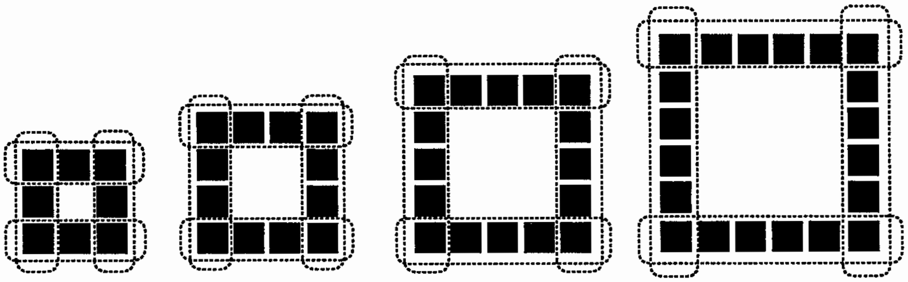
3. *Continued.*

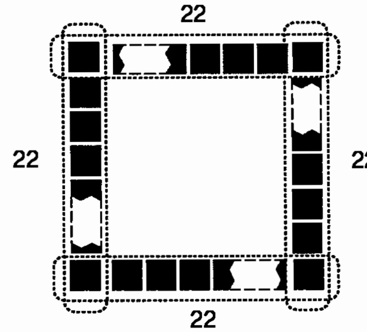
C. Arrangement

1st 2nd 3rd 4th



$4(3) - 4$ $4(4) - 4$ $4(5) - 4$ $4(6) - 4$

20th

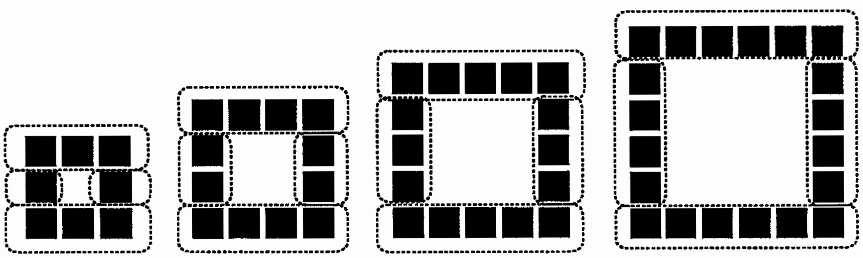


$4(22) - 4$

There are 22 tile on each side, counting each corner twice.

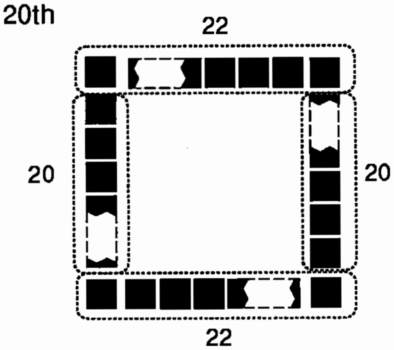
D. Arrangement

1st 2nd 3rd 4th



$2(3) + 2(1)$ $2(4) + 2(2)$ $2(5) + 2(3)$ $2(6) + 2(4)$

20th

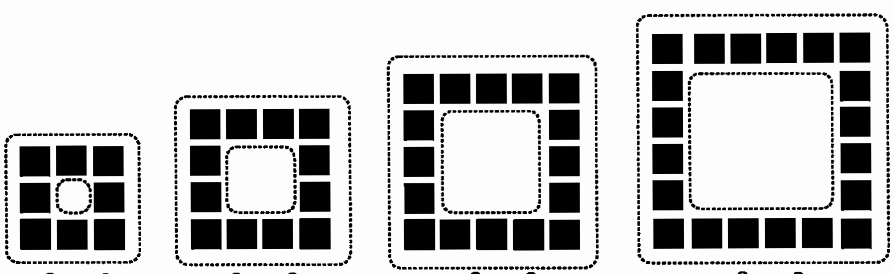


$2(22) + 2(20)$

There are 22 tile on the top and the bottom and 20 on each side between the top and the bottom.

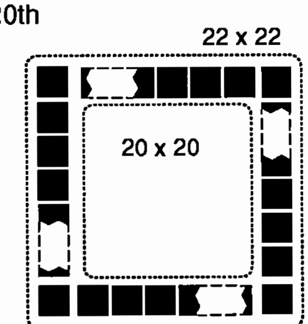
E. Arrangement

1st 2nd 3rd 4th



$3^2 - 1^2$ $4^2 - 2^2$ $5^2 - 3^2$ $6^2 - 4^2$

20th



$22^2 - 20^2$

There is a 22×22 square with a 20×20 square removed from inside it.

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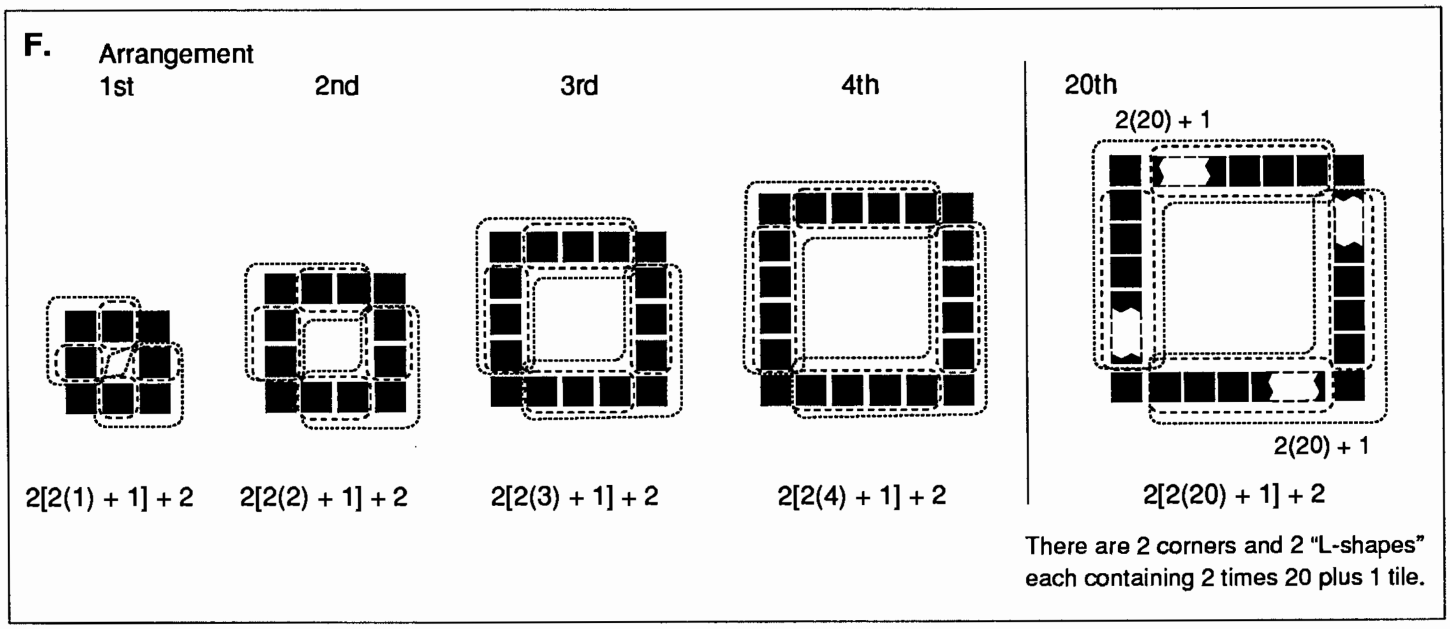
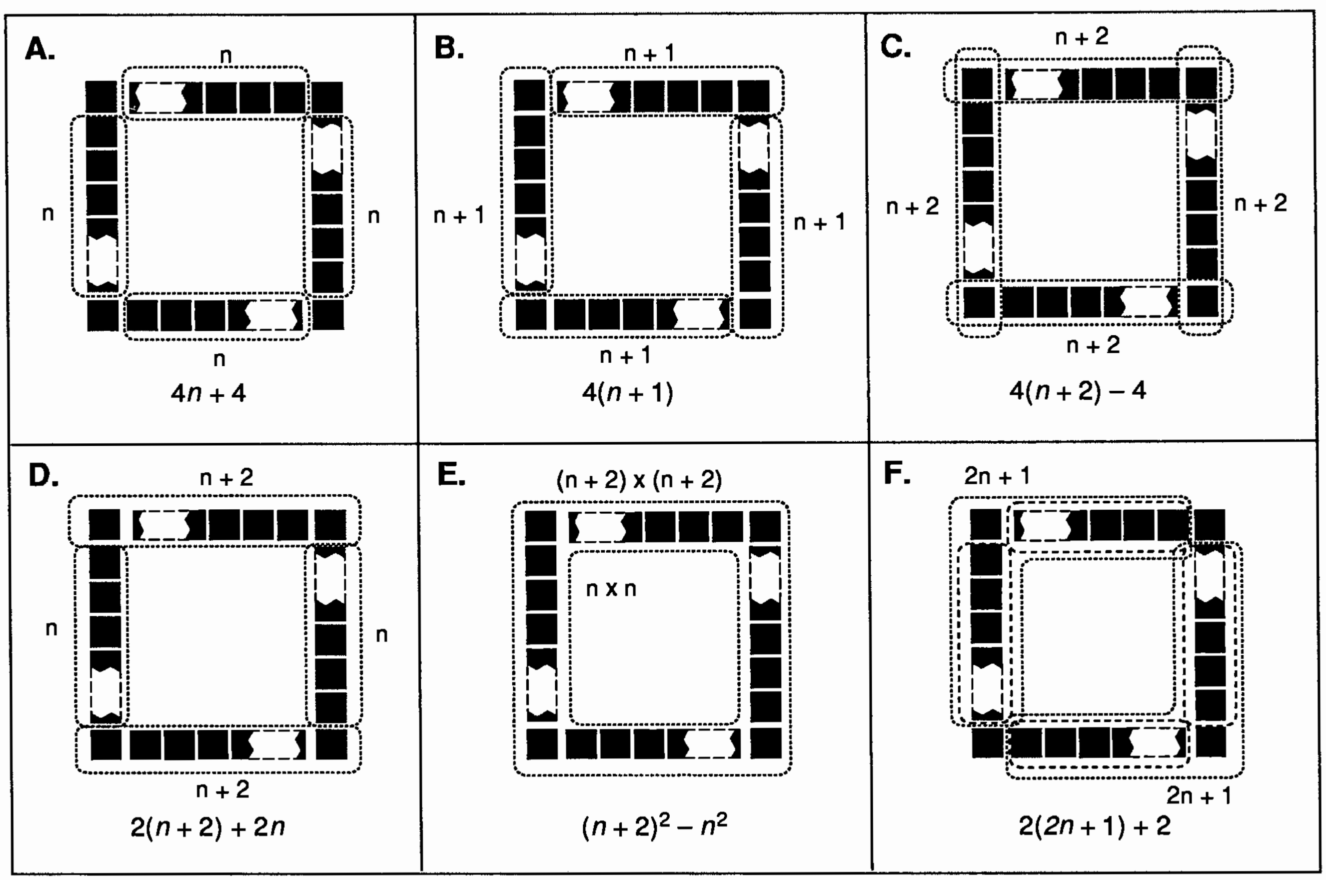
3. *Continued.*



4. For each of the illustrated methods, ask the students to write an expression for the number of tile in the *n*th arrangement suggested by that method. Illustrate.

4. A master from which an overhead transparency can be made to illustrate *n*th arrangements is attached (Master 2). Following are illustrations for the methods of viewing arrangements described in Comments 2 and 3.



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Actions

5. List all the expressions obtained for the number of tile in the nth arrangement. Discuss equivalent expressions.

Comments

4. *Continued.* If desired, the transparency can be cut apart so that, on the overhead, an nth arrangement can be placed alongside its corresponding 20th arrangement.

5. Here are expressions for the number of tiles in the nth arrangement, as illustrated in Comment 4.

$$4n + 4,$$

$$4(n + 1),$$

$$4(n + 2) - 4,$$

$$2(n + 2) + 2n,$$

$$(n + 2)^2 - n^2,$$

$$2(2n + 1) + 2.$$

Expressions, such as those listed, which give the same result when evaluated for a particular value of n, are said to be *equivalent*. They are also said to be *identically equal* or, simply, *equal*.

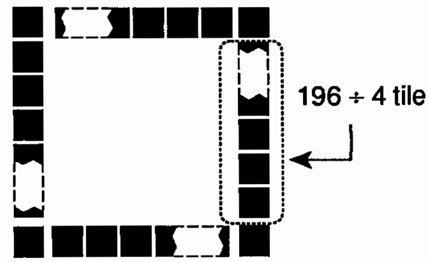
You may want to have the students evaluate each of their expressions for a particular value of n, say $n = 30$, and discuss the relative ease of these computations, e. g., evaluating $4(31)$ is simpler than evaluating $32^2 - 30^2$. Sometimes, in mathematical situations, it is advantageous to replace an expression by an equivalent expression that is simpler to evaluate.

Actions

6. Tell the students that one of the arrangements requires 200 tile to build. Ask them to determine which arrangement this is. Discuss the methods the students use. Relate the students' work to solving equations.

Comments

6. Various methods can be used. Viewing the arrangement as described in Comment 2 suggests removing the 4 corner tile and dividing the remaining 196 by 4. Thus, it is arrangement number 49 that contains 200 tile. A sketch may be helpful:



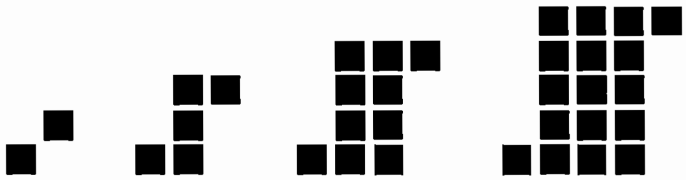
The above line of thought can be given an algebraic cast. The number of tile in the nth arrangement is $4 + 4n$. Thus, one wants the value of n for which $4 + 4n = 200$. Excluding the 4 corner tile reduces $4 + 4n$ to $4n$ and 200 to 196. Thus, $4n = 196$ and, hence, $n = 49$.

A statement of equality involving a quantity n, such as $4 + 4n = 200$, is called an *equation in n*. Determining the quantity n is called *solving the equation.*

Other ways of viewing the arrangement may lead to other methods of determining its number. For example, viewing the arrangement as described in the first method of Comment 3 may lead to dividing 200 by 4 and noting that the result, 50, is one more than the number of the arrangement. This, in effect, is solving the equation $4(n + 1) = 200$.

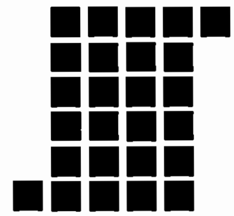
Actions

7. Repeat Actions 1 through 5 for the following sequence of tile arrangements.



Comments

7. Here is the next arrangement:

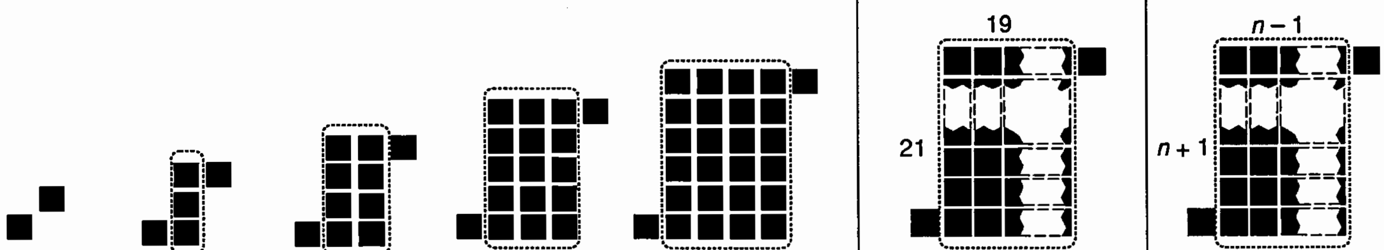


Masters for transparencies on which to illustrate the methods are attached (Masters 3 and 4). Illustrated below are ways of viewing the 20th arrangement, along with corresponding illustrations of the first 5 arrangements and the *n*th arrangement.

A.

Arrangement

1st | 2nd | 3rd | 4th | 5th | 20th | *n*th



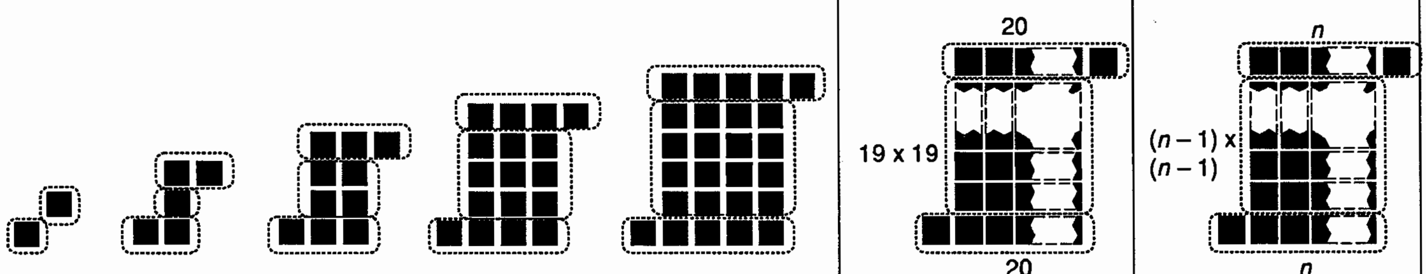
2	$(1 \times 3) + 2$	$(2 \times 4) + 2$	$(3 \times 5) + 2$	$(4 \times 6) + 2$	$(19 \times 21) + 2$	$(n-1)(n+1)+2$

A 19×21 rectangle with a single tile attached to each of the longer sides.

B.

Arrangement

1st | 2nd | 3rd | 4th | 5th | 20th | *n*th



$2(1)$	$1^2 + 2(2)$	$2^2 + 2(3)$	$3^2 + 2(4)$	$4^2 + 2(5)$	$19^2 + 2(20)$	$(n-1)^2 + 2n$

A 19×19 square with a row of 20 tile on the top and another row of 20 tile on the bottom.

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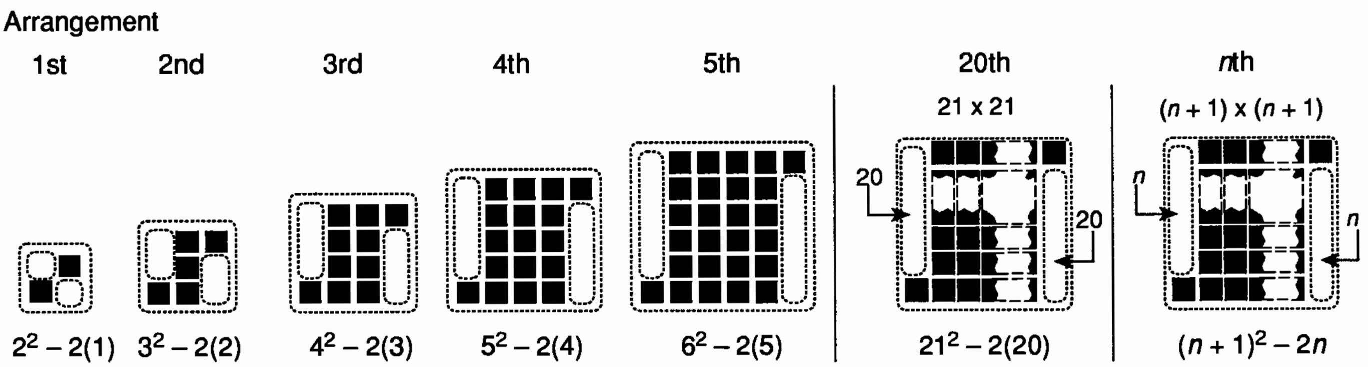
Actions

Comments

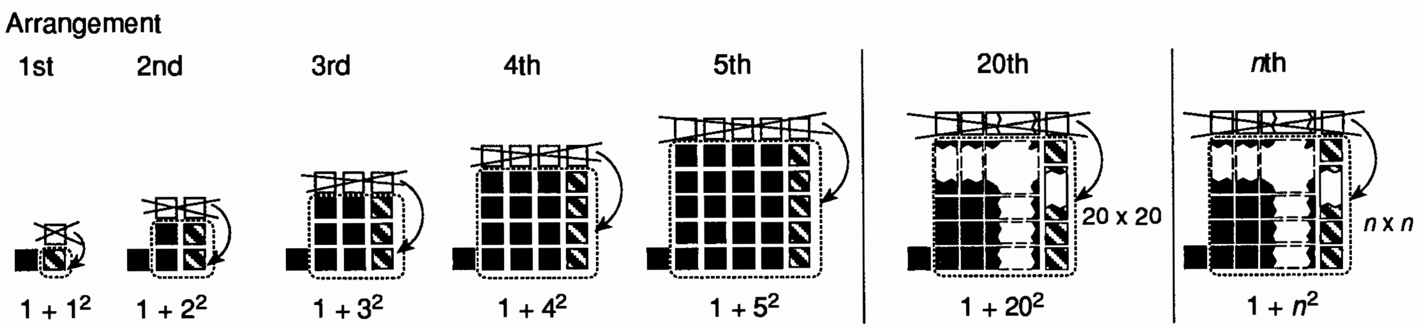
7. Continued.

C.



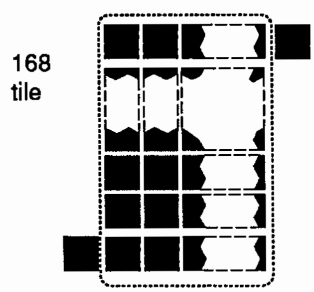
A 21 × 21 square with 20 tile removed from the first and last columns.

D.



Move the top row to the right side. The result is a 20 × 20 square with a single tile attached.

8. For the sequence in Action 7, ask the students to determine the number of the arrangement which contains 170 tile. Discuss the students' methods. Relate them to solving equations.



8. There are various ways to determine the number of the arrangement.

Thinking about the arrangement in the first manner described in Comment 7, 2 of the 170 tile are attached to a rectangle formed by the remaining 168. The dimensions of this rectangle differ by 2. Examining factors of 168, one finds the dimensions are 12 and 14. Since the number of the arrangement is 1 more than the smaller of these numbers (or 1 less than the greater), it is 13. Note that a number n has been found, namely 13, such that

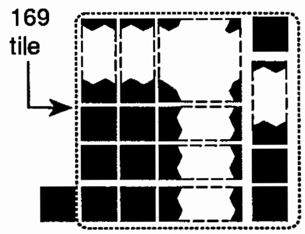
$$(n - 1)(n + 1) + 2 = 170.$$

Continued next page.

Actions

Comments

8. *Continued.* If the arrangement is thought of in the last manner described in Comment 7, one of the 170 tile would be attached to a square formed with the remaining 169. The side of this square, 13, is the number of the arrangement. A solution to the equation



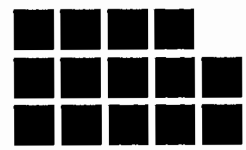
$$n^2 + 1 = 170$$

has been found.

9. Distribute a copy of Activity Sheet IX–2–A to each student. After the students have completed the activities, discuss their results and the methods used to arrive at them.

9. A master of the Activity Sheet is attached. The students can work singly or in groups. All or part of the activity can be assigned as homework.

(1) Here is the 4th arrangement:

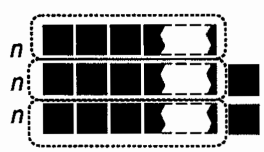


(2) There are various ways of describing the 25th arrangement. Here are two:

- Two rows of 26 tile each with a row of 25 tile on top.
- A 3 × 26 array of tile with one tile missing from the upper right hand corner.

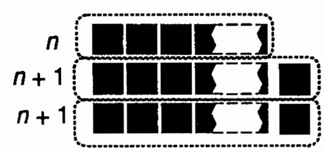
The 25th arrangement contains 77 tile.

(3) There are a number of ways of expressing T in terms of n, depending on how the nth arrangement is viewed. Below are some ways of viewing the nth arrangement and the corresponding formula relating T and n.



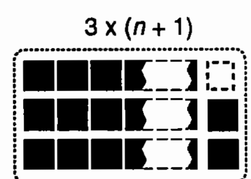
Three rows of n tile with 2 additional tile.

$T = 3n + 2$



A row of n tile and 2 additional rows of $n + 1$ tile.

$T = n + 2(n + 1)$



A 3 by $(n + 1)$ rectangle of tile with 1 tile removed.

$T = 3(n + 1) - 1$

A master for a transparency to use in describing various ways in which an arrangement can be viewed is attached (Master 5).

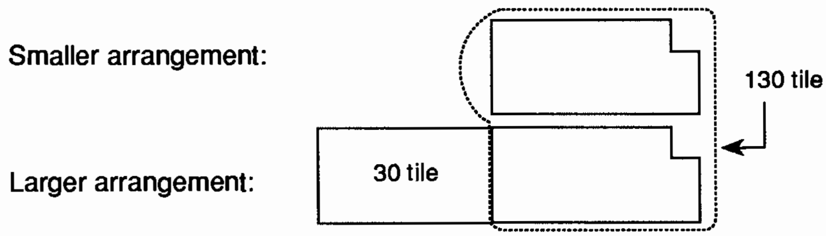
Continued next page.

Actions

Comments

9. *Continued.*

(4) If an arrangement contains 500 tile, the top row contains 498 ÷ 3, or 166, tile. The number of tile in the top row is the same as the number of the arrangement. (See the left hand figure in 9 (3).)



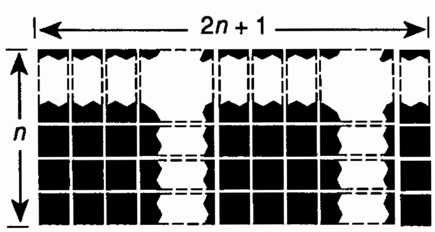
(5) As one sees in the figure on the left, the smaller arrangement contains 130 ÷ 2, or 65, tile. An arrangement with 65 tile has 21 tile in the top row and hence is the 21th arrangement. The larger arrangement has 10 more tile in the top row and, hence, is the 31st arrangement.

10. Repeat Action 9 for Activity Sheet IX–2–B.

10. (1) The dimensions of the next arrangement are 4 × 9.

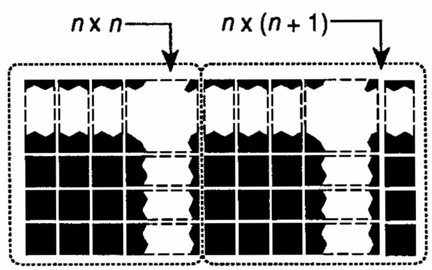
(2) The 30th arrangement is a 30 × 61 array of black tile. It contains 1830 tile.

(3) Below are some ways of viewing the *n*th arrangement, along with corresponding expressions for the number of tile they contain.



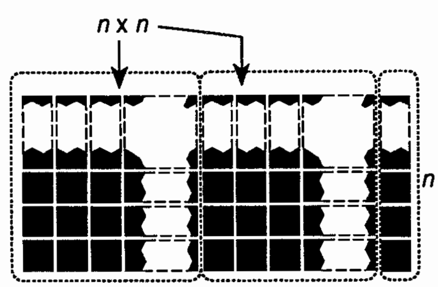
An n by $2n + 1$ rectangle of tile.

$n(2n + 1)$



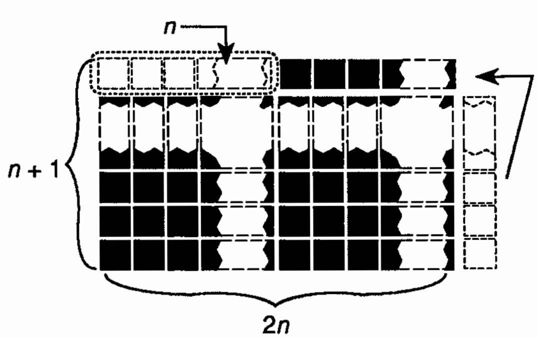
An n by n square of tile alongside an n by $n + 1$ rectangle of tile.

$n^2 + n(n + 1)$



Two n by n squares of tile alongside a column of n tile.

$2n^2 + n$



$(n + 1)2n - n$

Other equivalent expressions are suggested by rearranging tile. For example, taking the last column, turning it sideways and placing it on top of the arrangement results in an $n + 1$ by $2n$ rectangle with n tile missing, suggesting the expression $(n + 1)2n - n$.

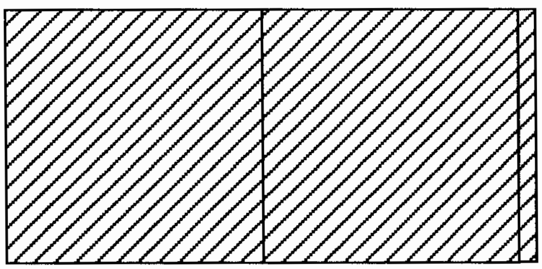
A master for a transparency to use in illustrating ways of viewing an arrangement is attached (Master 6).

Continued next page.

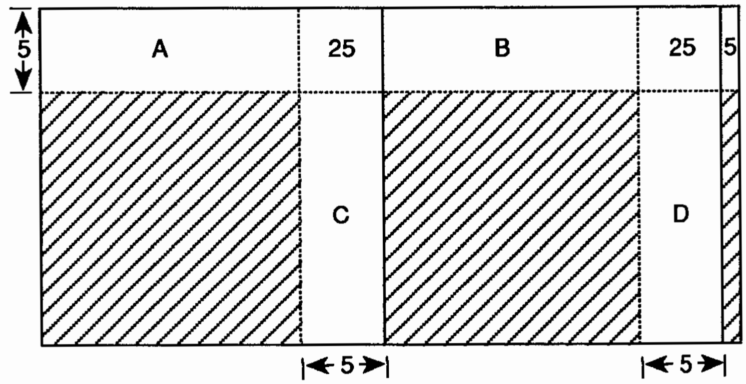
10. *Continued.*
(4) One way to determine which arrangement contains 1275 tile is to note that the number of tile in an arrangement is a bit larger than twice the square of the number of the arrangement. Since $\sqrt{1275 \div 2} \approx$ 25.25 (a calculator is helpful here), the number of the arrangement appears to be 25. Checking the number of tile in the 25th arrangement, one finds this is the case.

(5) Here is a sketch of the two arrangements:

Smaller arrangement:



Larger arrangement:

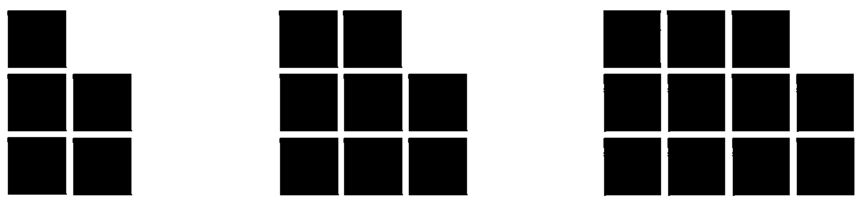


The shaded portion of the larger arrangement has the same number of tile as the smaller arrangement. The unshaded portion has 355 tile. Of these, 55 are accounted for as indicated. The remaining 300 comprise the 4 congruent rectangular regions *A*, *B*, *C* and *D*. Hence, each of these regions has 75 tile. Since one dimension is 5, the other is $75 \div 5$, or 15. Hence the side of a shaded square is 15. Thus, the smaller arrangement is the 15th and the larger is the 20th.

Name ______________________

Activity Sheet IX-2-A

1. Sketch the next arrangement in the following sequence of tile arrangements.



1st 2nd 3rd 4th

2. Describe the 25th arrangement. How many tile does it contain?

3. Let T be the number of tile in the nth arrangement. Write a formula relating T and n.

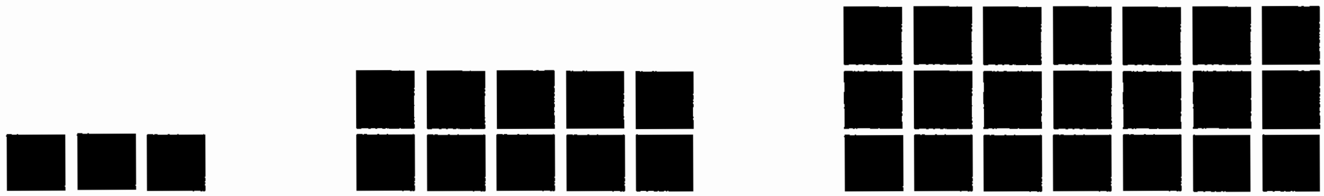
4. A certain arrangement contains 500 tile. Which arrangement is this?

5. Two arrangements together contain 160 tile. One of the arrangements contains 30 more tile than the other. Which two arrangements are these?

Activity Sheet IX-2-B

Name ______________________

1. Examine the following sequence of rectangular arrangements. Determine the dimensions of the next arrangement.



1st 2nd 3rd

2. Describe the 30th arrangement. How many tile does it contain?

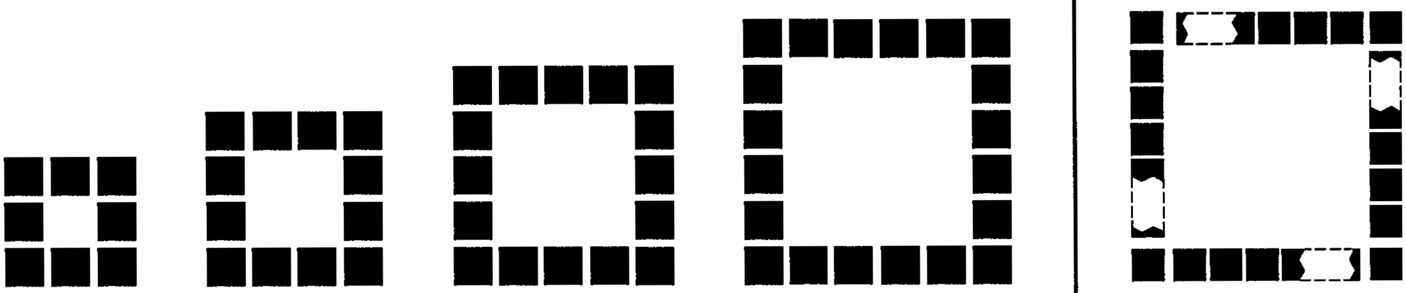
3. List some equivalent expressions for the number of tile in the *n*th arrangement.

4. A certain arrangement contains 1275 tile. Which arrangement is this?

5. The larger of two arrangements has 5 more rows and 355 more tile than the smaller. What two arrangements are these?

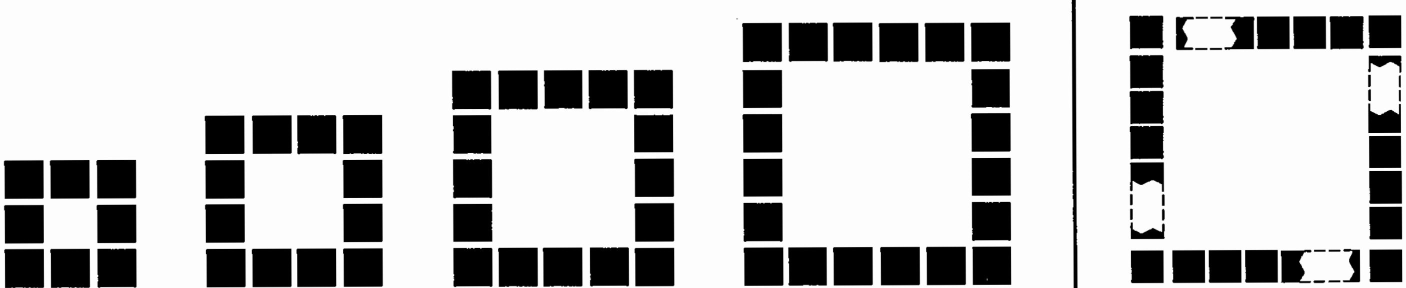
Arrangement

1st 2nd 3rd 4th 20th



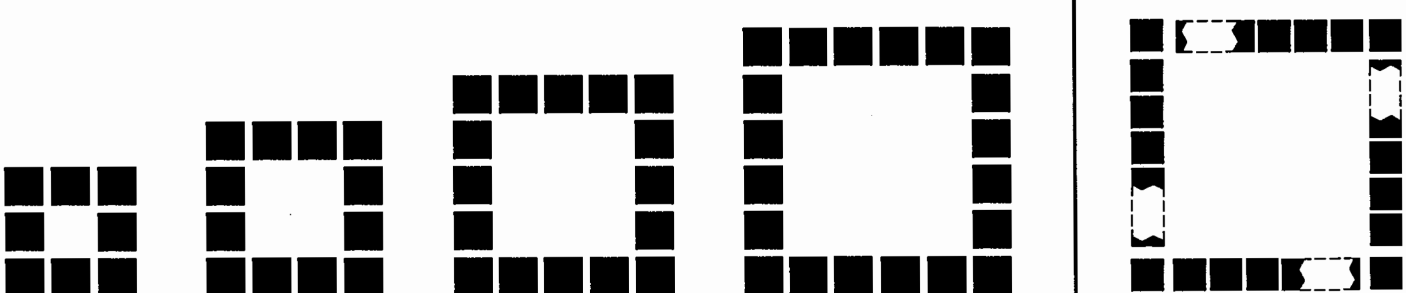
Arrangement

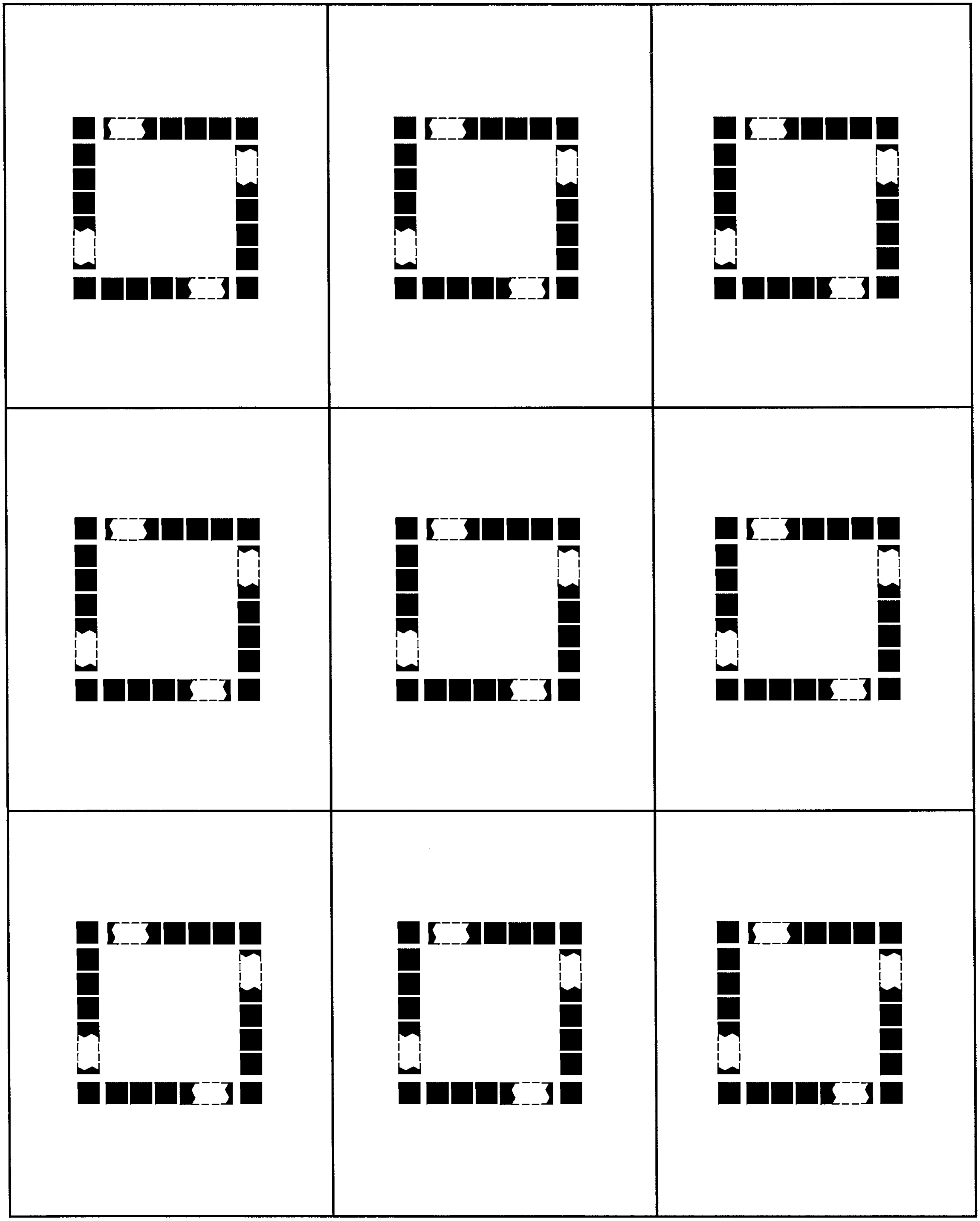
1st 2nd 3rd 4th 20th



Arrangement

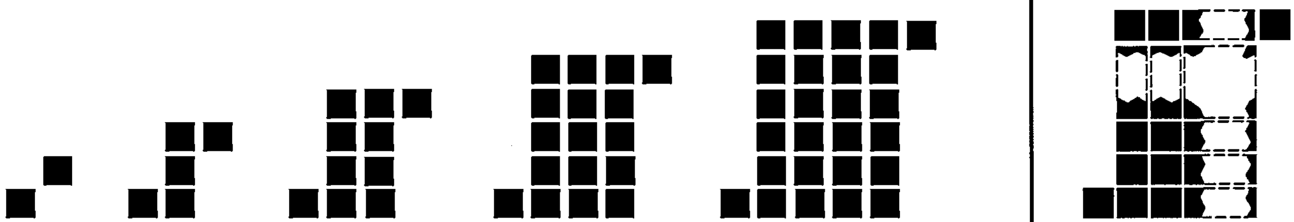
1st 2nd 3rd 4th 20th





Arrangement

1st 2nd 3rd 4th 5th 20th

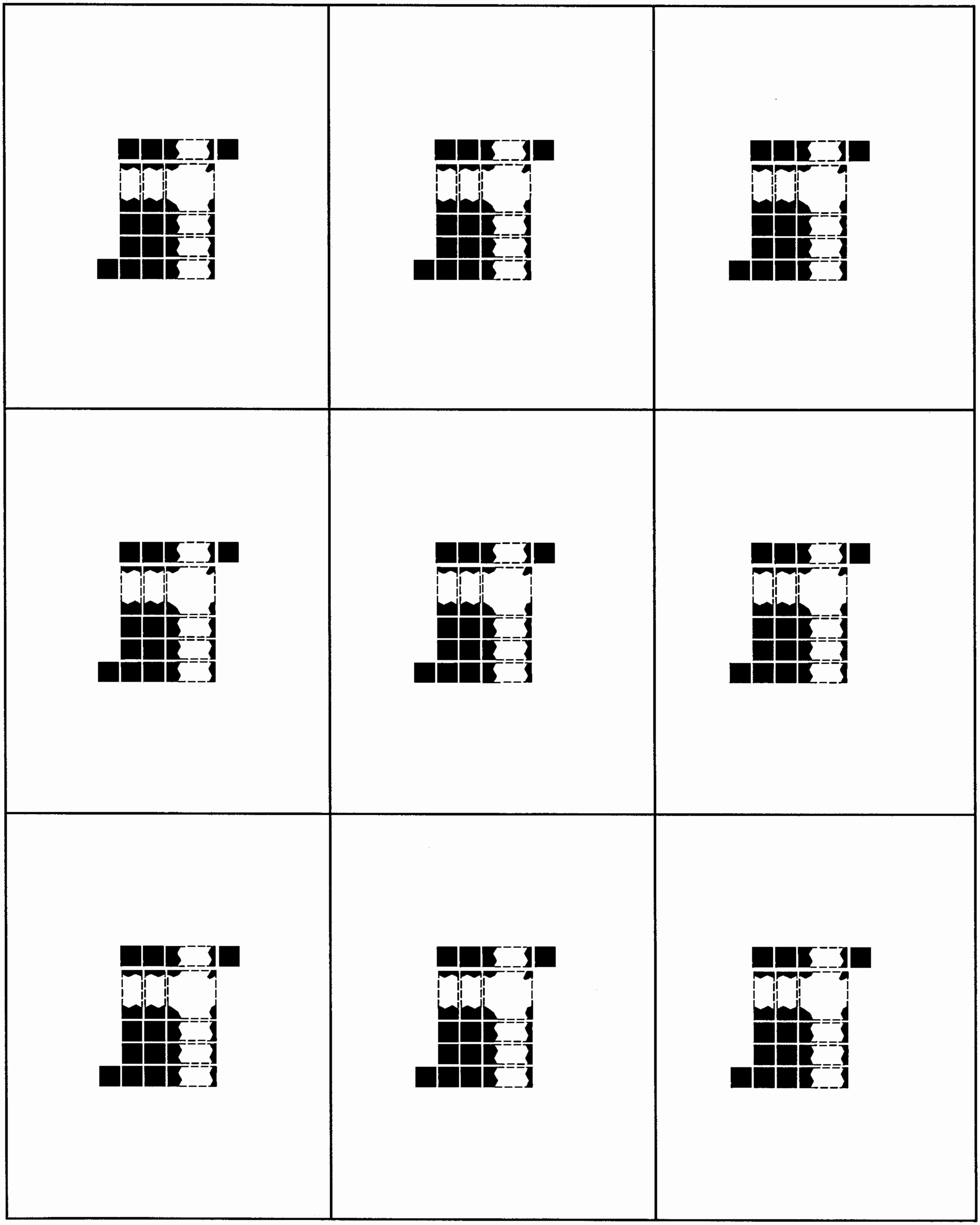


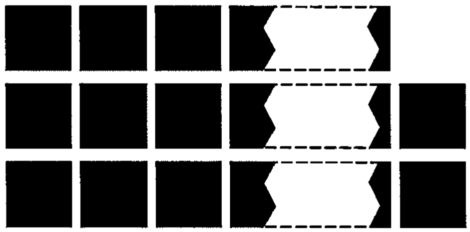
Arrangement

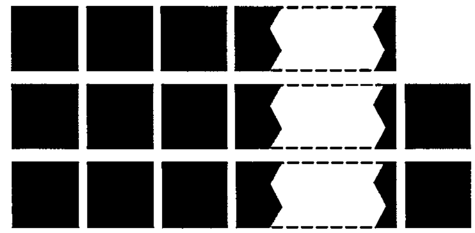
1st 2nd 3rd 4th 5th 20th

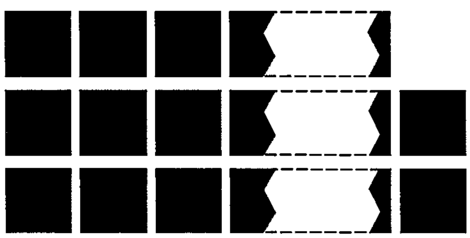
Arrangement

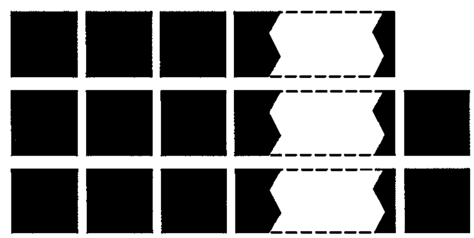
1st 2nd 3rd 4th 5th 20th

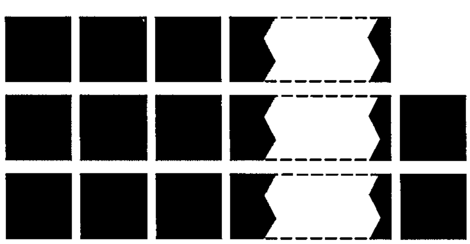


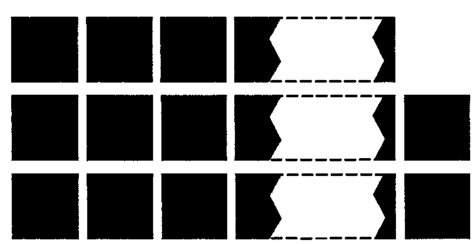


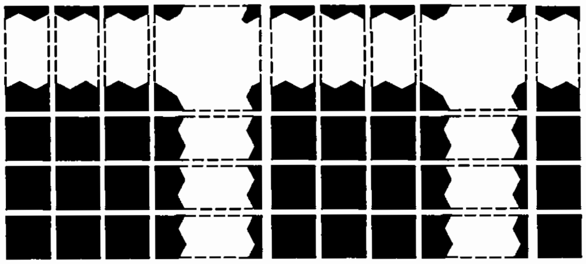


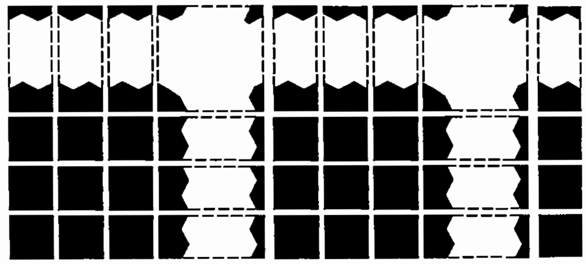


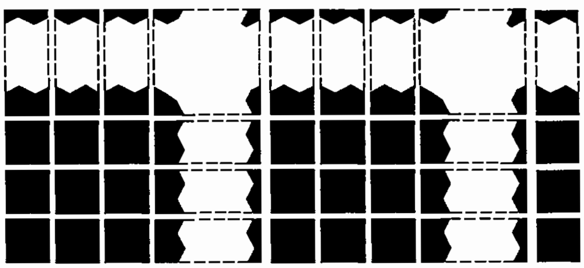


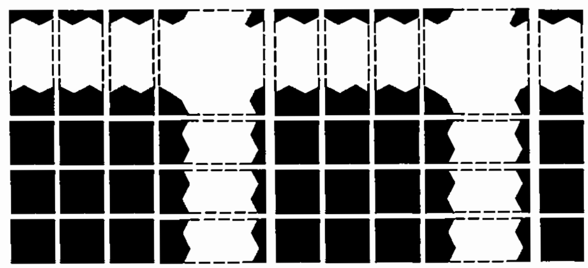


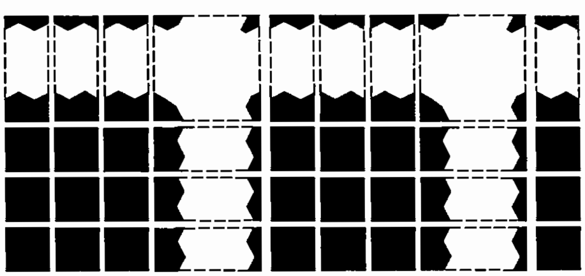


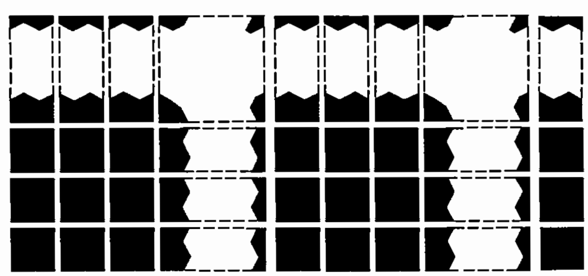












Tile Patterns, Part II

OVERVIEW

Algebraic expressions are represented as sequences of tile arrangements. Examining the properties of these arrangements leads to solving equations.

Prerequisite Activity
Unit IX, Activity 2, *Tile Patterns, Part I.*

Materials
Tile, scissors, tile pieces (see Comment 2) and copies of activity sheets as noted.

Actions

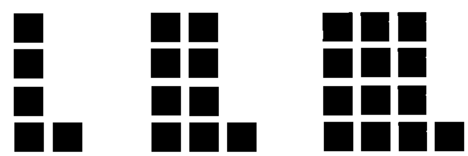
1. Distribute tile to each student. Write the expression $4n + 1$ on the overhead or chalkboard. Tell the students that the nth arrangement in a sequence of tile arrangements contains this many tile. Have the students build the first 3 arrangements of such a sequence.

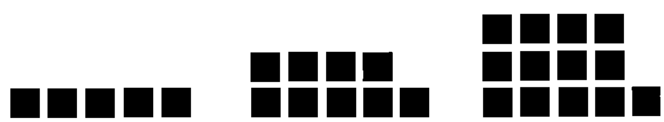
2. Have each student cut out a set of tile pieces. Discuss the pieces, then have the students use the pieces to build a representation of the nth arrangement of the sequence introduced in Action 1.

Comments

1. Each student or group of students will need 30 tile.

Any number of arrangements are possible. Below are two possibilities.





2. A master for tile pieces is attached. Each student or group of students will need two copies. Since these tile pieces are only used in this activity, paper copies will suffice. If you intend to do subsequent activities, you may wish to prepare and distribute algebra pieces as described in Comment 2 of Activity 5. The red side of these pieces can be ignored for this activity.

Continued next page.

Actions

3. Raise questions such as the following about the above sequence of tile arrangements:
- How many tile are required to build the 15th arrangement?
- Which arrangement requires 225 tile to build?
- What are some equivalent expressions for the number of tile in the nth arrangement?

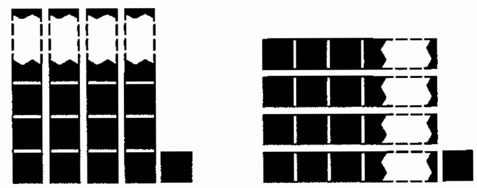
Discuss the students' responses.

Comments

2. *Continued.* There are three kinds of pieces: a unit square (representing a single tile, an *n-strip* (representing a strip of n tile) and an *n^2-mat* (representing an $n \times n$ array of tile).

n-strip n^2-mat

Two possible tile piece representations of the nth arrangement are shown below.



$4n + 1$ $4n + 1$

3. Answers to these questions can be related to the tile piece representations constructed in Action 2. If the 15th arrangement where constructed as suggested by either of the illustrations in Comment 2, each of the 4 n-strips would be replaced by 15 tile. Hence, a total of $(4 \times 15) + 1$, or 61, tile would be required.

If an arrangement contained 225 tile, then the 4 n-strips, in total, would contain 224 tile, so each n-strip would contain $224 \div 4$, or 56, tile. Hence, it is the 56th arrangement which contains 225 tile since, in this case, the number of an arrangement is the same as the number of tile in an n-strip.

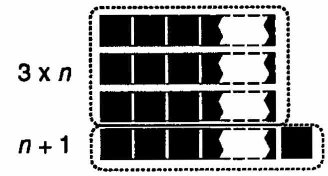
Continued next page.

Actions

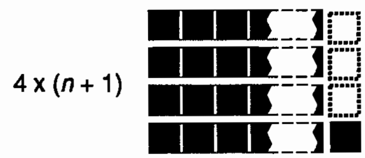
4. Distribute a copy of Activity Sheet IX-3-A to each student. Ask the students to complete the sheet and compare their results with those of other students. Discuss.

Comments

3. *Continued.* Equivalent expressions for the number of tile in the *n*th arrangement can be obtained by viewing tile piece representations in alternate ways. Here are two examples:

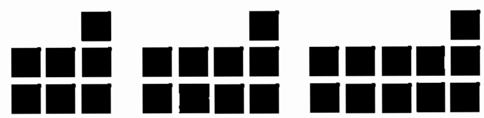


$3n + (n + 1)$

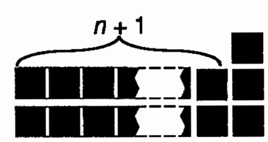


$4(n + 1) - 3$

4. (1). The first three tile arrangements can take a variety of forms. Here is one possibility:

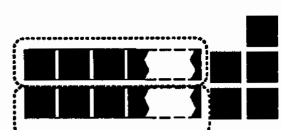


4. (2). This is one way tile pieces can be used to represent the *n*th arrangement:



*n*th arrangement

4. (3). If an arrangement contains 225 tile, then the two circled *n*-strips shown below contain a total of 220 tile. Hence, each contains 110 tile and it is the 110th arrangement which contains 225 tile.



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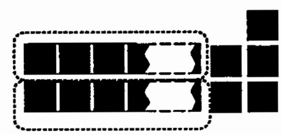
Actions

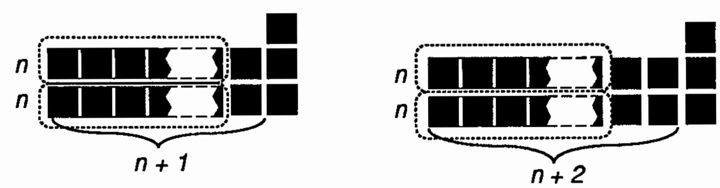
5. Repeat Action 4 for Activity Sheet IX-3-B.

Comments

4. (4). If the nth arrangement shown below contains at most 500 tile, the 2 circled n-strips contain at most 495 tile. Hence, the most tile possible in 1 n-strip is 494 ÷ 2, or 247. Thus the 247th arrangement is the largest that can be built with 500 or fewer tile.

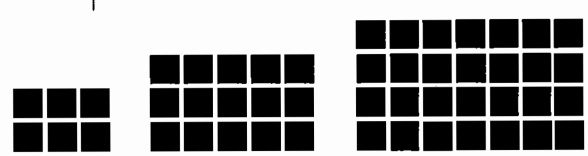


4. (5). The two successive arrangements shown below have a total of 400 tile. Of the 400 tile, all but 12 are in the 4 circled n-strips. Hence, there are 388 ÷ 4, or 97, tile in each n-strip. Thus the 97th and 98th arrangements require a total of 400 tile to build them.



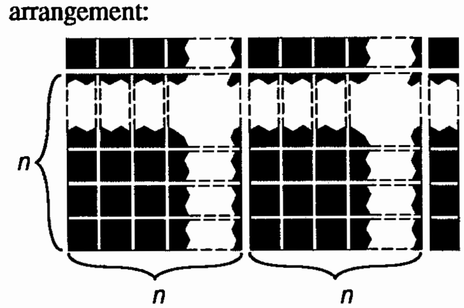
nth arrangement $(n + 1)$st arrangement

5. (1). Here is one possibility for the first three arrangements:



Note that, from arrangement to arrangement, the height increases by 1 and the width by 2.

5. (2). Here is one representation of the nth arrangement:



$(n + 1)(2n + 1)$

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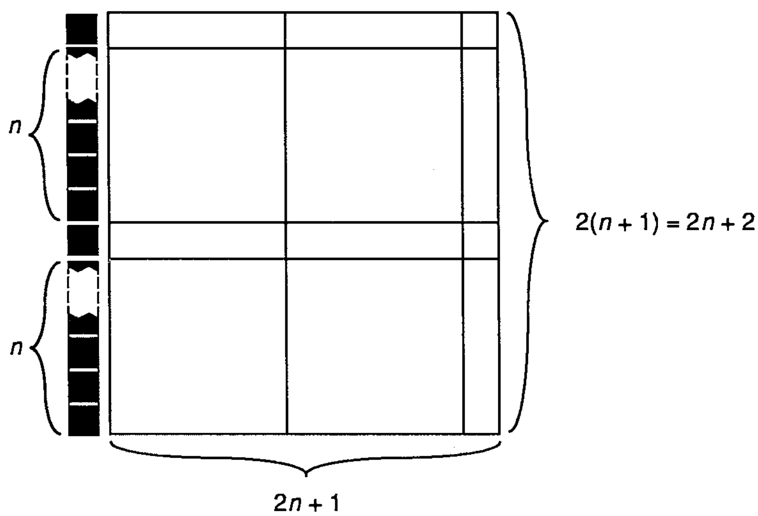
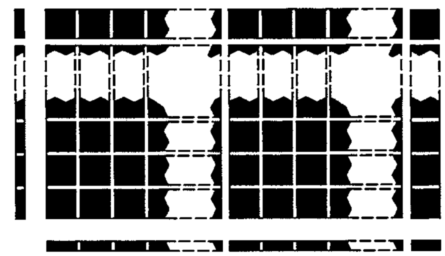
Actions



Comments

5. (2). *Continued.* Once one notices that the height of the nth arrangement is $n + 1$ and its width is $2n + 1$, edge pieces may be helpful in forming the nth arrangement. (See below.) Edge pieces can be obtained by cutting tile pieces in fourths.

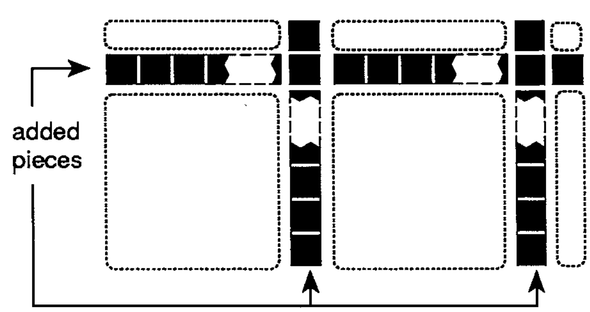


$(n + 1)(2n + 1)$

5. (3). If the nth arrangement contains 2145 tile, then $2n^2$ is fairly close to, but less than, 2145. Hence, it is a bit less than $\sqrt{2145 \div 2} \approx 32.75$. This suggests that $n = 32$ which can be verified by noting that $(32 + 1)(64 + 1) = 2145$.

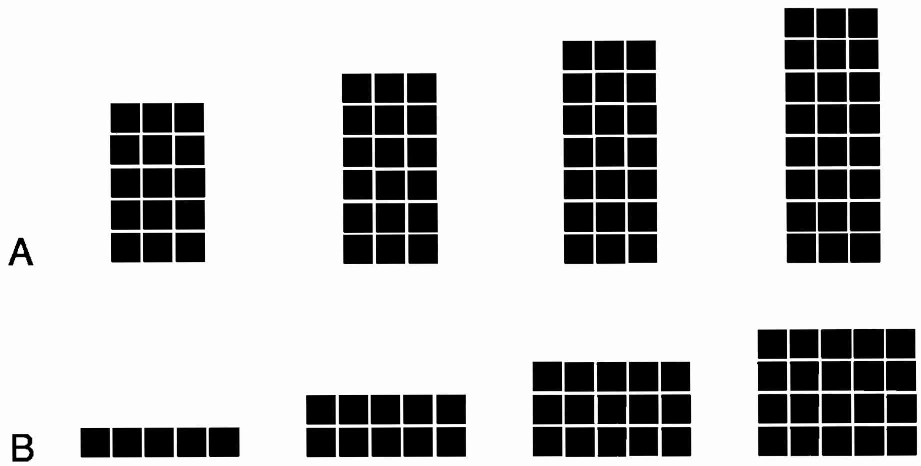
5. (4). If 2 copies of the nth arrangement are placed one above the other and 2 n-strips and 2 tile are added to the left edge, the result is a square. If the total number of tile added is 50, each n-strip contains 24 tile. Thus, the desired arrangement is the 24th.

5. (5). The pieces that must be added to the nth arrangement to obtain the $(n + 1)$st arrangement are shown below. The added portion contains 5 tile and 4 n-strips. If a total of 125 tile are added, each n-strip contains 30 tile. Thus, the arrangements are the 30th and 31st.



Actions

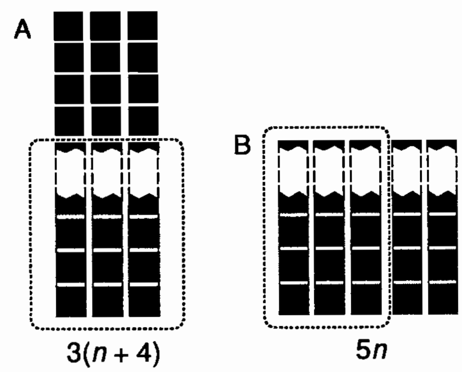
6. Show the students the first 4 tile arrangements in sequences A and B. Ask them to use their tile pieces to build, first, a representation of the nth arrangement of sequence A and, second, a representation of the nth arrangement of sequence B. Then ask them to determine for what n the nth arrangement in A will contain the same number of tile as the nth arrangement in B. Discuss.



Comments

6. Masters to prepare transparencies of sequences A and B (Master 1) and the sequences in Actions 7 and 8 (Masters 2 and 3) are attached.

Below are possible tile-piece representations of the nth arrangements for sequences A and B. Comparing these two arrangements, one sees they have 3 n-strips in common. Hence, they will have the same number of tile if the 2 remaining n-strips in B contain the same number of tile as the 12 remaining tile in A. Thus, each n-strip must contain 6 tile and the two arrangements will contain the same number of tile when n is 6.



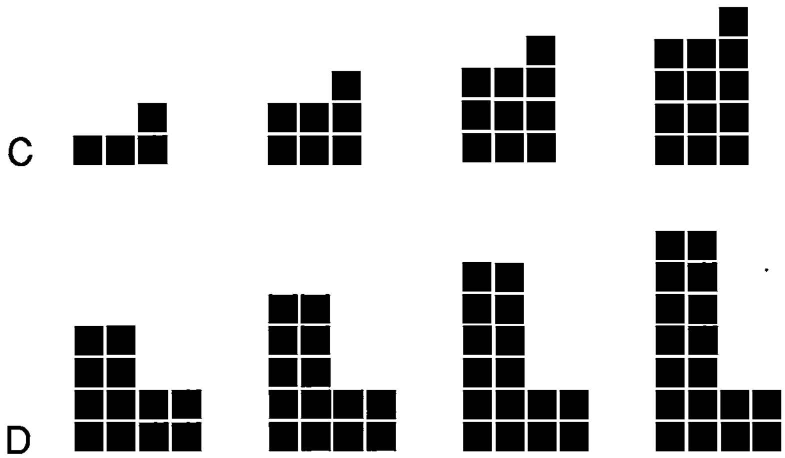
Notice we have determined that 6 is the solution of

$$5n = 3(n + 4).$$

Other representations and expressions for the nth arrangements are possible, which will lead to other formulations of the above equation.

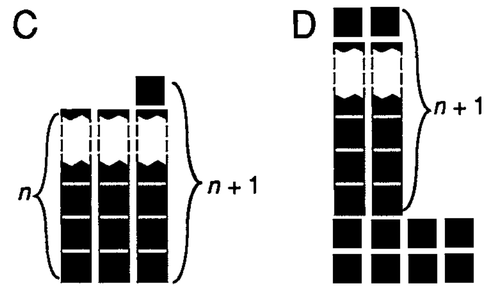
Actions

7. Repeat Action 6 for sequences C and D.



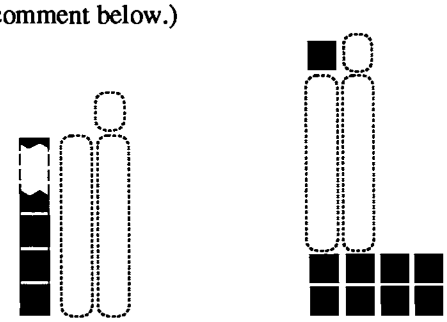
Comments

7. Here are possible tile piece representations of the *n*th arrangements:



$2n + (n + 1)$ $2(n + 1) + 8$

The *n*th arrangements in sequences C and D will contain the same number of tile if, when 2 *n*-strips and a single tile are removed from each, the remaining portions contain the same number of tile. This will happen if an *n*-strip contains 9 tile. (See the comment below.)



Note the equation

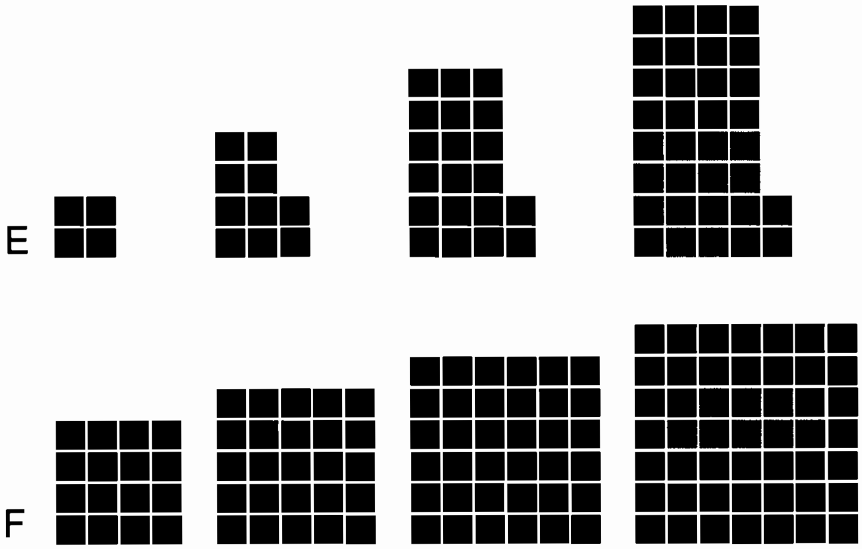
$$2n + (n + 1) = 2(n + 1) + 8$$

has been solved. Other expressions for the *n*th arrangement will lead to other formulations of this equation.

If an *n*-strip is placed alongside a row of 9 tile, it appears that the strip is too short to contain that many tile. It should be recognized that an *n*-strip can be mentally elongated (or shortened) to contain *n* tile, whatever *n* might be. To illustrate this, a master is attached for an *n*-strip that can be elongated.

Actions

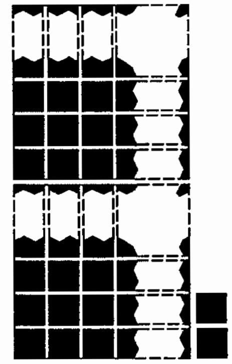
8. Repeat Action 6 for sequences E and F.



Comments

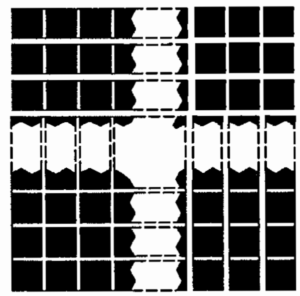
8. Here are possibilities for nth arrangements:

E



$2n^2 + 2$

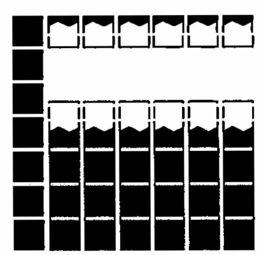
F



$(n + 3)^2$

Comparing the two representations of the nth arrangements, one sees they will contain the same number of tile provided an n^2-mat contains the same number of tile as 6 n-strips and 7 tile. This will be the case if n is 7.

Students may attempt to physically arrange 7 tiles and 6 n-strips into a square. This is not possible. However, one can mentally imagine elongating the strips to accommodate n being 7. Also the pieces can be cut and arranged as shown to help create this image.



Notice the positive solution of the equation

$$2n^2 + 2 = (n + 3)^2$$

has been found.

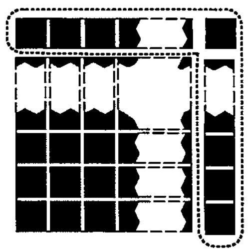
Actions

9. Ask the students to find a solution for each of the following equations. Call on volunteers to explain their methods.

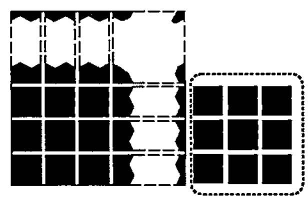
(a) $3n + 4 = 25$ (b) $4(n + 2) = 32$

(c) $3n + 1 = 2(n + 3)$ (d) $4n + 3 = 2n + 11$

(e) $(n + 1)^2 = n^2 + 9$ (f) $n(n + 2) = 168$



$(n + 1)^2$



$n^2 + 9$

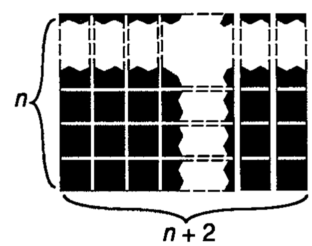
Comments

9. The students will use a variety of methods. If students experience difficulty, suggest they relate the equations to tile patterns. For example, solving equation (a) is equivalent to finding which arrangement in a sequence contains 25 tile if the nth arrangement contains $3n + 4$ tile.

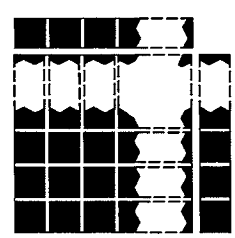
For what n does this arrangement contain 25 tile?

Equation (e) can be interpreted as asking: If the nth arrangement in one sequence contains $(n + 1)^2$ tile and that in another contains $n^2 + 9$ tile, when do corresponding arrangements in these sequences have the same number of tile? For this to happen, the circled portions in the arrangements to the left must contain the same number of tile. This will occur if each n-strip contains 4 tile.

To solve (f), one can determine for which n the following arrangement contains 168 tile. This amounts to finding two numbers which differ by 2 and whose product is 168. Since 12 and 14 are such numbers, $n = 12$.



Alternatively, the pieces can be arranged as shown below. Adding a tile in the upper right-hand corner creates an $(n + 1) \times (n + 1)$ square of 169 tile. Hence $n + 1 = 13$ and $n = 12$.



Historically, this method of solving an equation is called *completing the square.*

Name ______________________

Activity Sheet IX-3-A

1. Form the first three tile arrangements in a sequence for which the number of tile in the nth arrangement is $2(n + 1) + 3$. Sketch these three arrangements below.

2. Use tile pieces to form a representation of the nth arrangement. Sketch your representation here:

3. Which arrangement contains 225 tile?

4. Which is the largest arrangement that contains 500 or fewer tile?

5. A total of 400 tile is required to build two successive arrangements. Which arrangements are these?

Activity Sheet IX-3-B

Name ______________________

1. Form the first three tile arrangements in a sequence for which the number of tile in the nth arrangement is $(n + 1)(2n + 1)$. Sketch these three arrangements below.

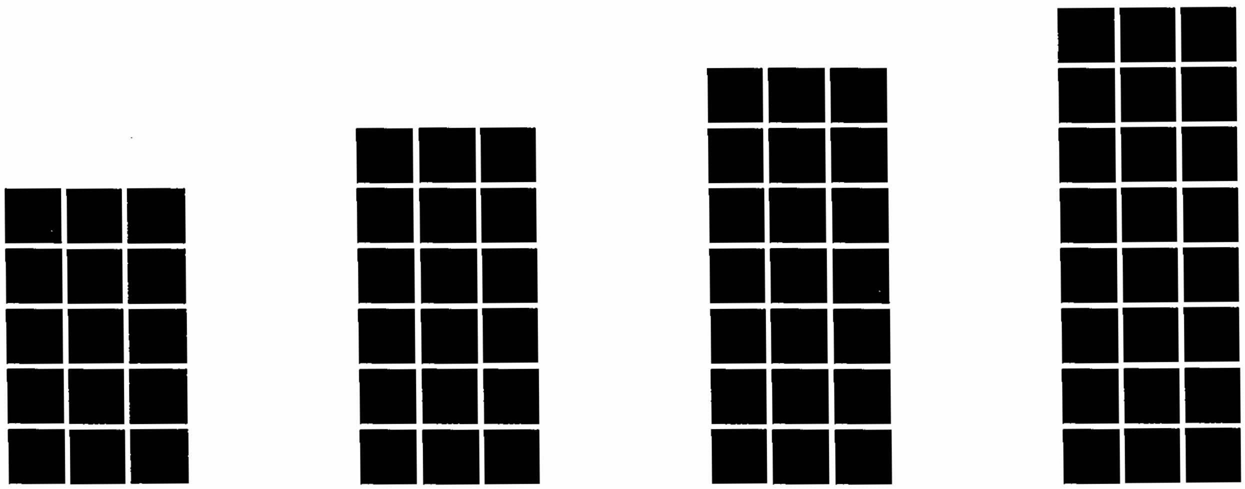
2. Use tile pieces to form a representation of the nth arrangement. Sketch your representation here:

3. Which arrangement contains 2145 tile?

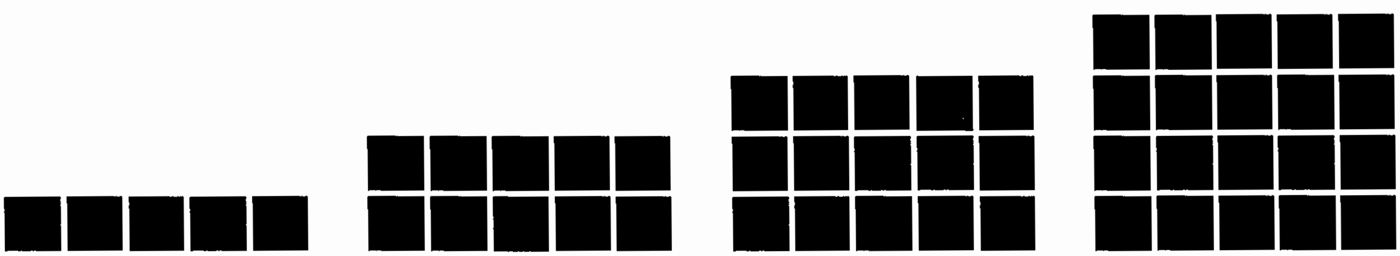
4. If the number of tile in a certain arrangement is doubled, 50 more tile are needed to form a 50 x 50 square. Which arrangement is this?

5. The larger of two successive arrangements contains 125 more tile than the smaller. Which two arrangements are these?

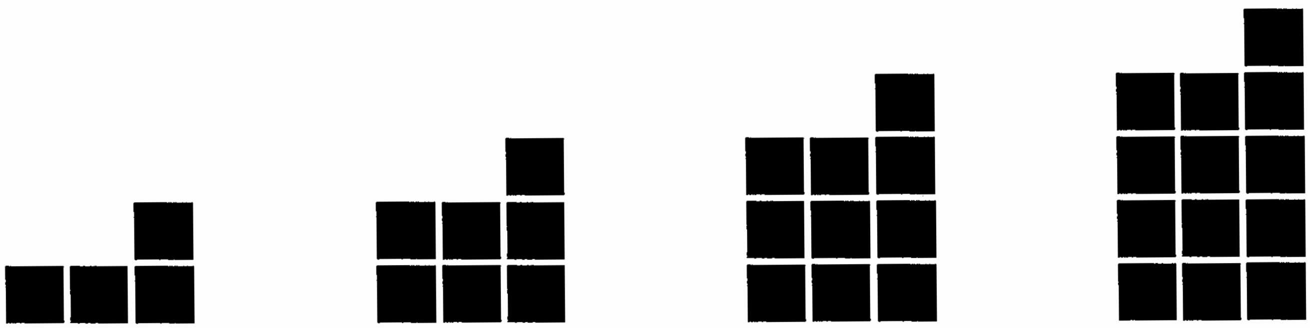
A



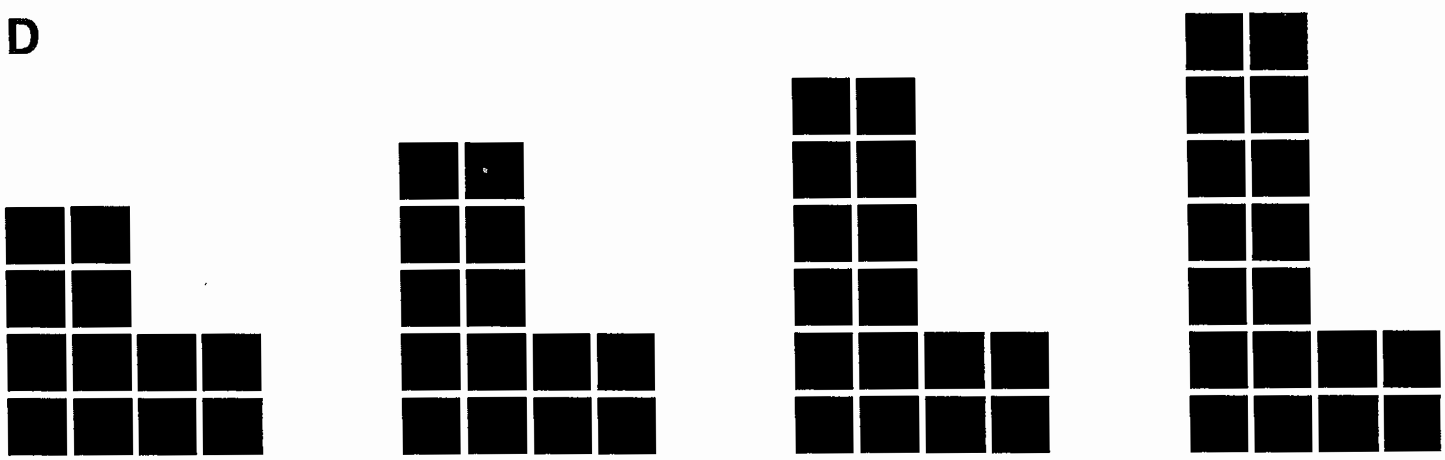
B



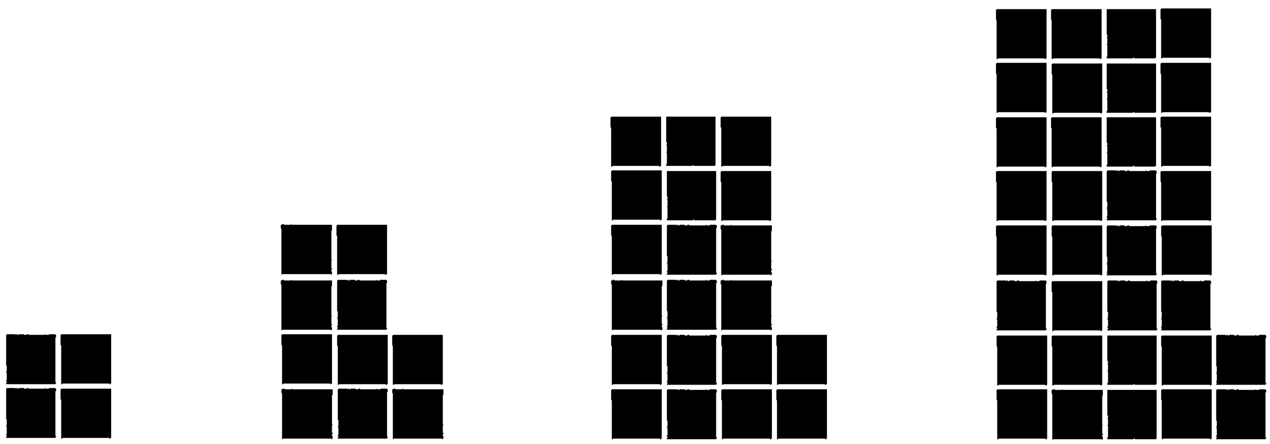
C



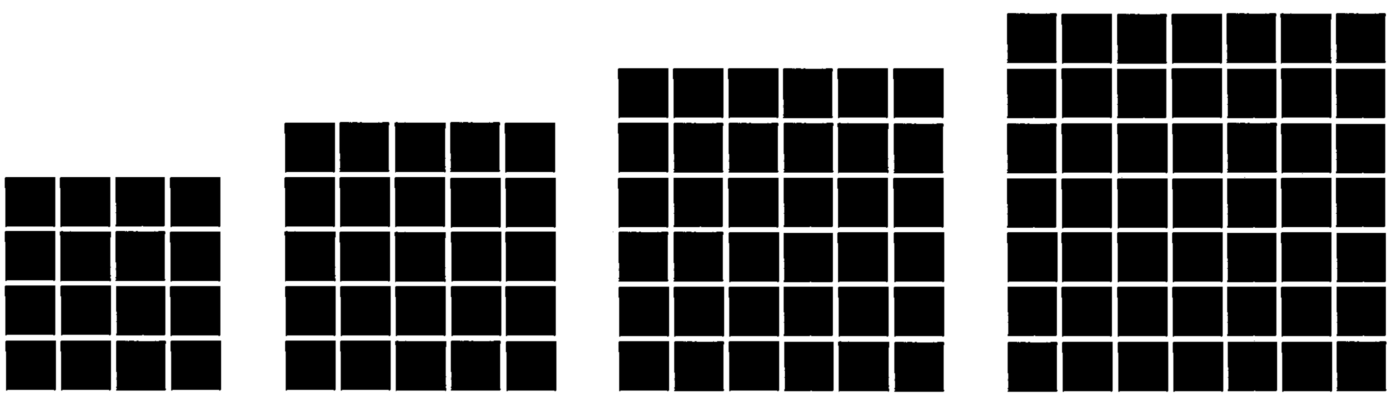
D



E

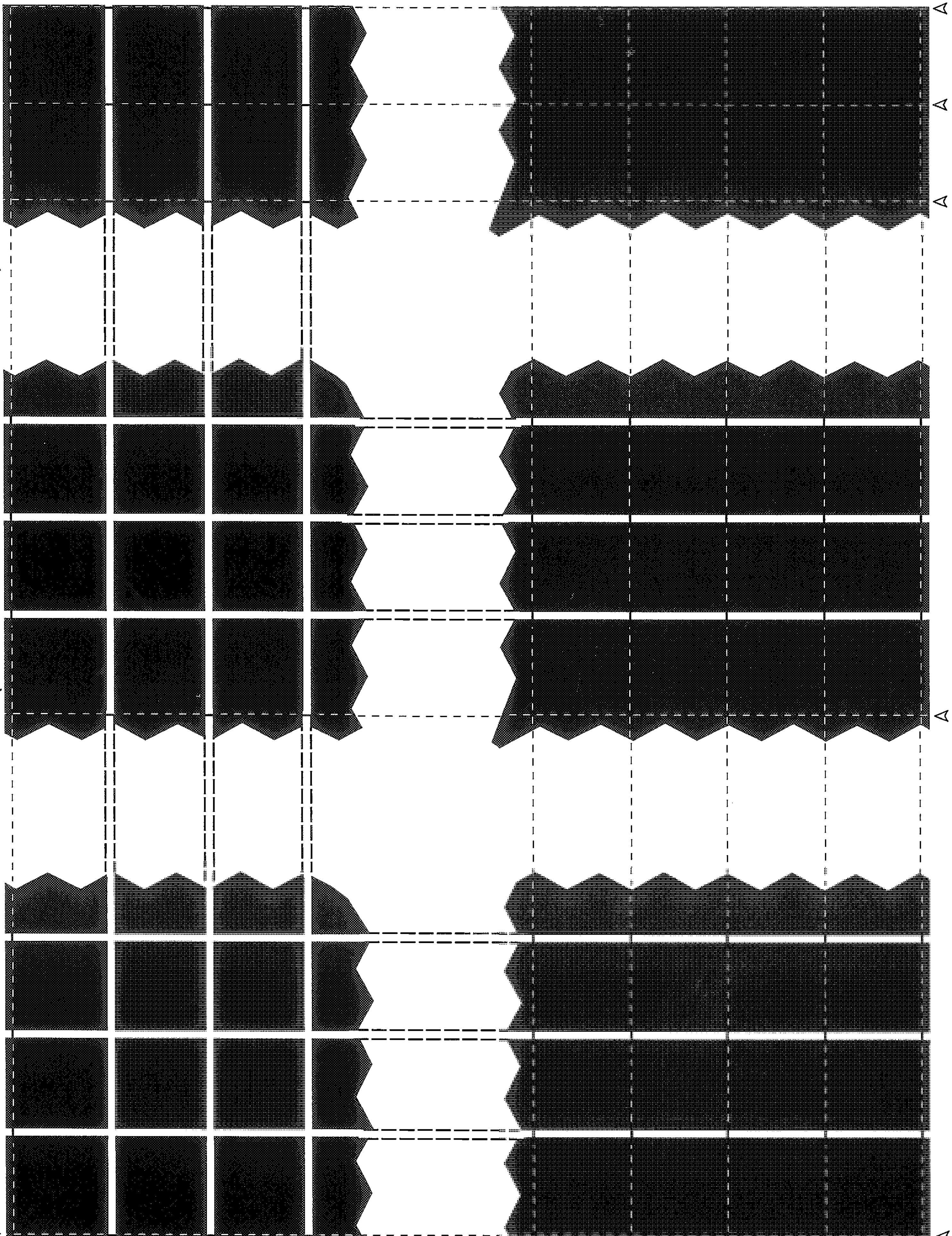


F

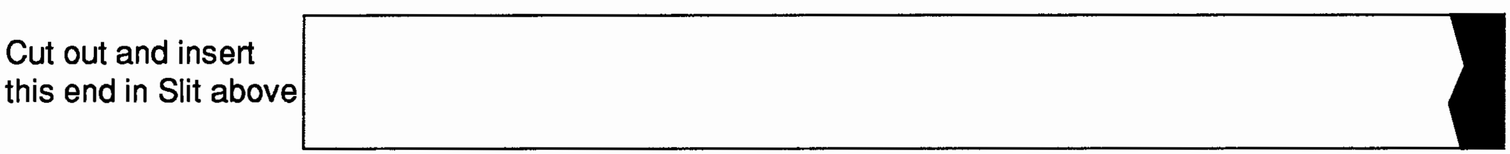


Tile Pieces

(cut on thin dotted lines at arrowheads)



CUT HERE



Counting Piece Patterns, Part I

O V E R V I E W

The net values of arrangements in counting piece patterns are determined. Functional notation for net values is introduced.

Prerequisite Activity

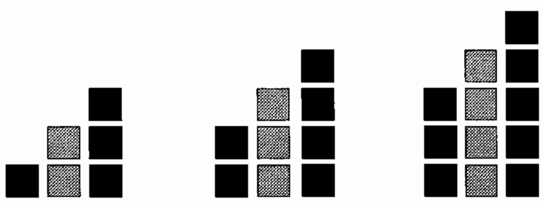
Unit VI, *Modeling Integers*; Unit IX, Activity 2, *Tile Patterns, Part 1.*

Materials

Bicolored counting pieces, black and red overhead pieces.

Actions

1. Distribute counting pieces to each student or group of students. Display the following sequence of counting piece arrangements on the overhead. Have the students form the next arrangement in the sequence.



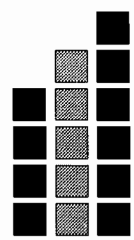
2. Ask the students to find the net values of each of the first four arrangements in the above sequence.

3. Ask the students to describe the 20th arrangement and find its net value. Discuss.

Comments

1. Each student or group of students will need about 75 counting pieces. Counting pieces are described in Comment 1 of Unit VI, Activity 1, *Counting Piece Collections.* Suggestions for preparing overhead counting pieces are also included in this Comment.

Here is the next arrangement:



2. You may want to review the meaning of the net value of a collection of counting pieces. (See Comments 1 and 4, Unit VI, Activity 1, *Counting Piece Collections.*)

The net values of the first four arrangements are 2, 3, 4 and 5, respectively.

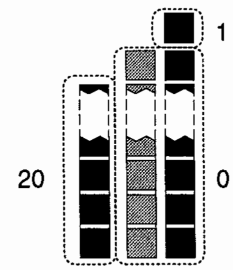
3. One possible description of the 20th arrangement is that it consists of a column of 21 red pieces with a column of 20 black pieces on one side and a column of 22 black pieces on the other side.

Continued next page.

Actions

Comments

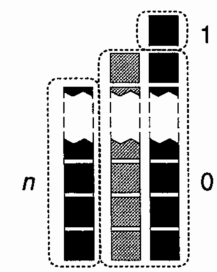
3. *Continued.* The net value of the 20th arrangement is 21, as indicated in the sketch.



20th arrangement
net value = 20 + 1 = 21

4. Repeat Action 3 for the *n*th arrangement.

4. The net value of the *n*th arrangement is $n + 1$:



*n*th arrangement
net value = $n + 1$

5. Tell the students that, henceforth, the "net value of an arrangement" will be referred to simply as "the value of the arrangement". Also, the string of symbols "$v(n)$" will be used as shorthand for the phrase "the value of the *n*th arrangement". Discuss this convention.

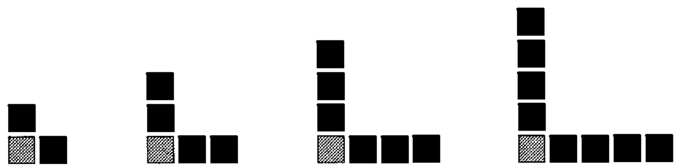
5. The statement in Comment 3, "the value of the 20th arrangement is 21," can be written, "$v(20) = 21$." The string of symbols "$v(20)$" is often read "*v* of 20". Similarly, Comment 4 can be written "$v(n) = n + 1$."

The string of symbols "$v(n)$" is sometimes read "*v* of *n*". If this reading is adopted, it is important to remember that "*v* of *n*" is a shortened version of the phrase "the value of the *n*th arrangement" and not the phrase "the value of *n*." Thus the statement "*v* of *n* is 30" means "the value of the *n*th arrangement is 30," not "the value of *n* is 30."

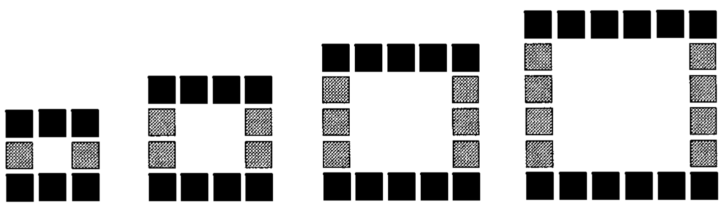
Actions

6. For each of the following sequences ask the students to describe the nth arrangement and find $v(n)$. Discuss the students' responses.

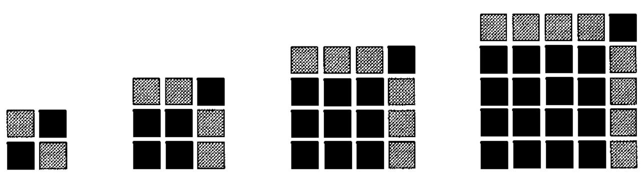
(a)



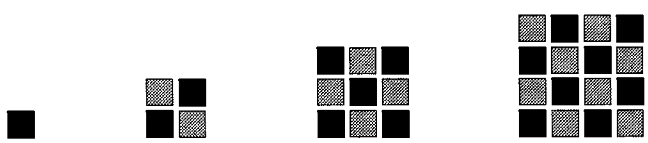
(b)



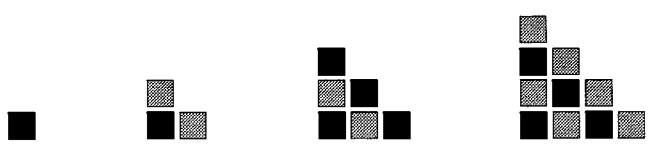
(c)



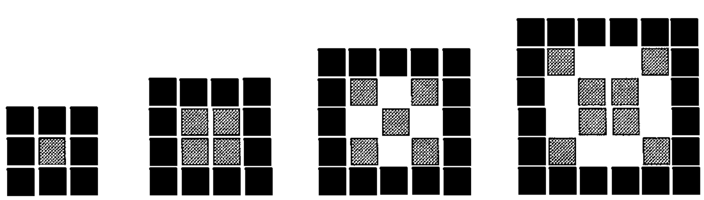
(d)



(e)



(f)

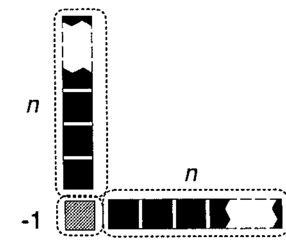


Comments

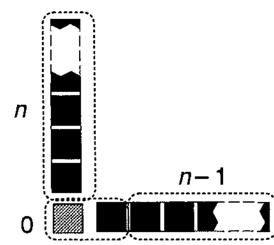
6. A master of the sequences is attached. For each sequence, you may wish to put the first three or four arrangements on the overhead and ask the students to construct the next arrangement. Sequences (e) and (f) are optional. The students may be interested in constructing other sequences to add to the collection given here.

In each case, the students may find a number of equivalent expressions for $v(n)$. Below are some possibilities.

(a)

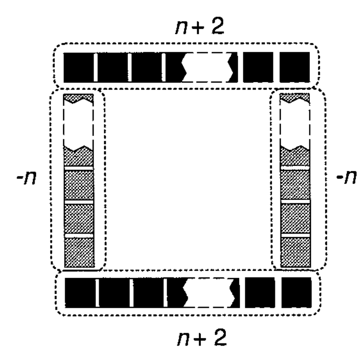


$v(n) = 2n + (-1)$

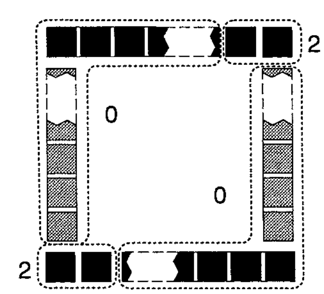


$v(n) = n + (n - 1)$

(b)



$v(n) = 2(n + 2) + 2(-n)$



$v(n) = 2 + 2 = 4$

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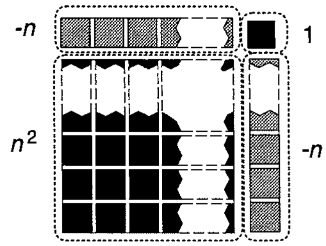
Actions

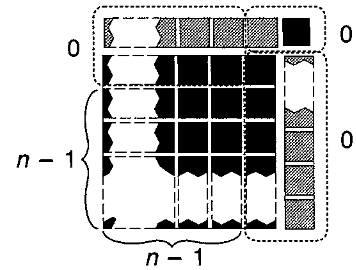
Comments

6. *Continued.*

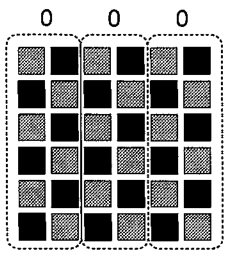
(c) Here are two possibilities:



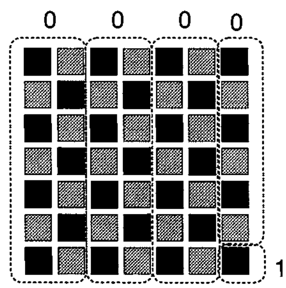
$v(n) = n^2 + 2(-n) + 1$



$v(n) = (n - 1)^2$



$v(6) = 0$

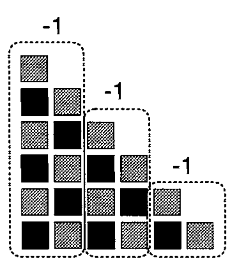


$v(7) = 1$

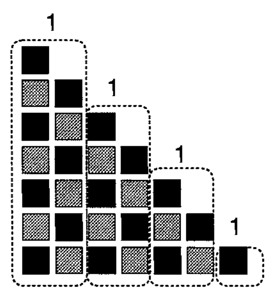
(d) Since two adjacent columns in the nth arrangement are opposites of each other, the net value of two adjoining columns is 0. Hence, if n is even, $v(n) = 0$. If n is odd, the remaining column, as shown in the example, has a net value of 1. So, if n is odd, $v(n) = 1$.

To summarize,

$$v(n) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$



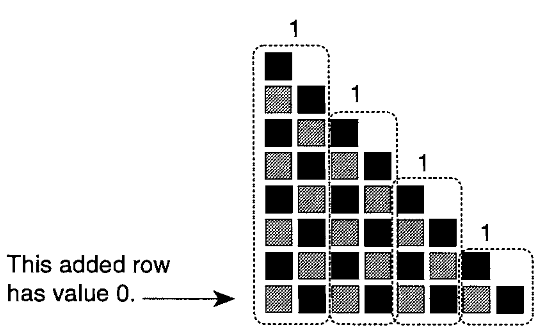
$v(6) = \frac{6}{2}(-1) = -3$



$v(7) = \frac{6}{2} + 1 = 4$

(e) If n is even, each two successive columns in the nth arrangement has a value of –1. There are $^n\!/_2$ pairs of columns. Thus, if n is even, $v(n)$ is $(^n\!/_2)(-1)$. If n is odd, each two successive columns has a value of 1. There are $^{(n-1)}\!/_2$ pairs of columns with an additional single black tile, which also has value 1. Hence, if n is odd, $v(n)$ is $(^{(n-1)}\!/_2) + 1$.

Alternatively, for odd n, note that adding a bottom row of alternating tile does not change the value of the arrangement. Now there are $^{(n+1)}\!/_2$ pairs of columns, each pair having value 1. Hence, an equivalent expression for $v(n)$, for odd n, is $^{(n+1)}\!/_2$.



$v(7) = \frac{7+1}{2} = 4$

Summarizing,

$$v(n) = \begin{cases} ^{(n+1)}\!/_2 & \text{if } n \text{ is odd,} \\ ^{n}\!/_2(-1) & \text{if } n \text{ is even.} \end{cases}$$

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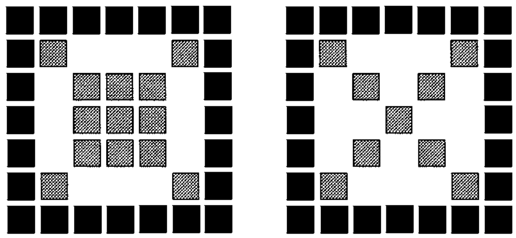
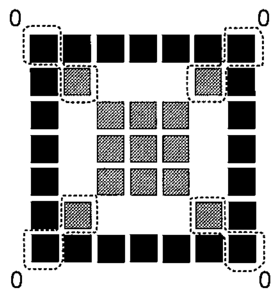
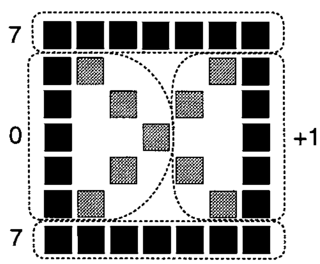
Actions



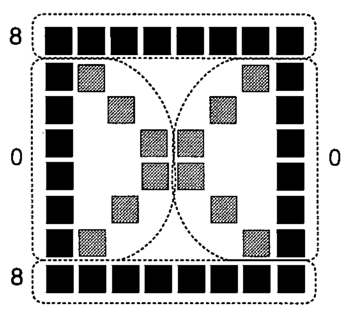
Possible 5th arrangements



$v(5) = 4(5) - 3^2$



$v(5) = 2(7) + 1$



$v(6) = 2(8)$

Comments

6. *Continued.*

(f) The students may disagree on how this sequence of arrangements continues. Some students may form the 5th arrangement as shown in the first figure, viewing the region within the square border of black tile as containing a red tile in each corner with an inner square of red tile. The dimension of this inner square increases by 1 from arrangement to arrangement. Other students may form the 5th arrangement as shown in the second figure, viewing the red tile as the diagonals of the square within the border of black tile.

If the sequence of arrangements is viewed as first described, in all but the first arrangement each red corner tile can be paired with a black corner tile as shown. Thus the nth arrangement (provided n is greater than 1) contains 4 pairs of corner tile whose net value is 0, 4 strips of n black tile connecting the corners and an $(n-2) \times (n-2)$ inner square of red tile.

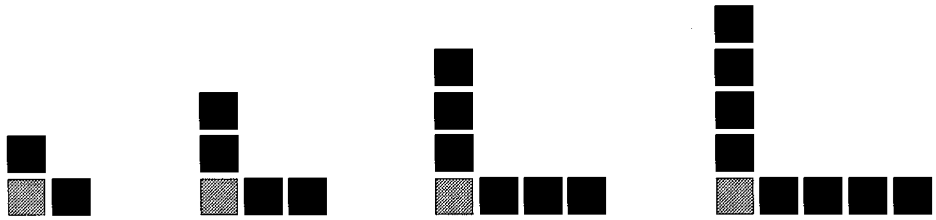
Thus,

$$v(n) = \begin{cases} 7 & \text{if } n = 1, \\ 4n - (n-2)^2 & \text{if } n > 1. \end{cases}$$

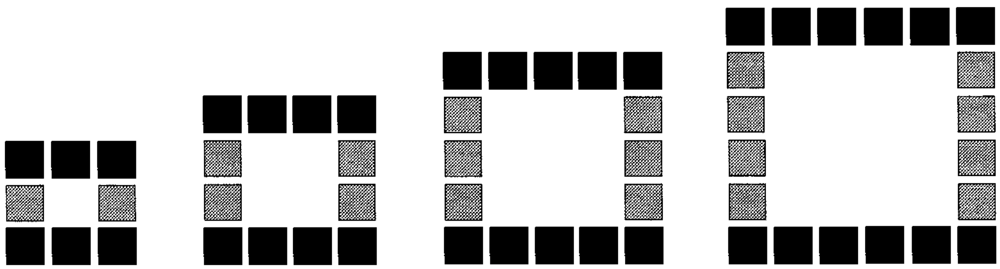
If an arrangement is viewed in the second manner, the portion of the arrangement between the top and bottom rows of black tile has a value of 1 if n is odd and a value of 0 if n is even. (See the examples.) Hence,

$$v(n) = \begin{cases} 2(n+2) + 1 & \text{if } n \text{ is odd}, \\ 2(n+2) & \text{if } n \text{ is even}. \end{cases}$$

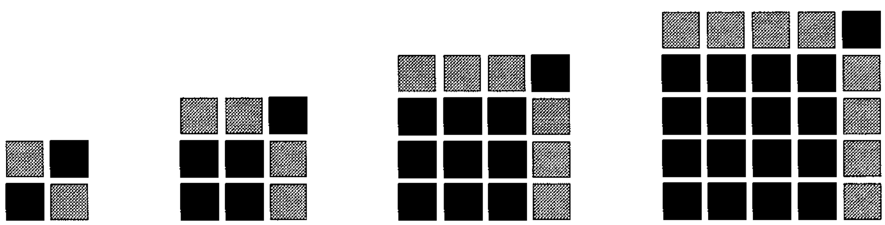
(a)



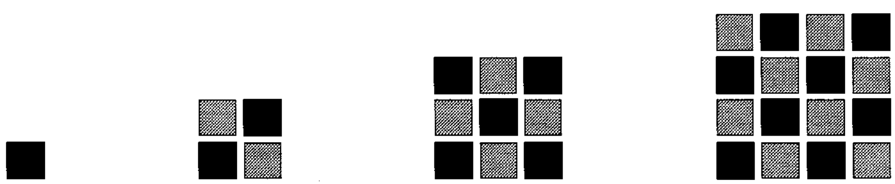
(b)



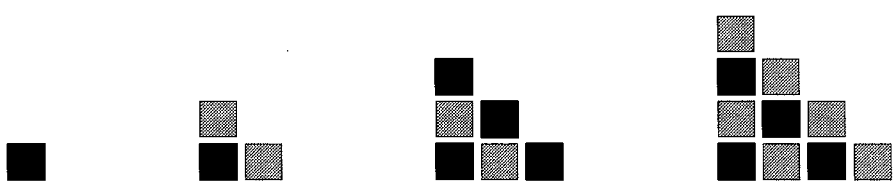
(c)



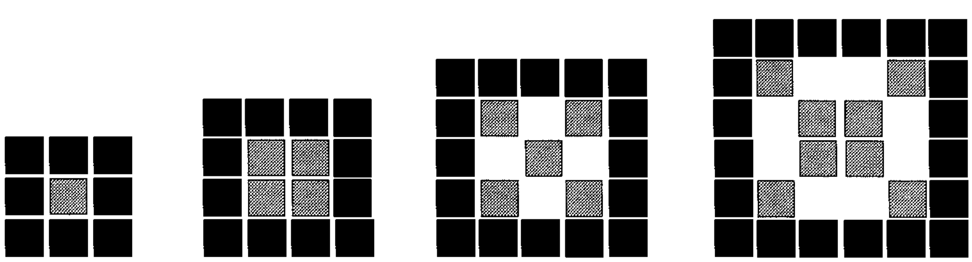
(d)



(e)



(f)



Counting Piece Patterns, Part II

OVERVIEW

Counting piece patterns are used to introduce equations involving negative integers.

Prerequisite Activity

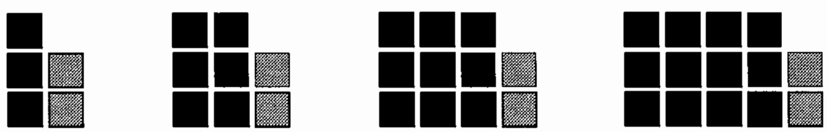
Unit IX, Activity 4, *Counting Piece Patterns, Part I.*

Materials

Algebra pieces (see Comment 2).

Actions

1. Show the students the following sequence of counting piece arrangements. Ask them to describe the 20th arrangement and determine its value.



2. Distribute algebra pieces to each student. Ask them to use their pieces to build a representation of the nth arrangement and then write an expression for $v(n)$.

Comments

1. There are various ways to describe the 20th arrangement. One way is to say it consists of 3 strips, each with 20 black tile, and 2 red tile. Following this description,

$$v(n) = 3(20) + (-2) = 58.$$

2. Algebra pieces consist of bicolored tile, n-strips (representing strips of n tile) and n^2-mats (representing $n \times n$ arrays of tile). These pieces are black on one side and red on the other.

Masters for algebra pieces are attached. If two-colored printing is not available, pieces can be printed on white cardstock or paper. In this case, red is represented by screened gray, as on this page. With reasonable care, two-sided copies can be made on standard copy machines. The two sides can be justified well enough to make usable pieces.

One master's worth of algebra pieces (2 n^2-mats, 10 n-strips, 8 individual tile) for each student or group of students will generally suffice.

Here is one representation of the nth arrangement:

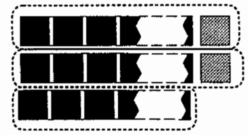
$$v(n) = 3n + (-2)$$

Actions

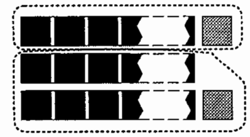
3. Ask the students to find equivalent expressions for $v(n)$. Discuss.

Comments

3. Grouping the algebra pieces in various ways leads to some equivalent expressions:

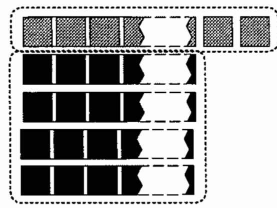


$v(n) = 2[n + (-1)] + n$

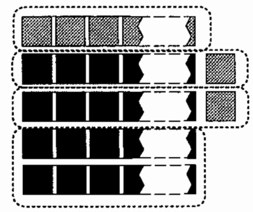


$v(n) = [n + (-1)] + [2n + (-1)]$

Other equivalent expressions for $v(n)$ can also be obtained by adding equal amounts of black and red pieces (a process which leaves the value of $v(n)$ unchanged) and grouping the resulting collection in various ways. In the following examples, a black and a red strip have been added.

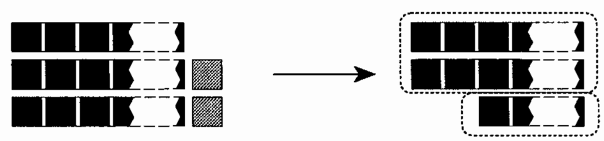


$v(n) = -(n + 2) + 4n$



$v(n) = -n + 2[n + (-1)] + 2n$

Equivalent expressions can also be obtained by removing equal amounts of black and red pieces. In the following sketch, 2 red and 2 black tile are removed, the black by cutting them off an n-strip.



$v(n) = 2n + (n - 2)$

Cutting strips can be avoided by placing blank tile over pieces to indicate they have been removed (see below). A master for blank tile and strips is attached.

Actions

4. Ask the students to determine which arrangement has a net value of 70. Discuss the students' methods. Relate their work to solving an equation.

Comments

4. The students will use various methods to determine the arrangement. They may find algebra pieces helpful in reaching a conclusion or explaining their thinking.

If the algebra piece representation of the nth arrangement shown below has value 70, then the 3 n-strips have a total value of 72 so each of them contains 24 tile. Since, in this case, the number of tile in a strip is also the number of the arrangement, it is the 24th arrangement which has value 70.

$v(n) = 70$

Another way of stating the conclusion is that 24 is the number n for which $v(n) = 70$ or, since $v(n) = 3n - 2$, 24 is the number n for which $3n - 2 = 70$. Thus in arriving at an answer, the students have solved the equation

$$3n - 2 = 70.$$

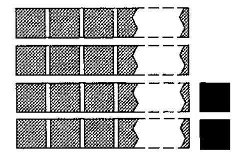
You can give the students other numbers and ask them to determine which arrangements, if any, have these numbers as values.

Actions

5. Distribute Activity Sheet IX-5 to the students. When they have completed the sheet, discuss their methods and conclusions.

Comments

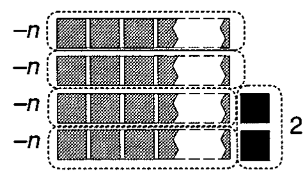
5. A. Here is one algebra piece representation of the nth arrangement for the sequence in part A:



$v(n) = -4n + 2$

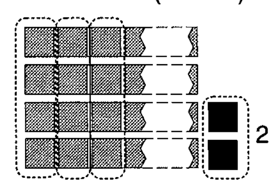
The 4 red n-strips can be thought of as the opposite of a collection of 4 black n-strips. Since 4 black n-strips have a value $4n$, the opposite collection has value $-(4n)$ or, dropping the parentheses, $-4n$.

Alternatively, since each red n–strip has value $-n$, the value of 4 of them can be written as $4(-n)$. The collection of 4 red n-strips can also be viewed as n collections of 4 red tile each, that is, n collections each of value -4, for a total value of $n(-4)$.



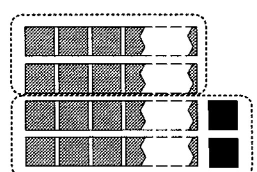
$v(n) = 4(-n)+ 2$

$-4\ -4\ -4\ \ldots$ (n times)

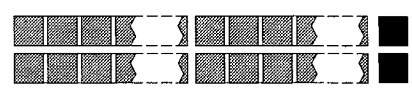


$v(n) = n(-4) + 2$

Below are two other variations for $v(n)$. In the sketch on the right, the pieces have been rearranged.



$v(n) = -2n + 2(-n + 1)$



$v(n) = 2(-2n + 1)$

(a) $v(15)$ can be determined by noting that the 15th arrangement will contain 4 rows of 15 red tile each and one column of 2 black tile, so its value is $4(-15) + 2$, or -58. It can also be determined by evaluating any one of the expressions for $v(n)$ when n is 15.

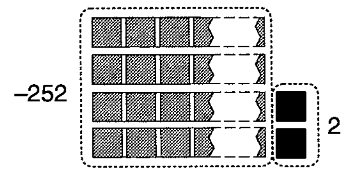
Continued next page.

Actions

Comments

5. A. *Continued.*

(b) If $v(n) = -250$, then the value of the nth arrangement is –250. Thus, the 4 red n-strips circled below must have a total value of –252. Hence, each strip must contain 63 red tile. Since the number of tile in a strip is the same as the number of the arrangement, $n = 63$. Note that the equation $-4n + 2 = -250$ has been solved.



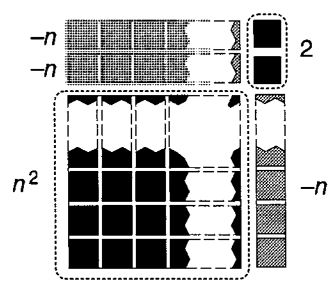
$v(n) = -250$

(c) Since the difference between successive arrangements is a column of 4 red tile, the $(n - 2)$nd arrangement can be obtained from the nth arrangement by removing 8 red tile. If $v(n) = -98$, then $v(n - 2) = -98 - (-8) = -90$.

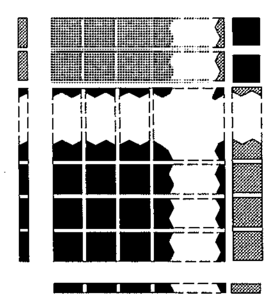
Alternately, one can first determine that if $v(n) = -98$ then $n = 25$. Thus, $n - 2$ is 23 and $v(23) = -4(23) + 2$ or –90.

B. Below are some expressions for $v(n)$ for the sequence in part B. Edge pieces, made by cutting strips and individual tile into fourths, may be helpful.

In writing expressions for $v(n)$, recall that adding red pieces to a collection has the same effect on its value as removing a like number of black pieces. Thus, for example, $n + (-2)$ and $n - 2$ are equivalent expressions.



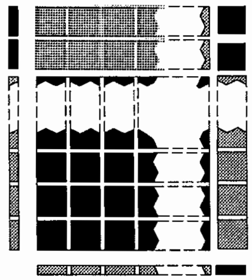
$v(n) = n^2 - 3n + 2$



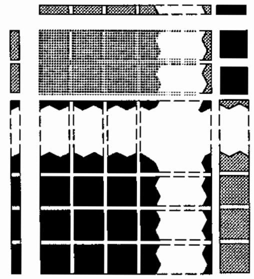
$v(n) = [n + (-2)]\,[n + (-1)]$
$= (n - 2)(n - 1)$

Continued next page.

5. B. *Continued.*
Edge pieces other than those shown on the previous page are possible. In the illustration to the side, the figure on the left shows a second possibility. The figure on the right shows an inappropriate selection of edge pieces. The students may need to be reminded that if edge pieces have the same color, the region associated with them is black, and if the edge pieces have opposite colors, the associated region is red.



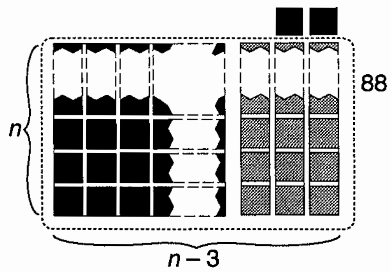
$v(n) = (-n + 1)(-n + 2)$



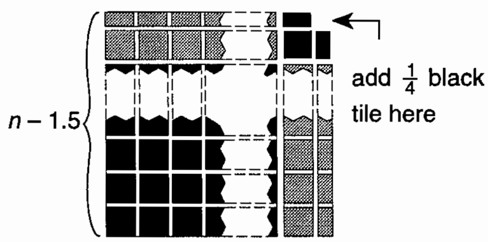
Incorrect edge pieces

(a) $v(20) = 342$.

(b) If $v(n) = 90$, the circled portion of the nth arrangement, rearranged as shown here, has value 88. This portion is a rectangle whose dimensions have value n and $n - 3$. Two numbers which differ by 3 and whose product is 88 are 11 and 8. Hence, $n = 11$.

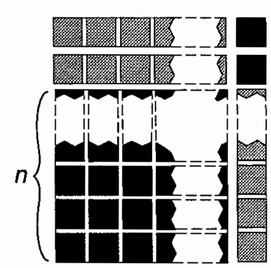


Some students may be interested in determining n by completing the square. If a red strip and a black tile are cut in half and the pieces rearranged as shown, adding ¼ black tile results in a $(n - 1.5) \times (n - 1.5)$ square whose value is 90.25. Using a calculator, $\sqrt{90.25} = 9.5$. Thus, $n - 1.5 = 9.5$ and hence $n = 11$.

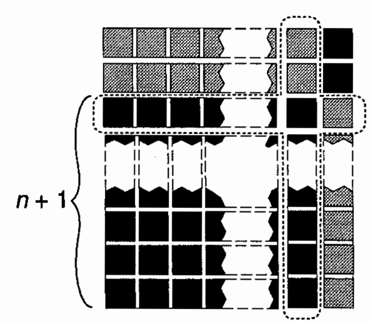


$v(n) = 90$

(c) If $v(n + 1) - v(n) = 50$, then the circled region of the $(n + 1)$st arrangement, shown here, has value 50. Hence, the 2 n-strips in this region have a total value of 52 so each contains 26 tile. Hence $n = 26$.



nth arrangement

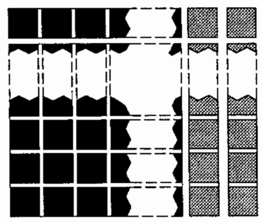


$(n + 1)$st arrangement

Actions

6. Show the students the algebra piece representations of the nth arrangements of Sequence 1 and Sequence 2 pictured below. Have the students find the values of these 2 arrangements.

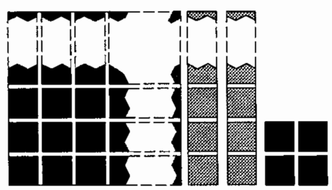
Sequence 1



nth arrangement

$v_1(n) = ?$

Sequence 2



nth arrangement

$v_2(n) = ?$

7. Ask the students to determine for what n the nth arrangement of Sequence 1 has the same value as the nth arrangement of Sequence 2. Discuss their methods.

Comments

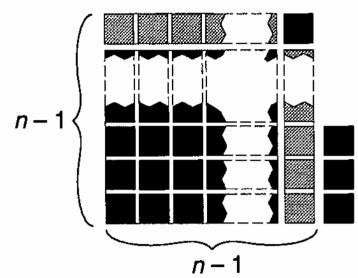
6. A master for an overhead transparency is attached. Subscripts have been used to distinguish $v(n)$ for Sequence 1 from $v(n)$ for Sequence 2.

There are a variety of equivalent expressions for $v_1(n)$. Here are three:

$$n^2 - n - 2,$$
$$(n + 1)(n - 2),$$
$$(n^2 + n) - (2n + 2).$$

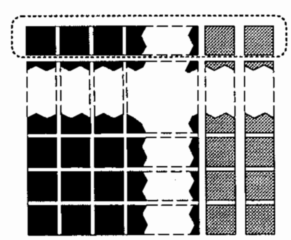
Following are some expressions for $v_2(n)$. The last one is suggested by rearranging the pieces as shown.

$$n^2 - 2n + 4,$$
$$n(n - 2) + 4,$$
$$(n - 1)^2 + 3.$$

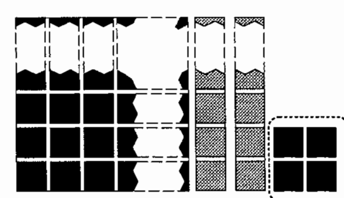


7. The two arrangements will have the same value if the circled portions shown below have the same value. This will be the case if the black n-strip contains 6 tile. Thus, $v_1(n) = v_2(n)$ when $n = 6$. That is, 6 is the solution of

$$n^2 - n - 2 = n^2 - 2n + 4.$$



$v_1(n) = n^2 - n - 2$



$v_2(n) = n^2 - 2n + 4$

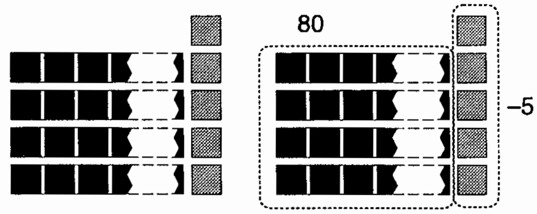
Actions

8. Ask the students to find a positive integer n such that:

(a) $4n - 5 = 75$ (b) $-3n + 7 = -92$
(c) $n^2 + 4 = 200$ (d) $5n - 4 = 3n + 6$
(e) $n^2 - 2n = 120$ (f) $(n - 1)(2n + 3) = 2n^2 + 5$

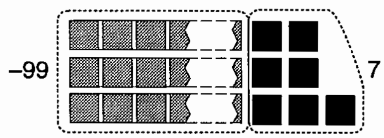
Comments

8. The equations can be solved by interpreting them as questions about counting piece patterns. For example, equation (a) can be thought of as asking which arrangement has value 75 in a sequence of tile arrangements for which $v(n) = 4n - 5$. The nth arrangement shown below has value 75 if the 4 n-strips contain a total of 80 tile. Hence, each n-strip contains 20 tile and thus $n = 20$.



$v(n) = 4n - 5$ $v(n) = 75$

(b) If the nth arrangement shown here has value –92, the 3 n-strips contain a total of 99 red tile. Hence, each n-strip contains 33 tile. So, $n = 33$.

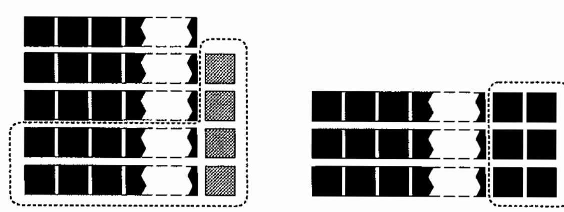


$v(n) = -92$

(c) If the value of the nth arrangement shown here is 200, the n^2-mat contains 196 tile. Thus $n^2 = 196$ and $n = 14$.

$n^2 + 4$

(d) The arrangements shown have the same value provided the circled portions have the same value. The portion on the right has value 6. The portion on the left will have value 6 if, together, the 2 n-strips have value 10. Hence, $n = 5$.



$5n - 4$ $3n + 6$

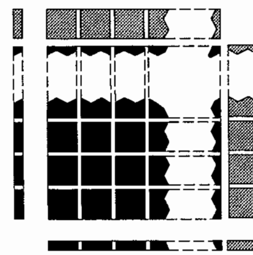
Continued next page.

Actions

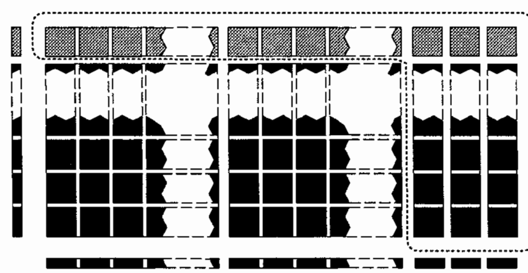
Comments

8. *Continued.*

(e) The arrangement shown here has value $n^2 - 2n$. Adding a black tile results in a square with edge $n - 1$. The original arrangement has value 120. So the completed square has value 121, that is, $n - 1$ is 11. Thus $n = 12$.



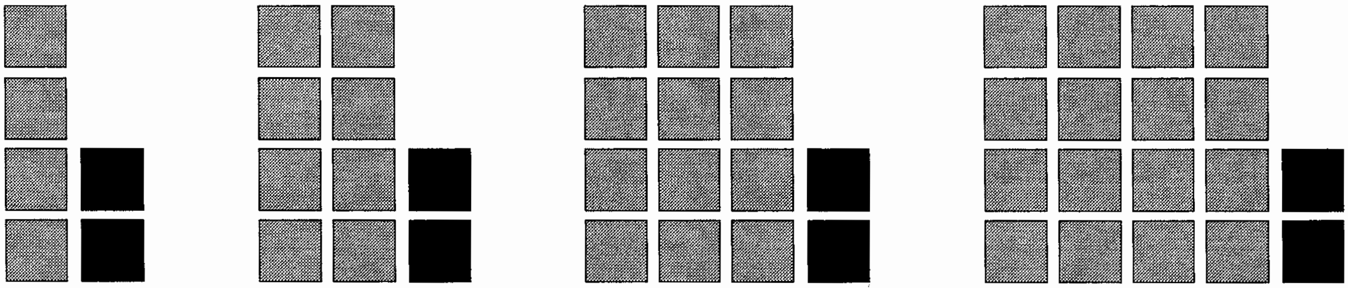
(f) An arrangement with value $(n - 1)(2n + 3)$ can be represented by a rectangular array whose edges have values $n - 1$ and $2n + 3$. This representation has value $2n^2 + 5$ if the circled portion has value 5. The collection in the circled portion has value $n - 3$. This equals 5 if n is 8.



$(n - 1)(2n + 3)$

Name ______________________________

A.



Shown above are the first 4 arrangements in a sequence of counting piece arrangements. List some equivalent expressions for $v(n)$.

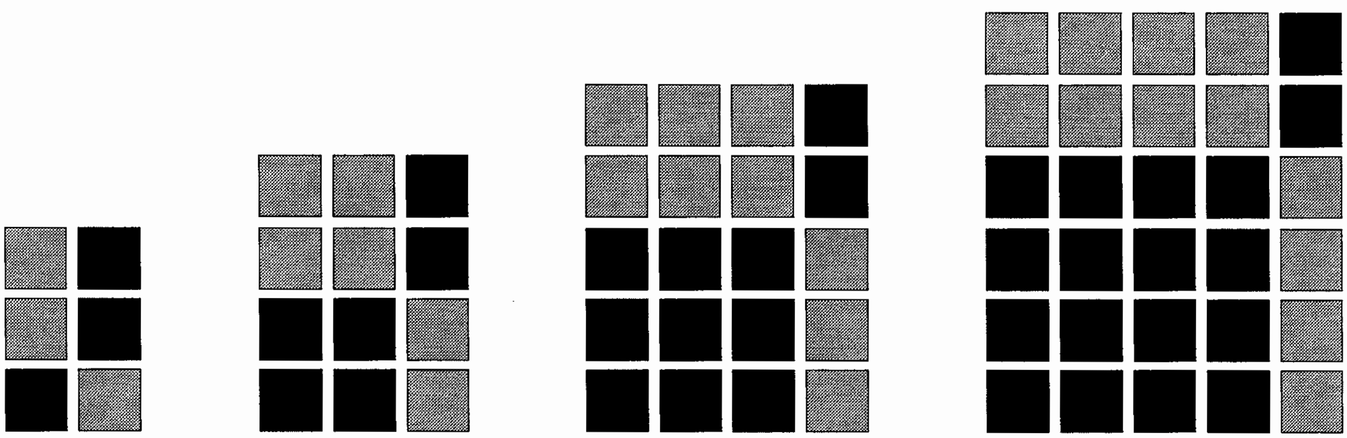
Complete the following statements:

(a) $v(15)$ = __________.

(b) If $v(n) = -250$, then n = __________.

(c) If $v(n) = -98$, then $v(n-2)$ = __________.

B.



Shown above are the first 4 arrangements in a sequence of counting piece arrangements. List some equivalent expressions for $v(n)$.

Complete the following statements:

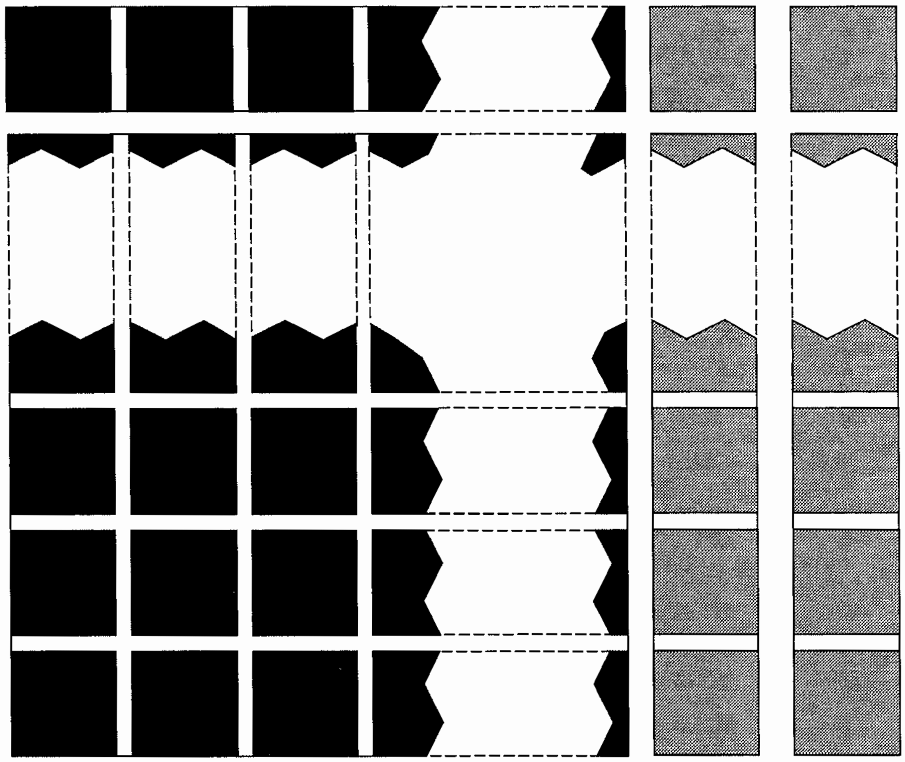
(a) $v(20)$ = __________.

(b) If $v(n) = 90$, then n = __________.

(c) If $v(n+1) - v(n) = 50$, then n = __________.

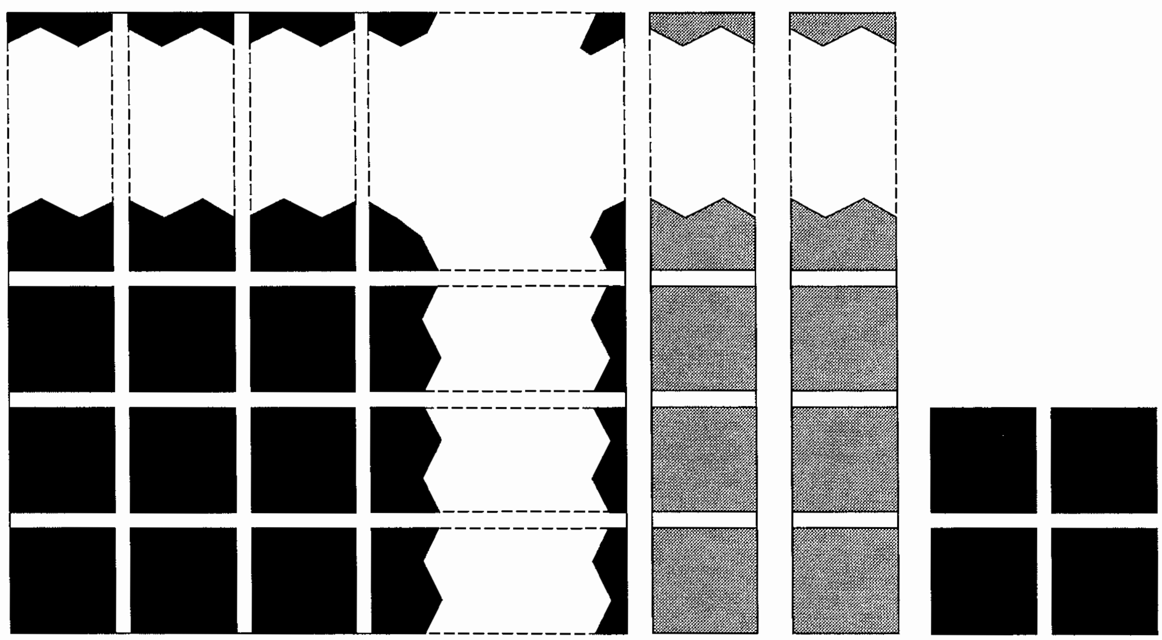
*n*th Arrangements

Sequence 1



$v_1(n) = ?$

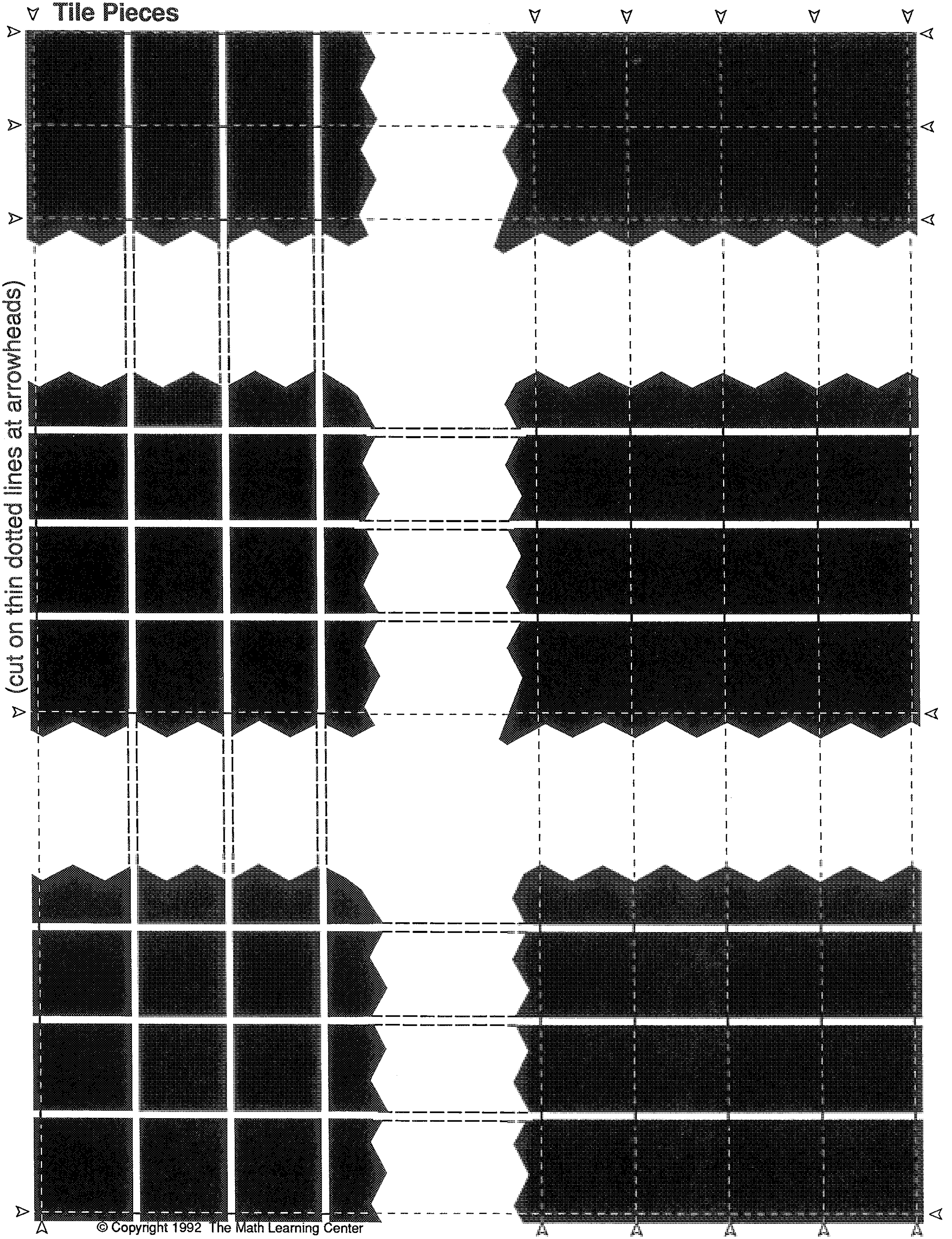
Sequence 2



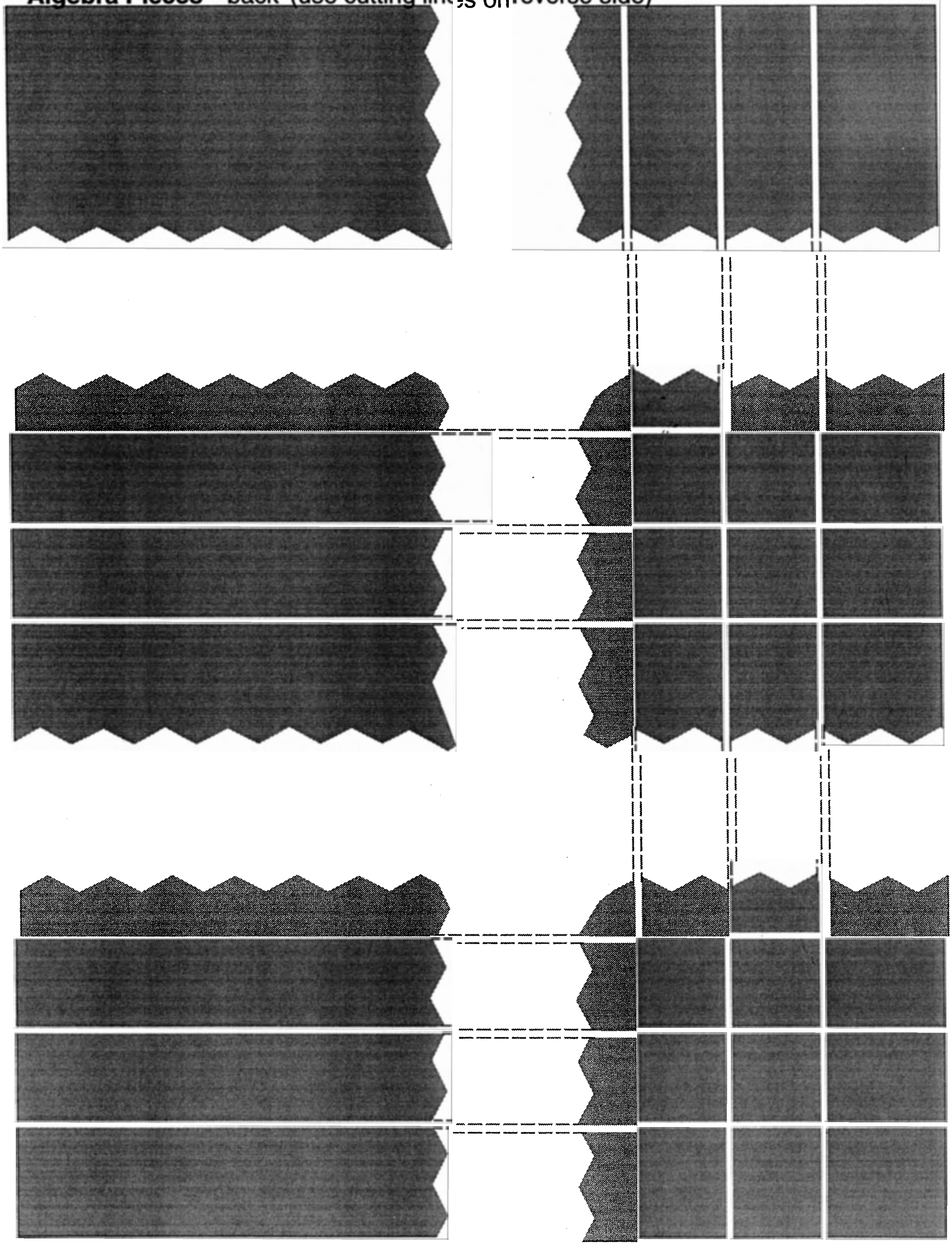
$v_2(n) = ?$

Tile Pieces

(cut on thin dotted lines at arrowheads)



Algebra Pieces—back (use cutting lines on reverse side)



cut on dotted lines

Blank Tile and Strips

Unit IX • Activity 6

Counting Piece Patterns, Part III

O V E R V I E W

Counting piece patterns are extended to include arrangements corresponding to non-positive, as well as positive, integers.

Prerequisite Activity

Unit IX, Activity 5, *Counting Piece Patterns, Part II.*

Materials

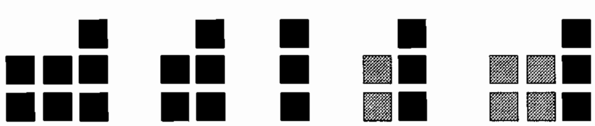
Counting pieces, n-strips and frames (see comment 8).

Actions

1. Distribute counting pieces to each student or group of students. Ask each student or group of students to form the following collection of counting piece arrangements. Then ask the students to add arrangements to the collection which maintain its pattern.

2. Discuss how the collection might be extended indefinitely in two directions.

3. Repeat Actions 1 and 2 for the following collection of arrangements.



Comments

1. Each student or group of students will need about 75 counting pieces and 10 or so n-strips.

The students may extend the pattern in one direction only. If that happens, ask them to extend the pattern in the other direction also.

2. Shown below is one way to extend the collection. Going to the right, a column of 3 black tile is added to an arrangement to get the next arrangement. Going to the left, a column of 3 red tile is added.

A master for an overhead transparency, the top half of which shows the extended collection, is attached (Master 1). The collection shown on the bottom half of the master is introduced in the next Action.

3. The collection can be extended to the right by adding a column of 2 red tile to an arrangement to get the next arrangement. It can be extended to the left by adding a column of 2 black tile to successive arrangements.

A master for the extended collection is attached (see Comment 2).

Actions

4. Discuss with the students how the arrangements in the extended collections might be numbered.

5. Ask the students to number the extended sequences as suggested in Comment 4 and then, in each case, write an expression for $v(n)$, the value of the arrangement numbered n.

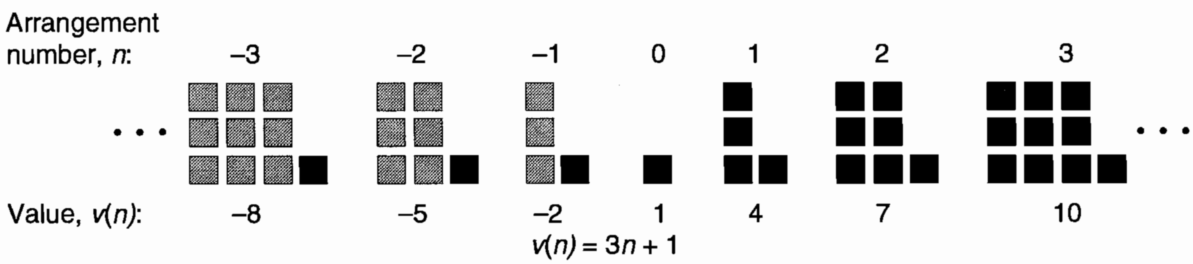
Comments

4. One way of numbering the arrangements is to select one of them and number it 0. Arrangements to the right of this arrangement are successively numbered 1, 2, 3, etc. Those to the left are successively numbered −1,−2, −3, etc.

A collection of arrangements which extends indefinitely in two directions and is numbered so there is an arrangement which corresponds to every interger, positive, negative and zero, will be called an *extended sequence*.

Mathematically speaking, a set of arrangement numbers is called an *index set* and an individual arrangement number is called an *index*. Thus, an arrangement whose number is −3 could be referred to as "the arrangement whose index is −3". Instead of using this language, we shall refer to this arrangement as "arrangement number −3" or "the −3rd arrangement". In the language of index sets, a *sequence* is a collection of arrangements whose index set is the set of positive integers and an *extended sequence*—a phrase coined for our purposes—is a set of arrangements whose index set is the set of all integers. On occasion, once a set of arrangements has been determined to be an extended sequence, it will be referred to simply as a sequence, the word "extended" being understood.

5. The expression for $v(n)$ will depend on the choice of numbering. If, in the first instance, the arrangement consisting of a single black tile is numbered 0, then $v(n) = 3n + 1$. Notice this formula holds for all integers n, positive, negative or zero.



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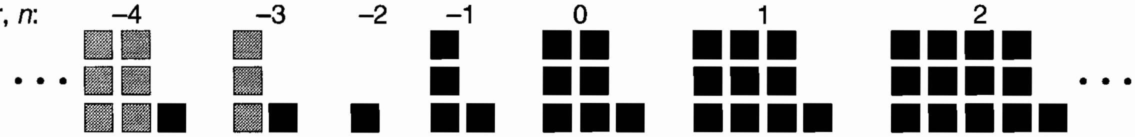
Actions

Comments

5. *Continued.* A different numbering will result in a different expression for $v(n)$. For example, if the arrangement which consists of 7 black tile is numbered 0, then $v(n) = 3n + 7$.

Arrangement number, n: −4 −3 −2 −1 0 1 2

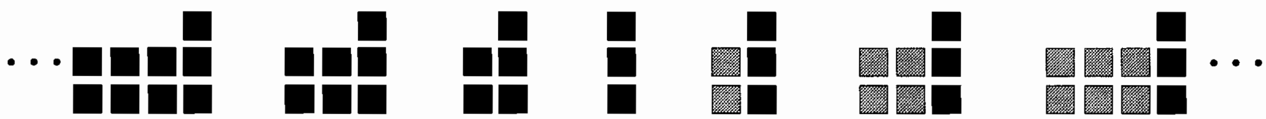


Value, $v(n)$: −5 −2 −1 4 7 10 13

$v(n) = 3n + 7$

For the second extended sequence, if the single column of 3 black tile is designated the 0th arrangement, then $v(n) = 2(-n) + 3 = -2n + 3$.

Arrangement number, n: −3 −2 −1 0 1 2 3



Value, $v(n)$: $2(3) + 3$ $2(2) + 3$ $2(1) + 3$ 3 $2(-1) + 3$ $2(-2) + 3$ $2(-3) + 3$

$v(n) = -2n + 3$

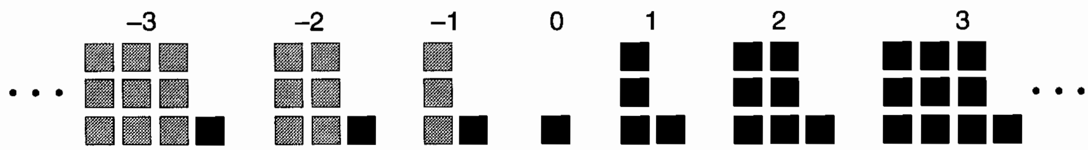
A master is attached for an overhead transparency on which numberings of the two extended sequences and the associated expressions for $v(n)$ can be shown (Master 2).

Actions

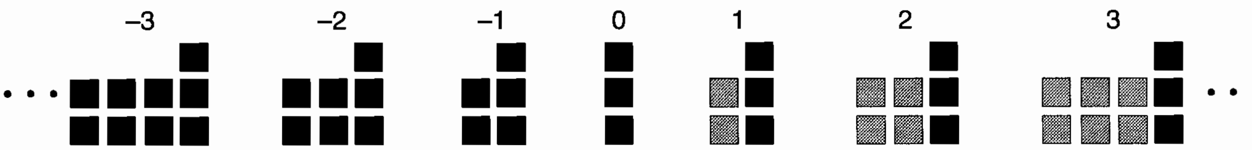
6. In this Action and Actions 7–9, assume the arrangements are numbered as shown below. For each extended sequence, ask the students to describe (a) the 50th arrangement and (b) the –100th arrangement.

Comments

6. For the first extended sequence, the 50th arrangement has 50 columns of 3 black tile with an adjoining black tile. Alternatively, it can be described as 3 rows of 50 black tile with an adjoining black tile. Other descriptions are possible.



$v(n) = 3n + 1$



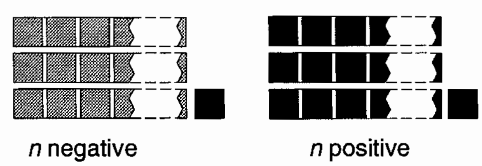
$v(n) = -2n + 3$

The –100th arrangement (that is, the arrangement numbered –100) has 3 rows of 100 red tile each and an adjoining black tile. Note that the number of red tile in each row is the opposite of the number of the arrangement.

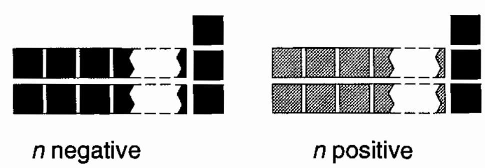
For the second extended sequence, the 50th arrangement has 2 rows of 50 red tile each and a column of 3 adjoining black tile. The –100th arrangement has 2 rows of 100 black tile each and an adjoining column of 3 black tile. Other descriptions are possible.

7. Distribute n-strips to the students. For each extended sequence, ask the students to build a representation of the nth arrangement for n positive. Repeat for n negative.

7. One way of forming the arrangements is shown below.



$v(n) = 3n + 1$



$v(n) = -2n + 3$

Actions

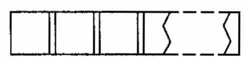
8. Introduce *n-frames* and *–n-frames*. Ask the students to use them to build representations of the *n*th arrangement for the extended sequences shown in Action 6.

Comments

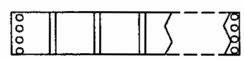
8. In Action 7, the strips used to form the arrangements for positive *n* are a different color than those used for negative *n*, e. g., in the second sequence, red strips are used for positive *n* and black strips for negative *n*.

Frames are introduced to provide pieces that are sometimes red and sometimes black. An *n-frame* contains black tile if *n* is positive, red tile if *n* is negative, and no tile if *n* is 0. In all cases, the total value of the tile it contains is *n*. Thus, if *n* is positive, it contains *n* black tile and if *n* is negative, it contains $-n$ (or $|n|$)red tile. (E.g., if $n = -50$, an *n*-frame contains $-(-50)$, or 50, red tile. The value of 50 red tile is -50.)

A *–n-frame* is the opposite of an *n*-frame. It contains red tile if *n* is positive, black tile if *n* is negative and no tile if *n* is 0. In all cases, the total value of the tile it contains is $-n$. Thus, if *n* is positive, it contains *n* red tile and if *n* is negative, it contains $-n$ (or $|n|$) black tile. A $-n$-frame is distinguished from an *n*-frame by the small o's on each end.



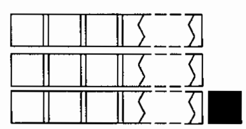
n-frame
Has value *n* for all *n*, positive, negative or zero.



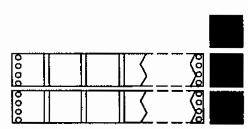
$-n$-frame
Has value $-n$ for all *n*, positive, negative or zero.

Masters for *n*-frames and $-n$-frames are attached. They are intended to be printed back-to-back so that turning over an *n*-frame results in a $-n$-frame, and conversely.

Below are *n*th arrangements for each of the two extended sequences.



$v(n) = 3n + 1$



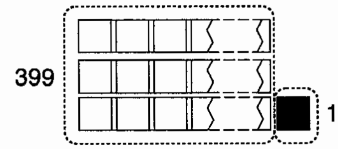
$v(n) = -2n + 3$

Actions

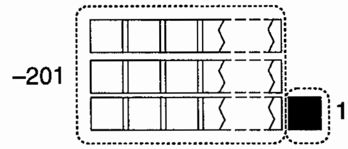
9. Still assuming the numberings in Action 6, ask the students to determine, in the first extended sequence, the number of the arrangement which has value (a) 400, (b) –200. Ask them to determine, in the second extended sequence, the number of the arrangement which has value (a) 165, (b) –75.

Comments

9. If an arrangement in the first extended sequence has value 400, the 3 *n*-frames in the *n*th arrangement shown below have a total value of 399. Hence, each has value 133, Thus, $n = 133$ and it is the 133rd arrangement which has value 400.

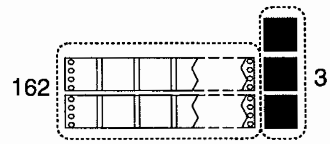


$v(n) = 3n + 1 = 400$



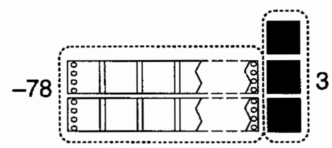
$v(n) = 3n + 1 = -200$

If an *n*th arrangement has value –200, the 3 *n*-frames in the figure have a total value of –201. Hence, each has value –67 and thus $n = -67$.



$v(n) = -2n + 3 = 165$

If the value of the *n*th arrangement of the second extended sequence is 165, then each of the two –*n*-frames in the figure has a value of 81. Hence, $-n = 81$ in which case, $n = -81$. So it is the –81st arrangement which has value 165.

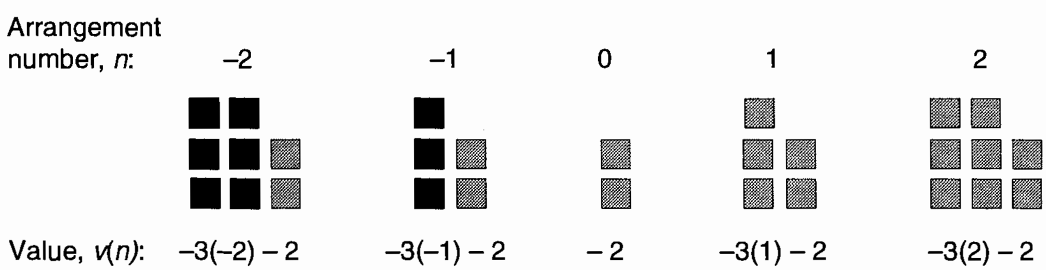


$v(n) = -2n + 3 = -75$

If the value of the *n*th arrangement shown on the right is –75, then each –*n*-frame in the figure has value –39. Hence, $-n = -39$. Thus, $n = 39$ and it is the 39th arrangement which has value –75.

10. Ask the students to build the –2nd, –1st, 0th, 1st and 2nd arrangements of an extended sequence for which $v(n) = -3n - 2$.

10. Here is one possibility for the requested arrangements:

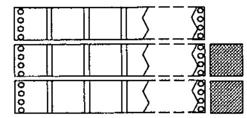


Actions

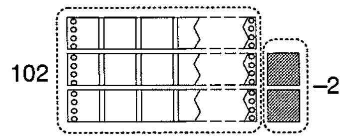
11. Ask the students to determine which arrangement, if any, of the extended sequence in Action 10 has value (a) 100, (b) 200, (c) –200.

Comments

11. It may be useful for the students to first form a representation of the nth arrangement. Since $v(n) = -3n - 2 = 3(-n) - 2$, one possible representation is the following:

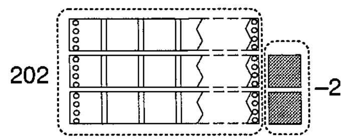


$v(n) = -3n - 2$



$v(n) = -3n - 2 = 100$

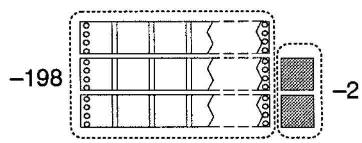
(a) If an nth arrangement has value 100, each $-n$-frame is $102 \div 3$ or 34. Thus $n = -34$.



$v(n) = -3n - 2 = 200$

This is not possible.

(b) This situation is not possible, since 202 is not a multiple of 3.

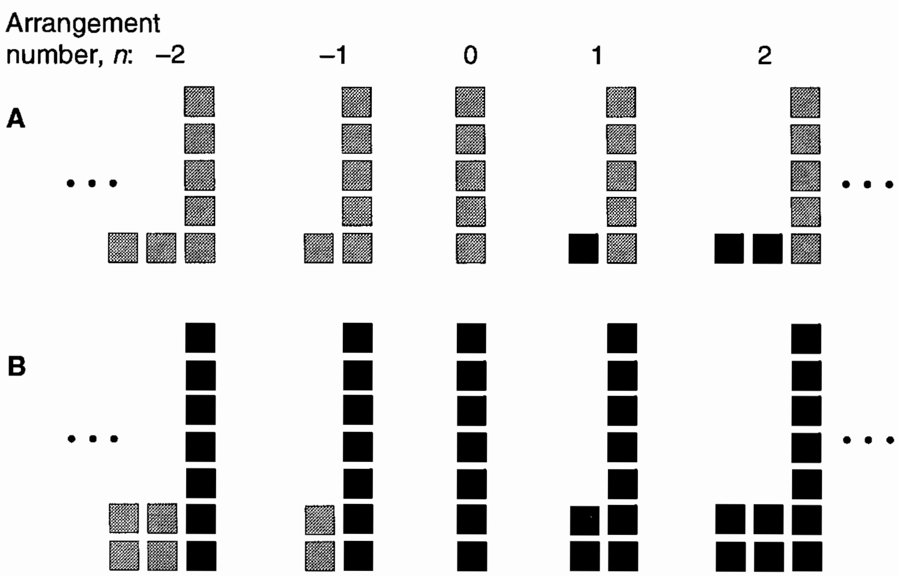


$v(n) = -3n - 2 = -200$

(c) In this case, $-n = -198 \div 3 = -66$. Hence $n = 66$.

Actions

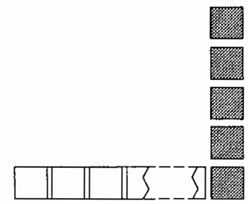
12. Show the students the following portions of two extended sequences, A and B. For each sequence, ask the students to build an algebra piece representation of the nth arrangement. Then ask them to determine for which n these two arrangements have the same value.



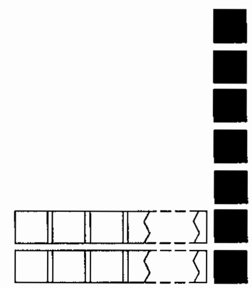
Comments

12. A master for an overhead transparency of the two extended sequences is attached (Master 3).

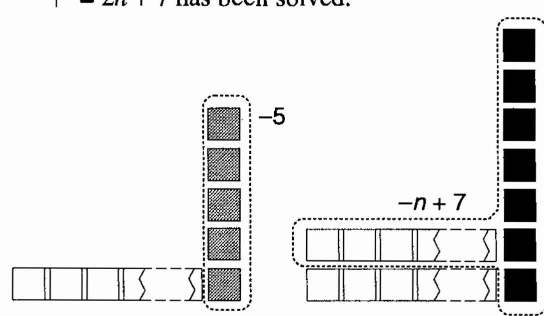
A representation for the nth arrangement of A:



A representation for the nth arrangement of B:

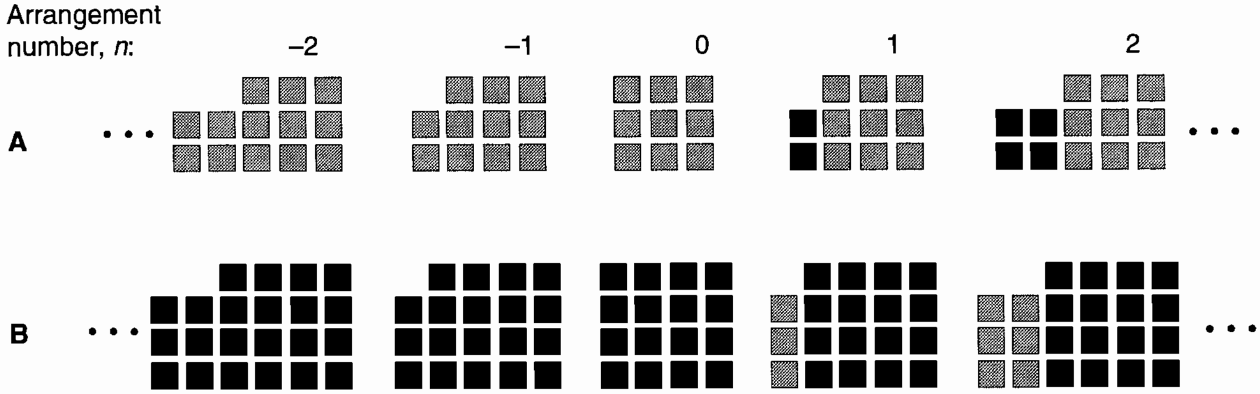


The two arrangements have the same value if the circled portions shown below have the same value. The portion on the left has value −5. The portion on the right will have this value if the enclosed n-frame has value −12, i.e., if $n = -12$. Note the equation $n - 5 = 2n + 7$ has been solved.

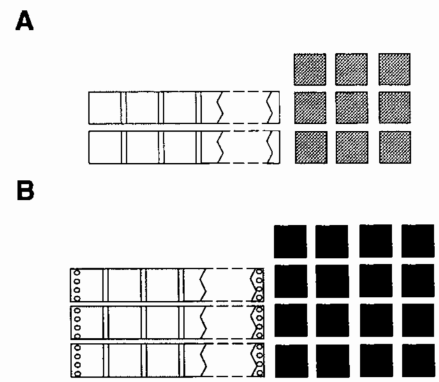


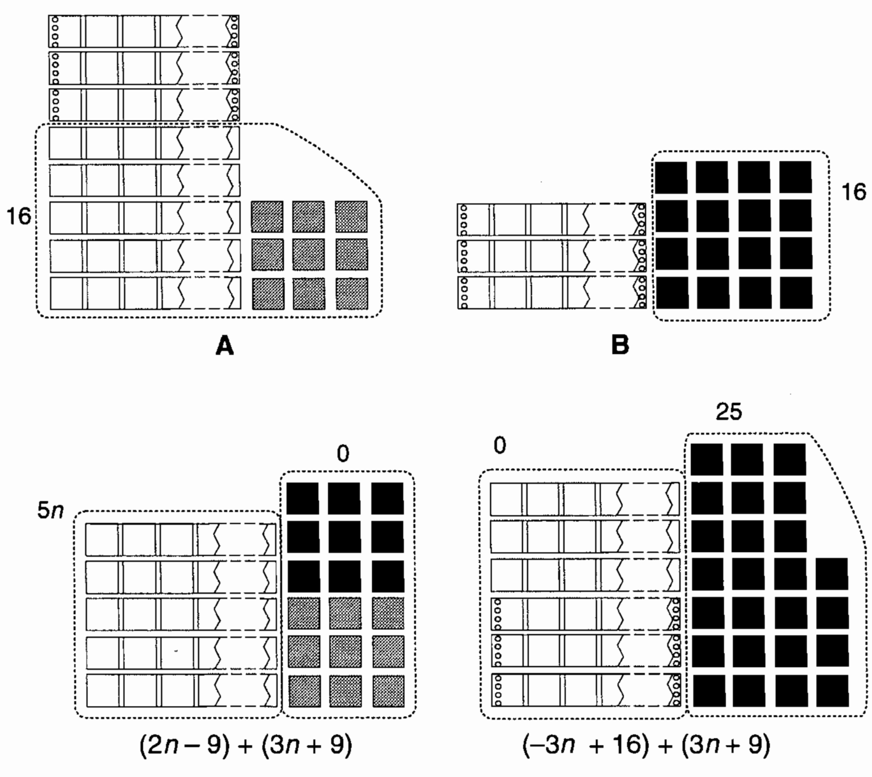
13. Repeat Action 12 for the following two extended sequences:

13. A master for the two extended sequences is attached (Master 4).



The nth arrangements are shown below.





The values of the arrangements, as they appear above, are difficult to compare. However, adding 3 n-frames and 3 $-n$-frames to the nth arrangement for A, as shown, does not change its value. The two arrangements have the same value if the circled portions have the same value. This happens if the 5 n-frames in the circled portion on the left have total value 25. This is the case when $n = 5$.

An alternative solution is based on the observation that if the same value is added to two arrangements, the resulting arrangements will have equal values provided the originals did, and vice versa. Shown here, values $3n + 9$ have been added to arrangements with values $2n - 9$ and $-3n + 16$. The resulting arrangements have equal values provided $5n = 25$ or, simply, $n = 5$

Note that the equation $2n - 9 = -3n + 16$ has been solved.

Actions

14. Ask the students to use algebra pieces or sketches to help them solve the following equations:

(a) $4n + 7 = -133$ (b) $8 - 5n = -142$

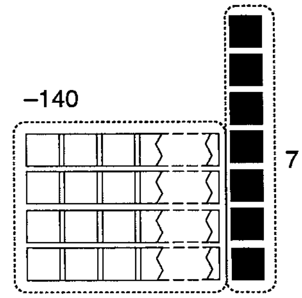
(c) $4n + 5 = 3n - 8$ (d) $8n - 4 = 6n + 10$

(e) $n + 10 = 1 - 2n$ (f) $6n - 64 = 2n$

(g) $3n - 81 = 6n + 84$ (h) $5n - 170 = 190 - 4n$

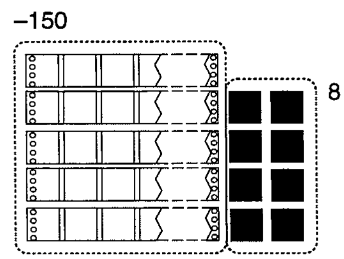
Comments

14. (a) An arrangement whose value is $4n + 7$ has value -133 if the 4 circled n-frames have total value -140. This is the case if $n = -35$:



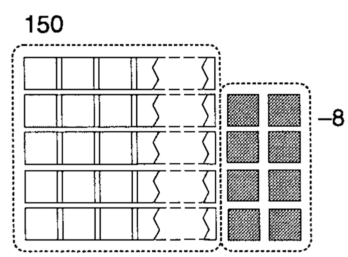
$$4n + 7 = -133$$
$$n = (-140) \div 4 = -35$$

(b) An arrangement with value $8 - 5n$ has value -142 when each $-n$-frame has value -30, that is, when $n = 30$:



$$8 - 5n = -142$$
$$-n = (-150) \div 5 = -30$$

Alternatively, an arrangement with value $8 - 5n$ has value -142 when the opposite arrangement has value 142:

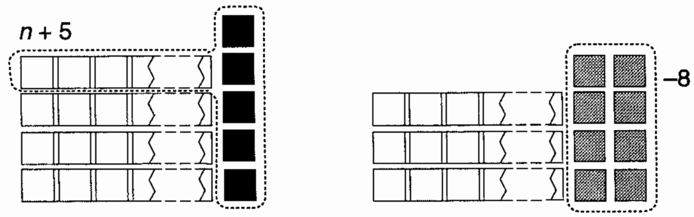


$$5n - 8 = 142$$
$$n = 150 \div 5 = 30$$

Continued next page.

Actions

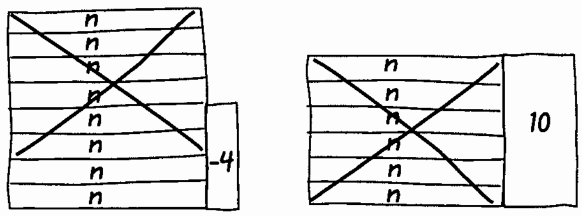
Comments



$4n + 5$ $3n - 8$

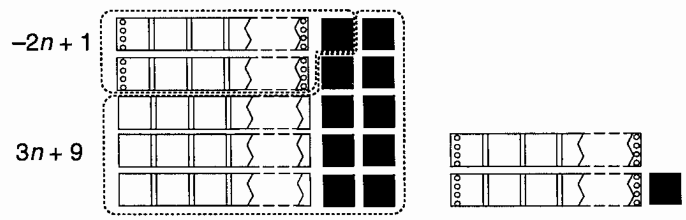
14. *Continued.*

(c) Arrangements with values $4n + 5$ and $3n - 8$ have the same value when the two circled portions each have value −8, in which case $n = -13$.



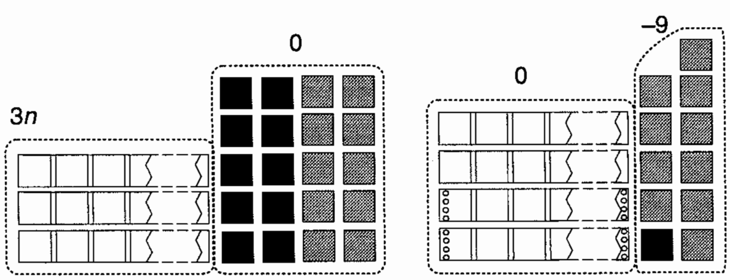
$8n - 4$ $6n + 10$

(d) As the sketches show, $8n - 4$ and $6n + 10$ have the same value when $2n - 4$ has value 10. This is so if $2n$ has value 14, in which case $n = 7$.



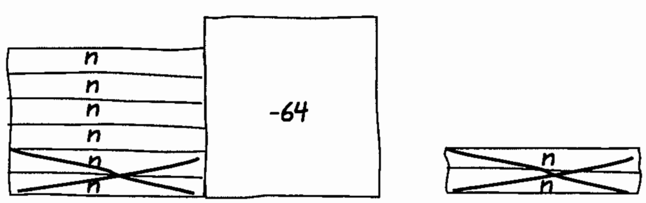
$n + 10$ $-2n + 1$

(e) Adding 2 $-n$-frames and 2 n-frames to an arrangement with value $n + 10$ doesn't change its value. Comparing it with an arrangement with value $1 - 2n$, shows the two arrangements have the same value if $3n + 9 = 0$ or $n = -3$.



$(n + 10) + (2n - 10)$ $(-2n + 1) + (2n - 10)$

An alternative solution is based on the last method of Comment 13. Starting with collections with values $n + 10$ and $1 - 2n$, respectively, and then adding 2 n-frames and 10 red tile to each collection results in two collections with values $3n$ and −9, respectively. Hence $n + 10 = 1 - 2n$ provided $3n = -9$. This is so if $n = -3$.



$6n - 64$ $2n$

(f) One sees that sketches for $6n - 64$ and $2n$ have the same value if $4n - 64 = 0$, which happens when $n = 16$.

Continued next page.

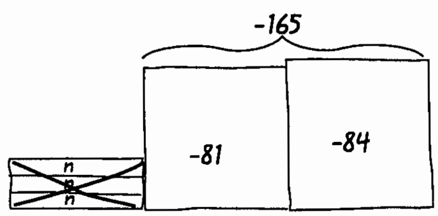
Actions

Comments

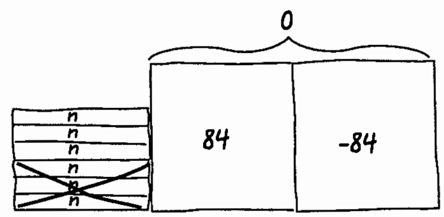
14. *Continued.*

(g) A section representing –84 has been added to sketches for $3n - 81$ and $6n + 84$. The sketches have equal values if $3n = -165$ or $n = (-165) \div 3 = -55$.

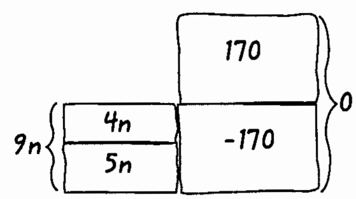
(h) As shown in the sketch below, if $4n + 170$ is added to $5n - 170$ and $190 - 4n$, the results have equal values provided $n = 40$.



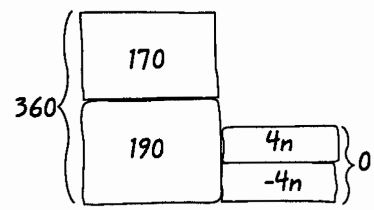
(3n – 81) – 84



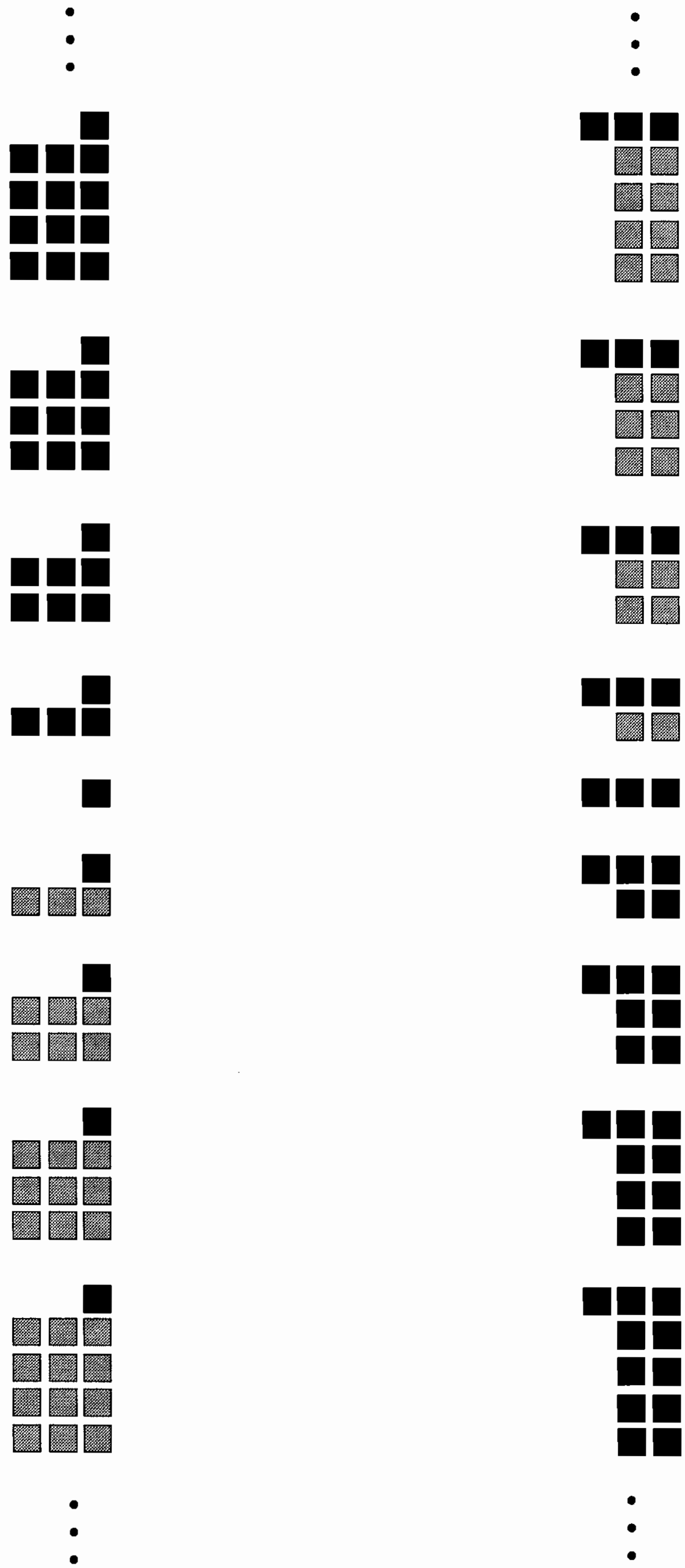
(6n + 84) – 84



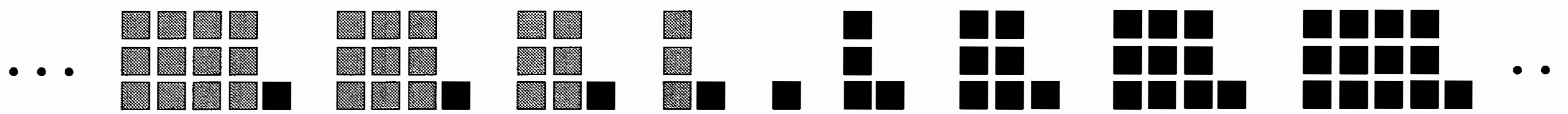
(5n – 170) + (4n +170)



(190 – 4n) + (4n +170)



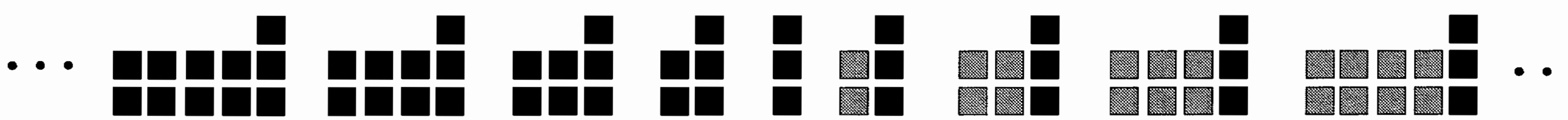
Arrangement number, n



Value, $v(n)$ –11   –8   –5   –2   1   4   7   10   13

$v(n) =$

Arrangement number, n

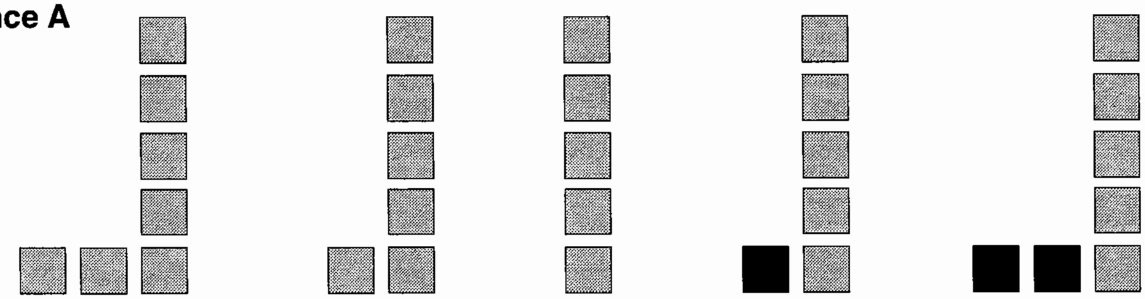


Value, $v(n)$ 11   9   7   5   3   1   –1   –3   –5

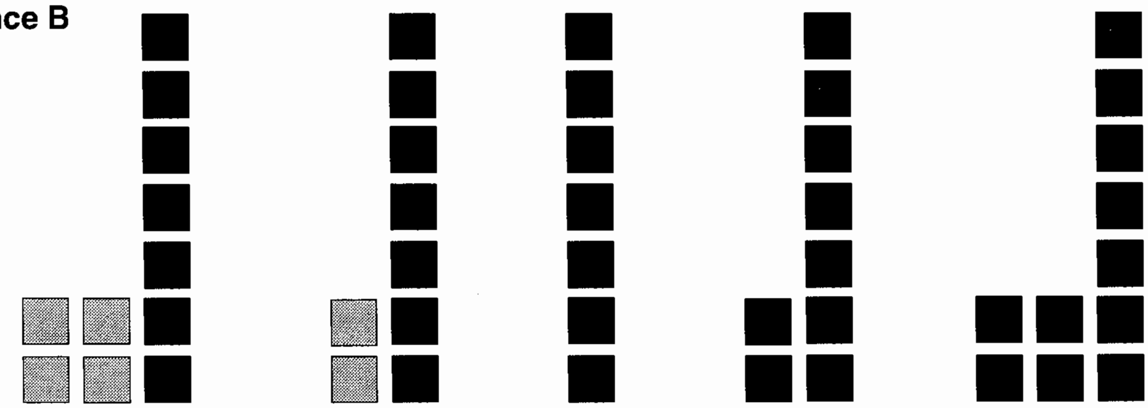
$v(n) =$

Arrangement number, n: −2 −1 0 1 2

Sequence A

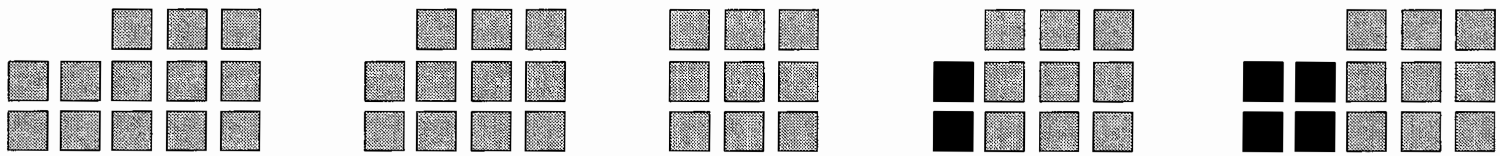


Sequence B

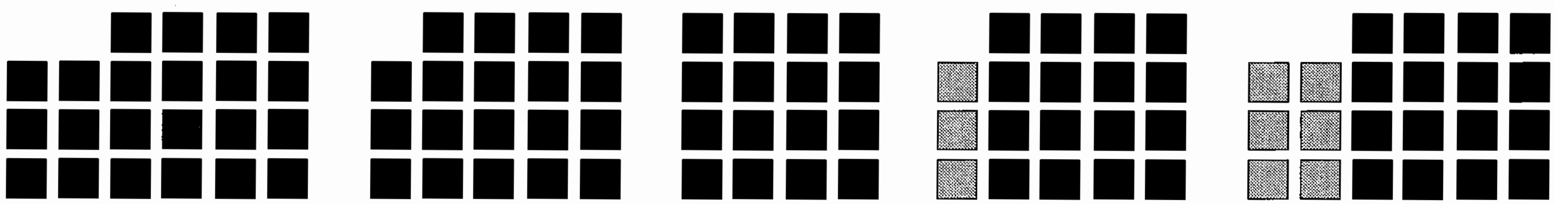


Arrangement number, *n*: −2 −1 0 1 2

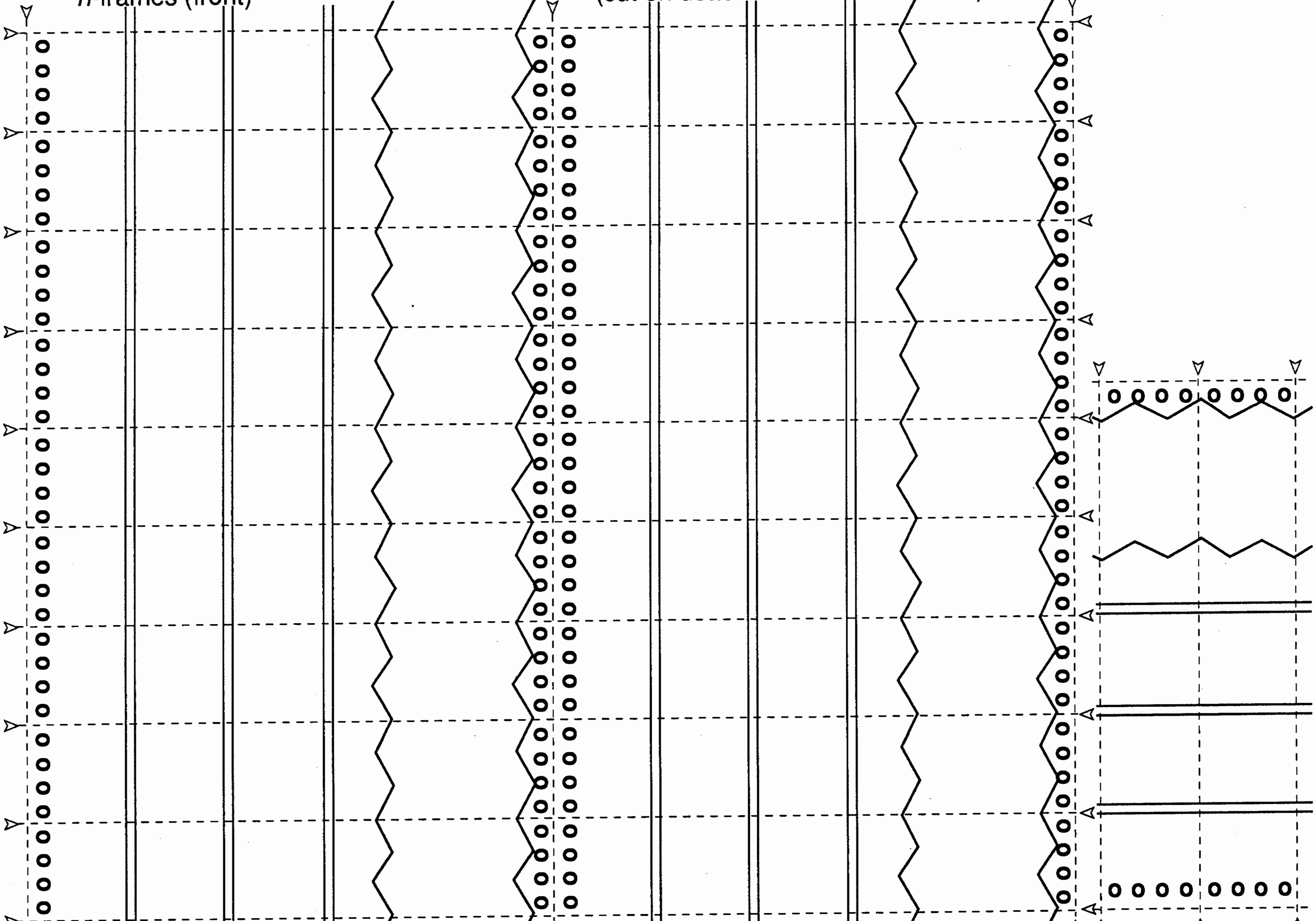
Sequence A



Sequence B



n-frames (front) (cut on dotted lines at arrowheads)



n-frames (back)

(cutting lines on reverse side)

Counting Piece Patterns, Part IV

OVERVIEW

Counting piece patterns are used to introduce quadratic equations.

Prerequisite Activity

Unit IX, Activity 6, *Counting Piece Patterns, Part III.*

Materials

Counting pieces including edge pieces, n^2-mats and frames.

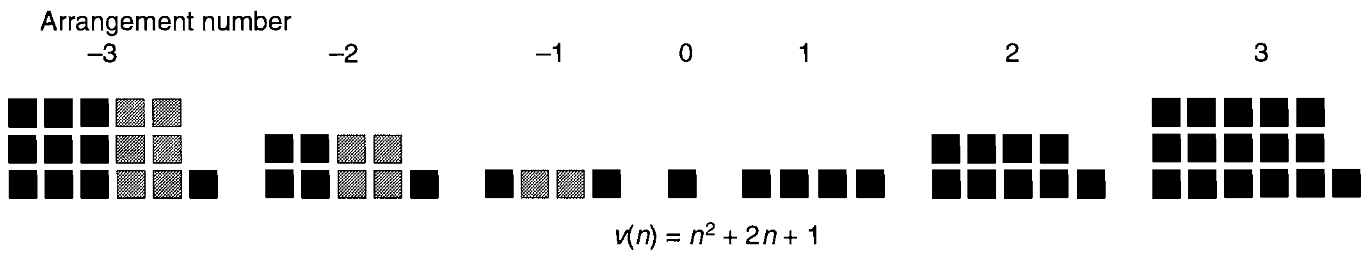
Actions

Comments

1. Distribute counting pieces to each student or group of students. Ask the students to form arrangements number –3, –2, –1, 0, 1, 2, 3 from an extended sequence of tile patterns for which $v(n) = n^2 + 2n + 1$.

1. Each student or group of students will need about 75 counting pieces, 50 edge pieces, 2 n^2-mats and 10 frames.

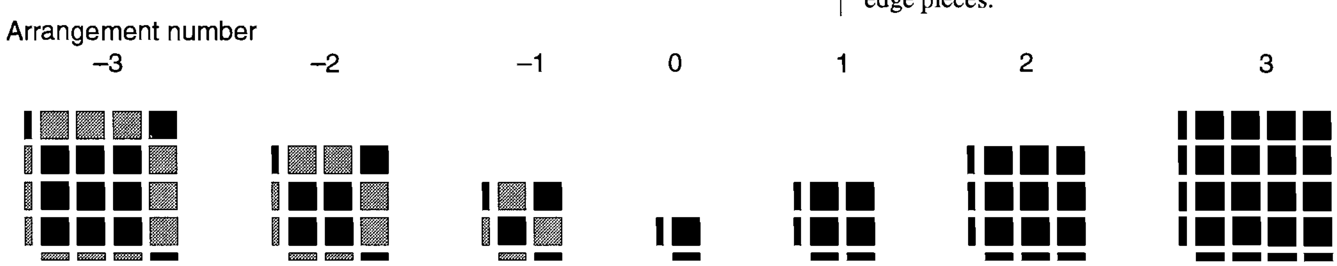
Shown below is one possible set of arrangements.



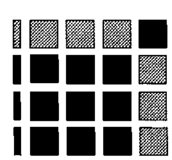
2. Point out that there exists a set of square arrangements which fit the criterion of Action 1. Ask the students to form such arrangements, using edge pieces to show the sides of the squares.

2. One or more students may form square arrangements in Action 1. If so, you may call the other students' attention to these arrangements.

Below is a set of square arrangements with edge pieces.



Other edges are possible. Here is another possibility for the –3rd arrangement:

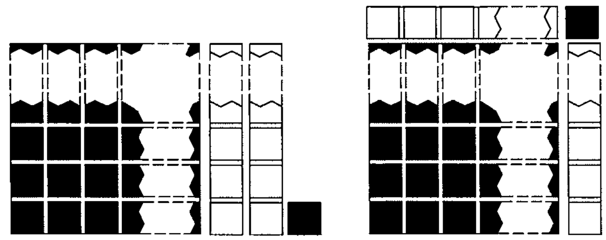


Actions

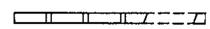
3. Ask the students to form the *n*th arrangement for the extended sequence of Action 1. Discuss how this arrangement can be displayed as a square with edge pieces.

Comments

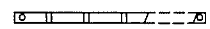
3. One possible *n*th arrangement is shown on the left below. The pieces can be rearranged to form a square as shown on the right.



It will be useful to have edge pieces whose color, like that of frames, differs for positive and negative *n*. Such pieces—referred to as *edge frames*—are obtained by cutting frames into fourths:



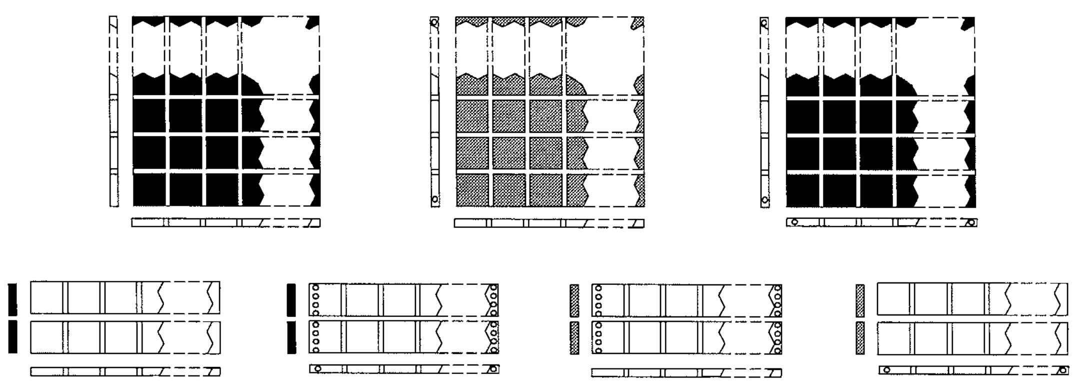
Has value *n* for *n* positive, negative or 0.



Has value –*n* for *n* positive, negative or 0.

Edge frames

The use of edge frames is illustrated below. Recall that black arrays have edges of the same color and red arrays have edges of opposite color. Also, an array with a black edge has the same color as its other edge while an array with a red edge has color which is opposite the color of its other edge.



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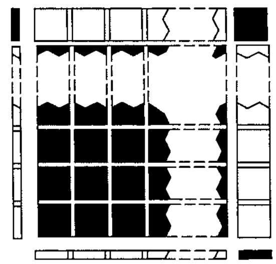
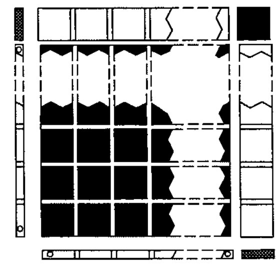
Actions

Comments

3. *Continued.* Below are two possibilities for edges of the nth arrangement shown above. Notice that the figures show that $(n + 1)^2$ and $(-n - 1)^2$ are equivalent expressions for $v(n)$.



$(n + 1)^2$



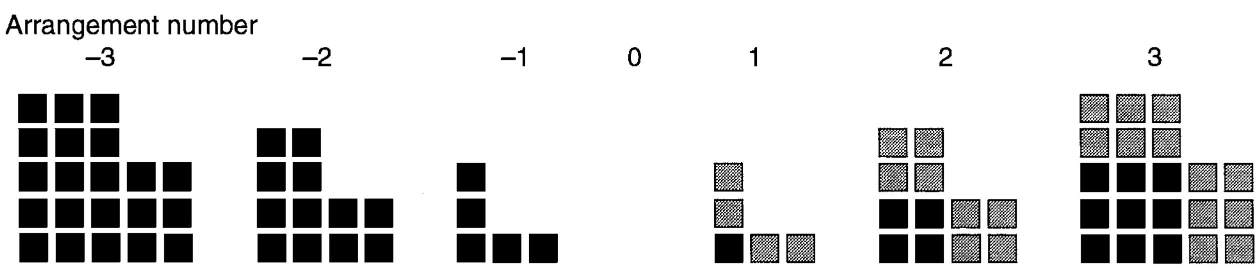
$(-n - 1)^2$

4. Ask the students to determine which arrangements in the extended sequence of Action 1 have a value of 400.

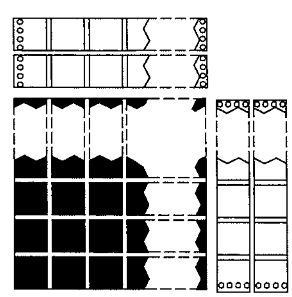
4. A square has value 400 provided its edge has value 20 or –20. Hence the nth arrangement, viewed as a square whose edge has value $n + 1$ (see the figure on the left above), has value 400 provided $n + 1$ has value 20 or -20. Since $n + 1$ is 20 when n is 19 and $n + 1$ is –20 when n is –21, the 19th and –21st arrangements have value 400.

5. Show the students the following extended sequence. Ask them to form the nth arrangement and write an expression for $v(n)$.

5. A master for an overhead transparency of the sequence is attached.



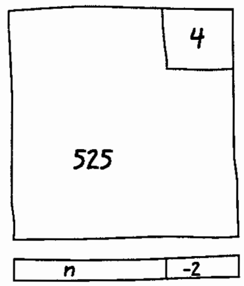
The nth arrangement contains a black n^2-mat and 4 $-n$-frames:



$v(n) = n^2 - 4n$

Actions

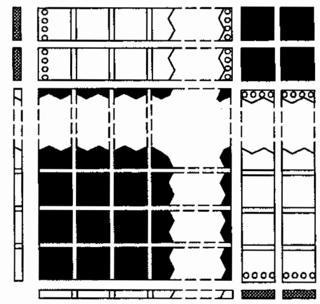
6. Ask the students to determine for what n the extended sequence in Action 5 has $v(n) = 525$.



23 or –23

Comments

6. Adding 4 black tile to an nth arrangement results in a square array whose edges have value $n - 2$, as shown below.

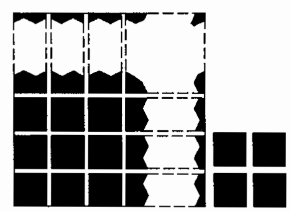


$v(n) + 4 = (n - 2)^2$

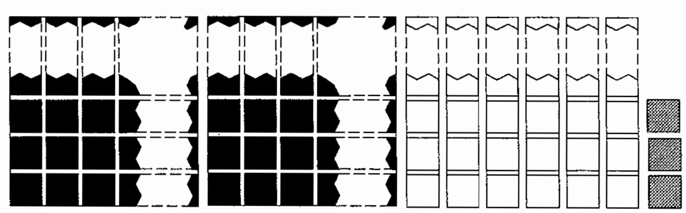
A square whose value is 525 + 4, or 529, has an edge whose value is 23 or –23 (a calculator with a square root key is helpful here). If $n - 2$ is 23 then n is 25 and if $n - 2$ is –23 then n is –21. Hence, the 25th and –21st arrangements have value 525.

7. Ask the students to form the nth arrangement of an extended sequence for which $v(n) = n^2 + 4$ and then have them do the same for an extended sequence for which $v(n) = 2n^2 + 6n - 3$. Have the students determine for which n the nth arrangements of these two extended sequences have the same value.

7. One way of representing the two nth arrangements is shown below.



$n^2 + 4$



$2n^2 + 6n - 3$

These two arrangements will have the same value if, after an n^2-mat has been removed from each of them, the remaining portions have the same value, i.e., if $n^2 + 6n - 3$ has value 4 or, what is the same, $n^2 + 6n$ has value 7.

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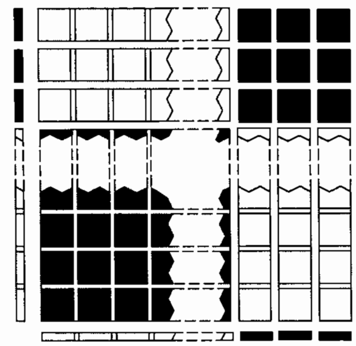
Actions

Comments

7. *Continued.* As shown below, adding 9 black tile to $n^2 + 6n$ provides a square array whose edge has value $n + 3$.



$n^2 + 6n + 9$

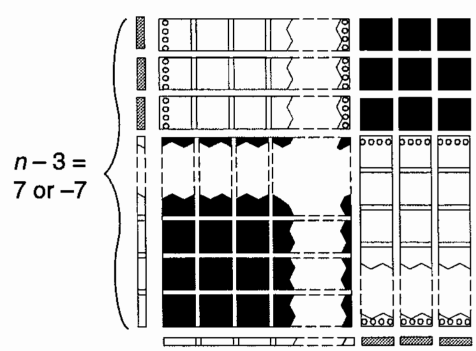
Thus $n^2 + 6n$ has value 7 if the square array has value 16, i.e., if its edge has value 4 or –4. If $n + 3$ is 4 then n is 1 and if $n + 3$ is –4, then n is –7. Hence the 1st arrangements of the two sequences have the same value, as do the –7th arrangements.

8. Ask the students to use algebra pieces or sketches to help them find all solutions of the following equations:

(a) $n^2 - 6n = 40$
(b) $2n^2 + 38 = 4n^2 - 12$
(c) $(n - 1)(n + 3) = 165$
(d) $4n^2 + 4n = 2600$
(e) $n^2 - 5n + 6 = 0$

8. Various methods for solving the equations are illustrated below. The students may use other methods.

(a) If 9 black tile are added to a collection for $n^2 - 6n$, the resulting collection can be formed into a square array with edge $n - 3$ (see below). If $n^2 - 6n$ has value 40, the square array has value 49, i.e., its edge has value 7 or –7. If $n - 3$ is 7 then n is 10 and if $n - 3$ is –7 then n is –4. So the equation has two solutions: 10 and –4.



$(n^2 - 6n) + 9 = 40 + 9 = 49$

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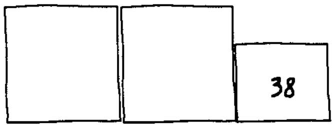
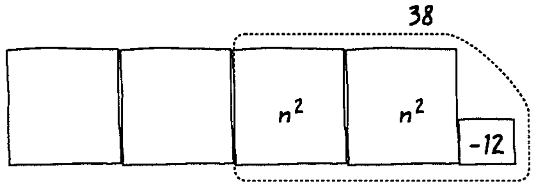
Actions

Comments



$2n^2 + 38$

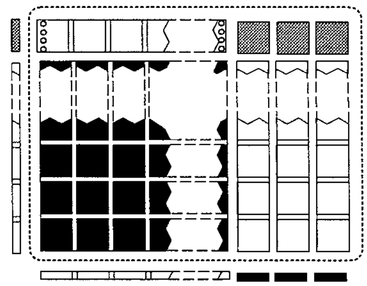


$4n^2 - 12$

8. *Continued.*
(b) Comparing sketches for $2n^2 + 38$ and $4n^2 - 12$, one sees they have the same value if $2n^2 - 12$ is 38. This is the case if n^2 is 25, that is, if n is 5 or n is –5.

165

11 or −15

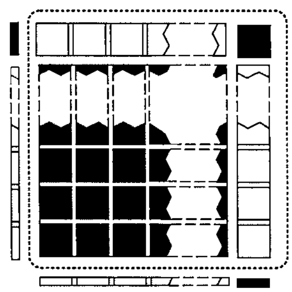


15 or −11

(c) Shown here is a representation of $(n - 1)(n + 3)$. Note the values of the edges differ by 4 and their product is 165. Since 11 and 15 differ by 4 and $11 \times 15 = 165$, the array will have value 165 if the edges have values 11 and 15 or –11 and –15. The edges have values 11 and 15 if n is 12; they have values –11 and –15 if n is –14. (Finding the pair 11 and 15 is facilitated by noting that one of the pair should be smaller and one larger than $\sqrt{165} \approx 13$.)

165 + 4

13 or −13

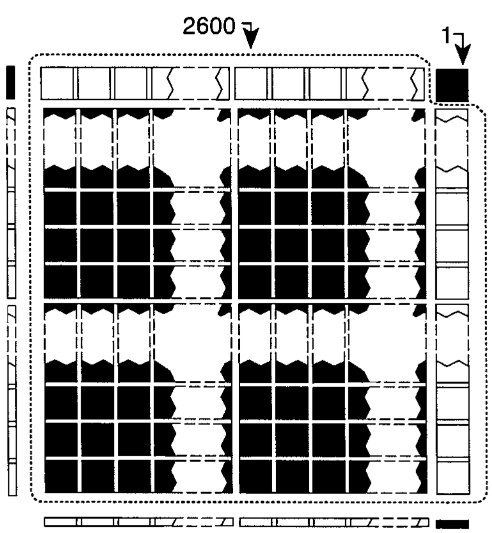


Alternatively, adding 4 black tile to the above array and removing collections whose value is 0 leaves a collection of pieces that can be arranged in a square array that has value 169 and edge $n + 1$. Hence $n + 1$ is 13 or –13, in which case $n = 12$ or $n = -14$.

2600

1

51 or −51



(d) If 1 black tile is added to a collection for $4n^2 + 4n$, a square array with edge $2n + 1$ can be formed. If the value of the original collection is 2600, the value of the square array is 2601. Using a calculator, one finds $\sqrt{2601} = 51$. Hence $2n + 1$ is 51 or –51. Thus $n = 25$ or $n = -26$.

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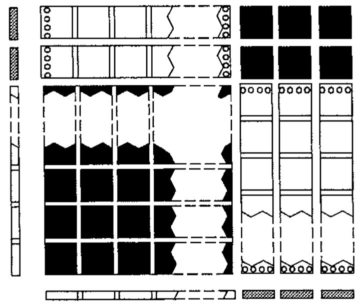
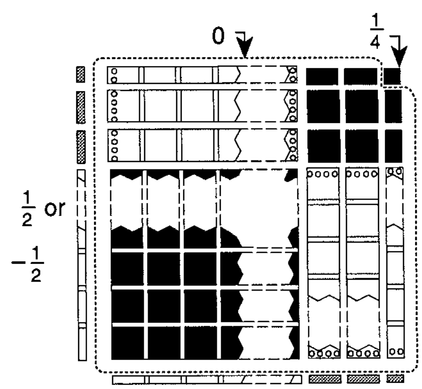
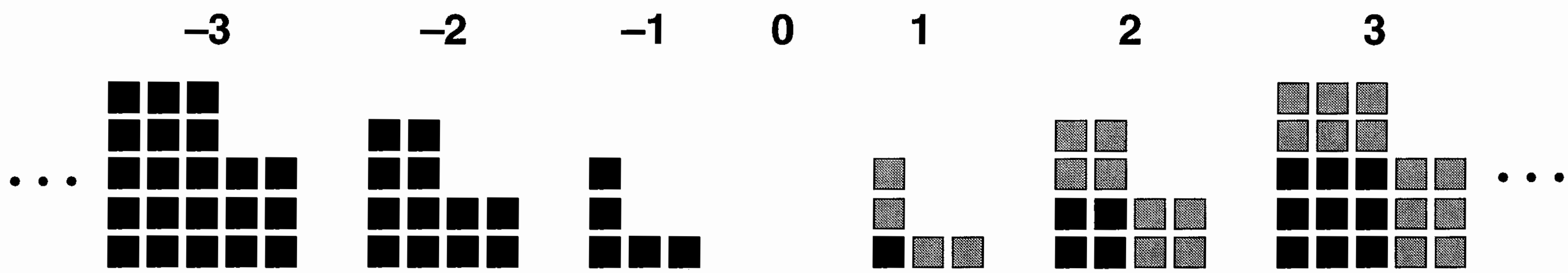


8. *Continued.*
(e) A collection for $n^2 - 5n + 6$ can be formed into a rectangular array with edges $n - 2$ and $n - 3$. The array has value 0 if at least one edge has value 0. This is the case if $n = 2$ or $n = 3$.



Alternatively, by cutting a $-n$-frame and two black tile in halves and adding a quarter of a black tile to a collection for $n^2 - 5n + 6$, the resulting collection can be formed into a square with edge $n - 2\frac{1}{2}$. If the original collection has value 0, the square array has value $\frac{1}{4}$ and its edge has value $\frac{1}{2}$ or $-\frac{1}{2}$. Now, $n - 2\frac{1}{2}$ is $\frac{1}{2}$ when $n = 3$ and it is $-\frac{1}{2}$ when $n = 2$.

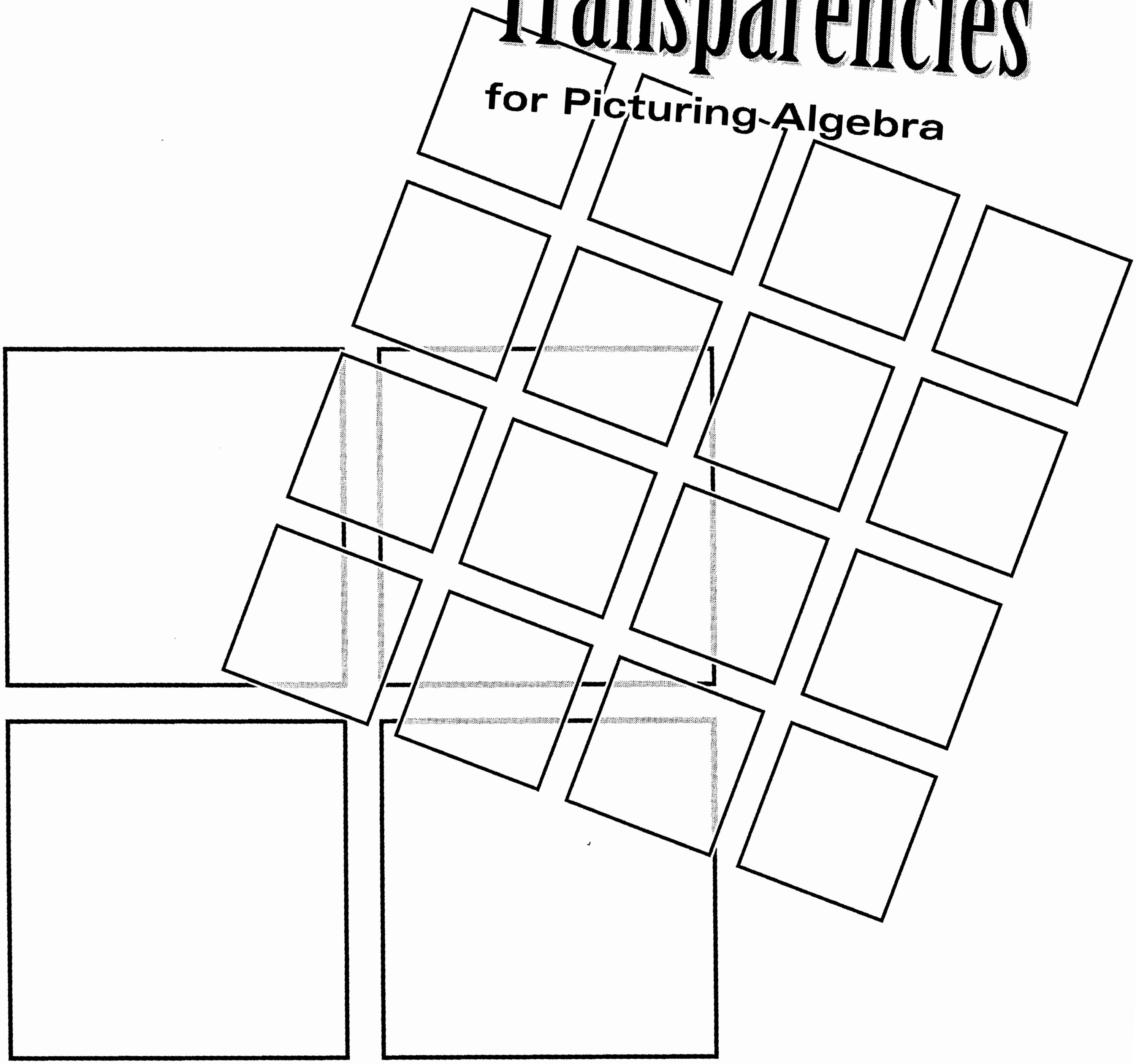
Arrangement number



Unit IX / Math and the Mind's Eye

Transparencies

for Picturing Algebra



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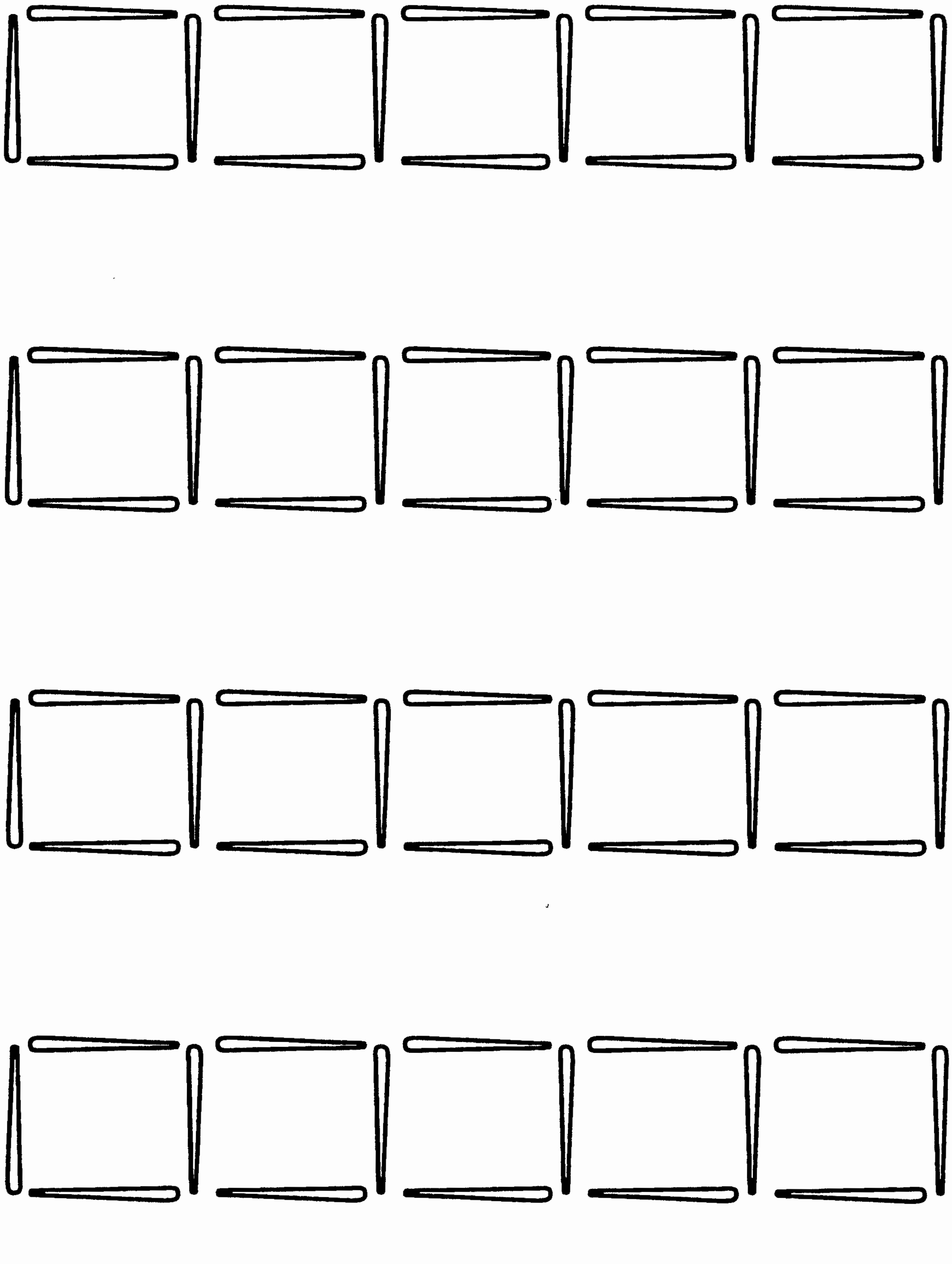
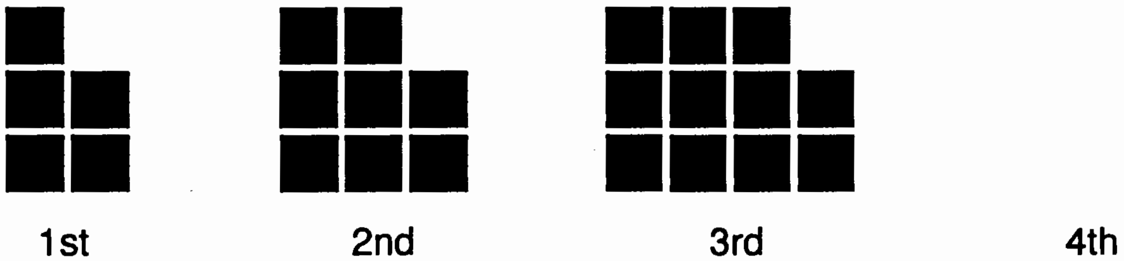


Name ____________________

1. Sketch the next arrangement in the following sequence of tile arrangements.



2. Describe the 25th arrangement. How many tile does it contain?

3. Let *T* be the number of tile in the *n*th arrangement. Write a formula relating *T* and *n*.

4. A certain arrangement contains 500 tile. Which arrangement is this?

5. Two arrangements together contain 160 tile. One of the arrangements contains 30 more tile than the other. Which two arrangements are these?

To determine the number of toothpicks,

To determine the number of toothpicks,

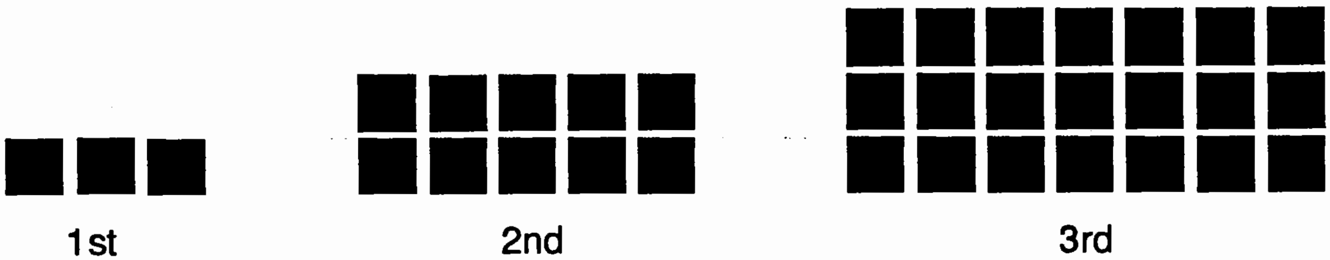
To determine the number of toothpicks,

To determine the number of toothpicks,

Activity Sheet IX-2-B

Name ______________________

1. Examine the following sequence of rectangular arrangements. Determine the dimensions of the next arrangement.



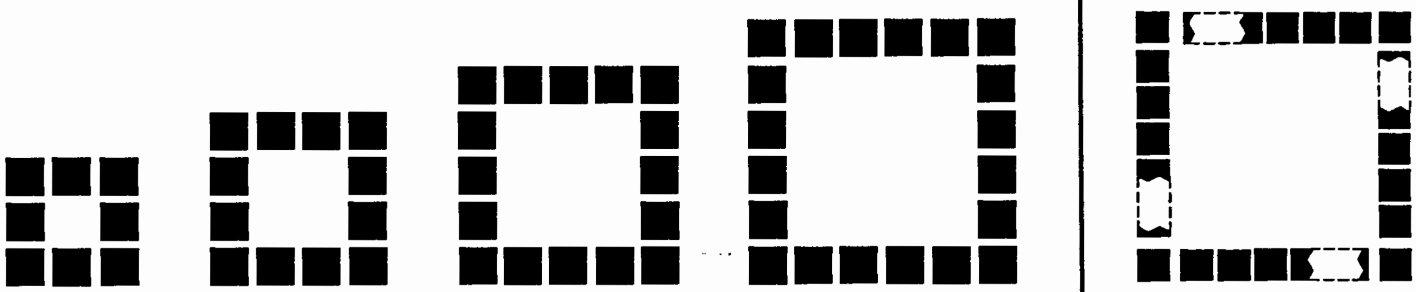
2. Describe the 30th arrangement. How many tile does it contain?

3. List some equivalent expressions for the number of tile in the *n*th arrangement.

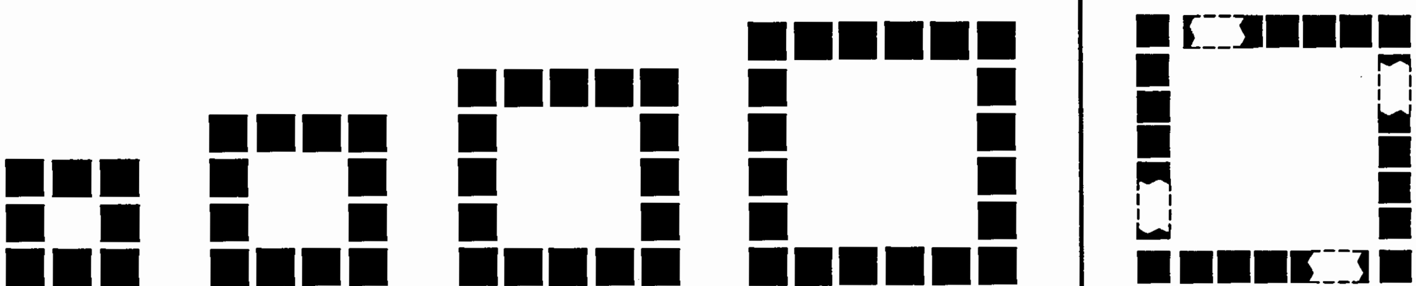
4. A certain arrangement contains 1275 tile. Which arrangement is this?

5. The larger of two arrangements has 5 more rows and 355 more tile than the smaller. What two arrangements are these?

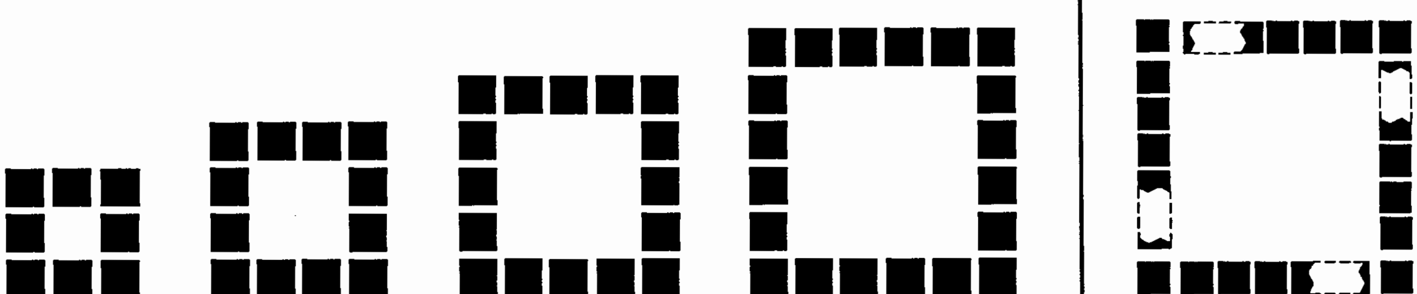
Arrangement
1st 2nd 3rd 4th 20th



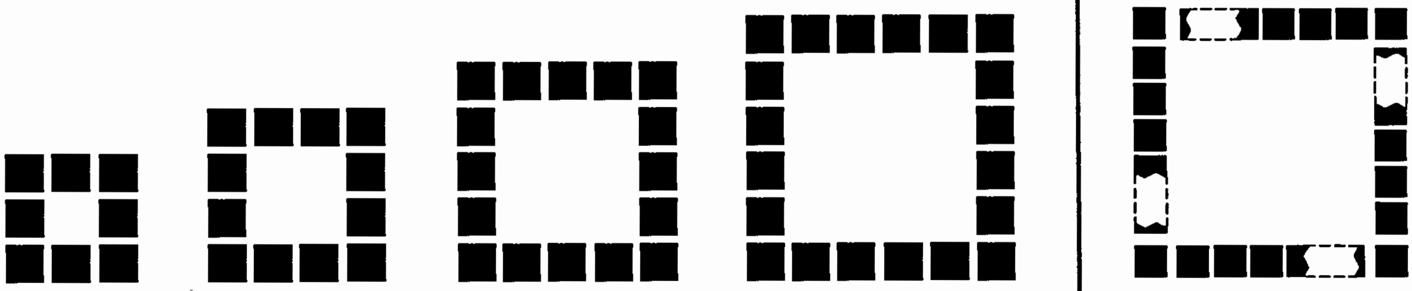
Arrangement
1st 2nd 3rd 4th 20th



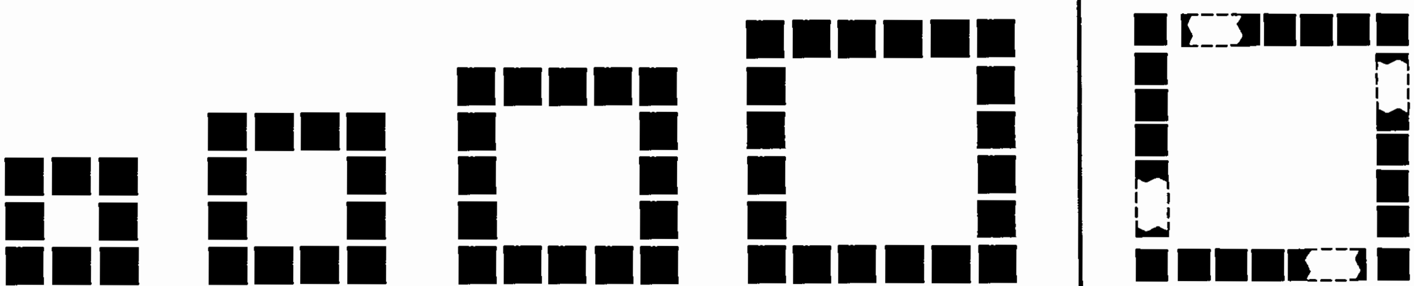
Arrangement
1st 2nd 3rd 4th 20th



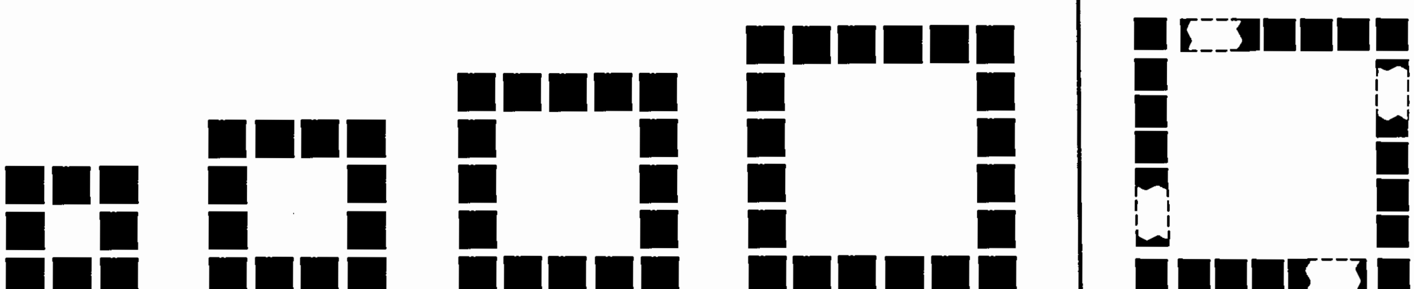
Arrangement
1st 2nd 3rd 4th 20th

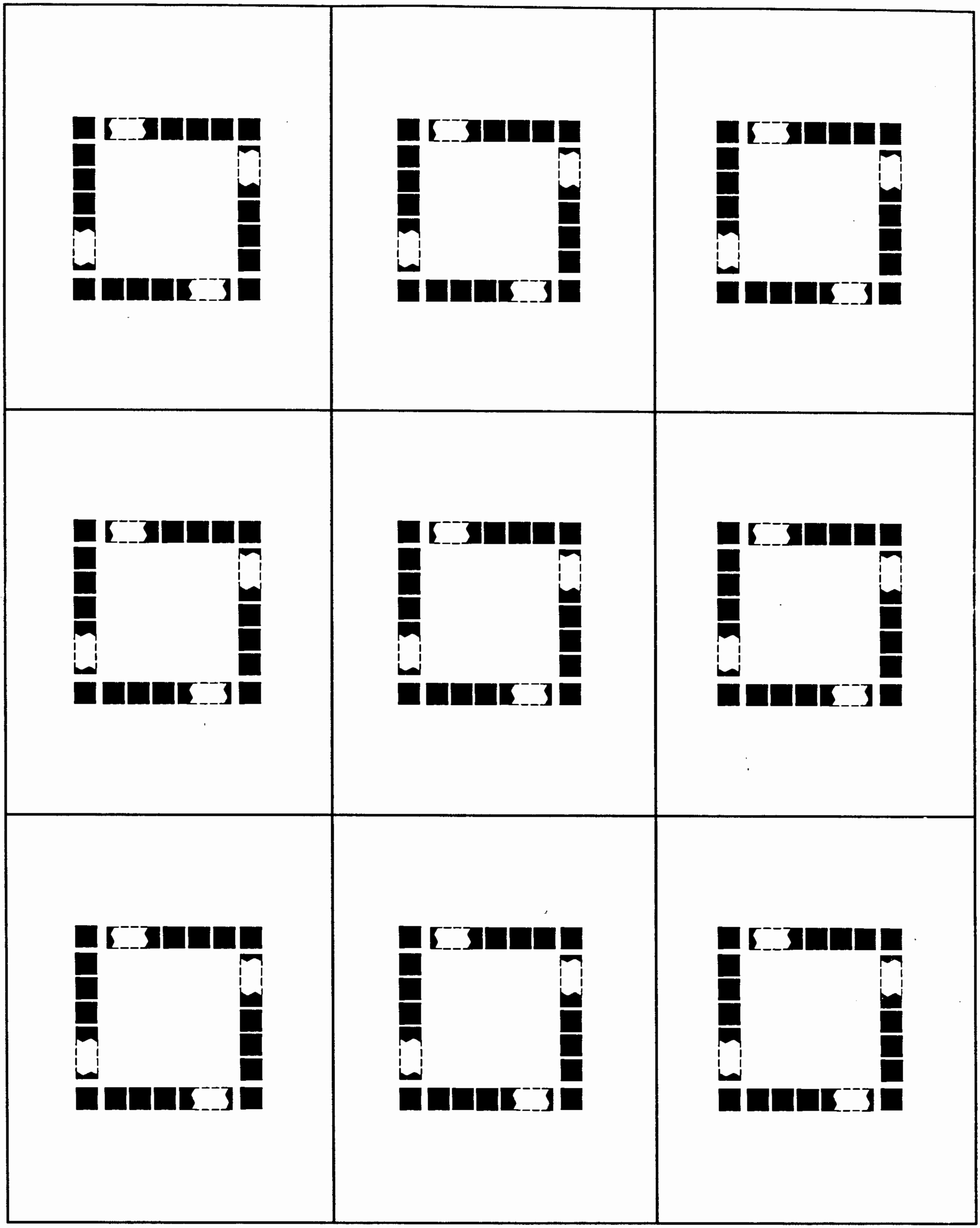


Arrangement
1st 2nd 3rd 4th 20th



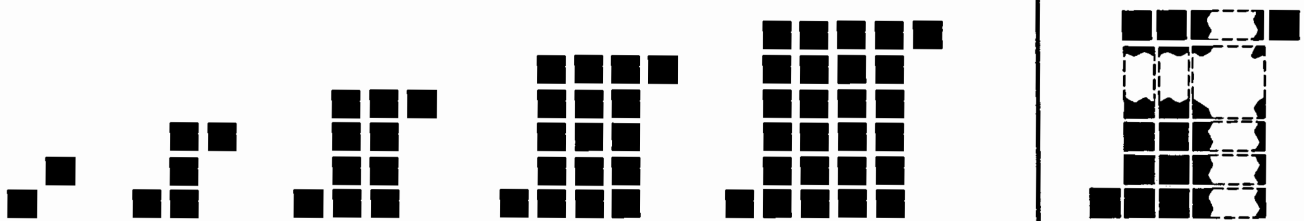
Arrangement
1st 2nd 3rd 4th 20th





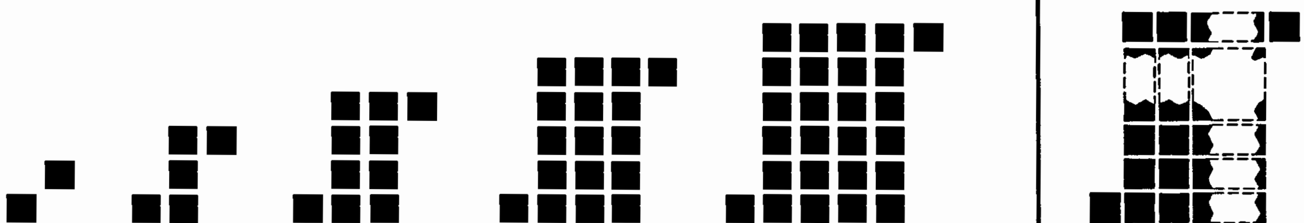
Arrangement

1st 2nd 3rd 4th 5th 20th



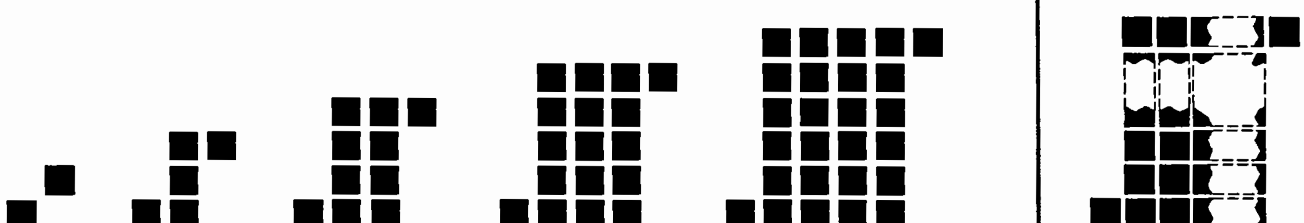
Arrangement

1st 2nd 3rd 4th 5th 20th



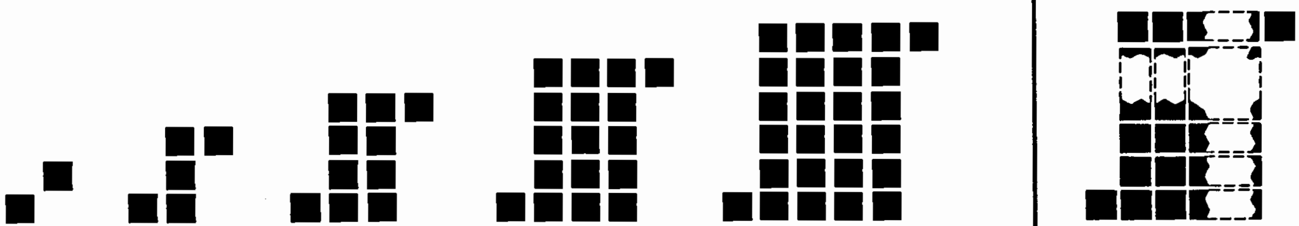
Arrangement

1st 2nd 3rd 4th 5th 20th



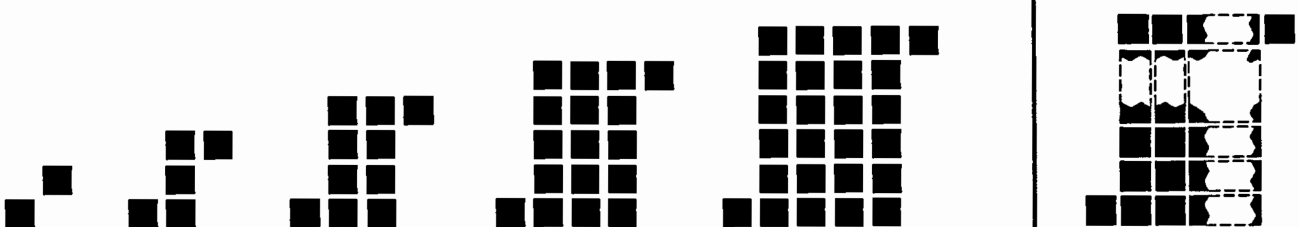
Arrangement

1st 2nd 3rd 4th 5th 20th



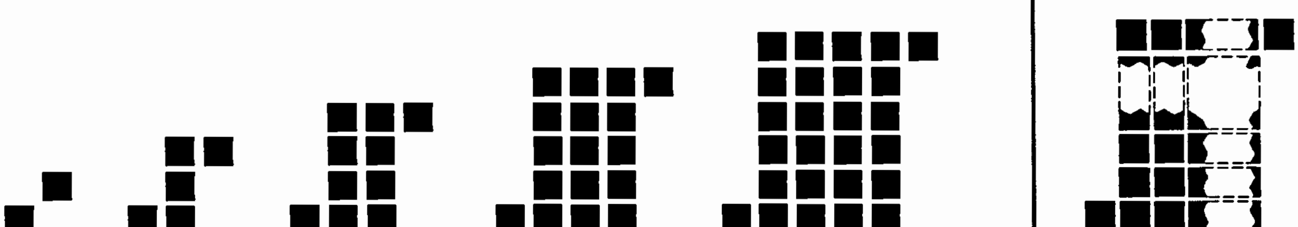
Arrangement

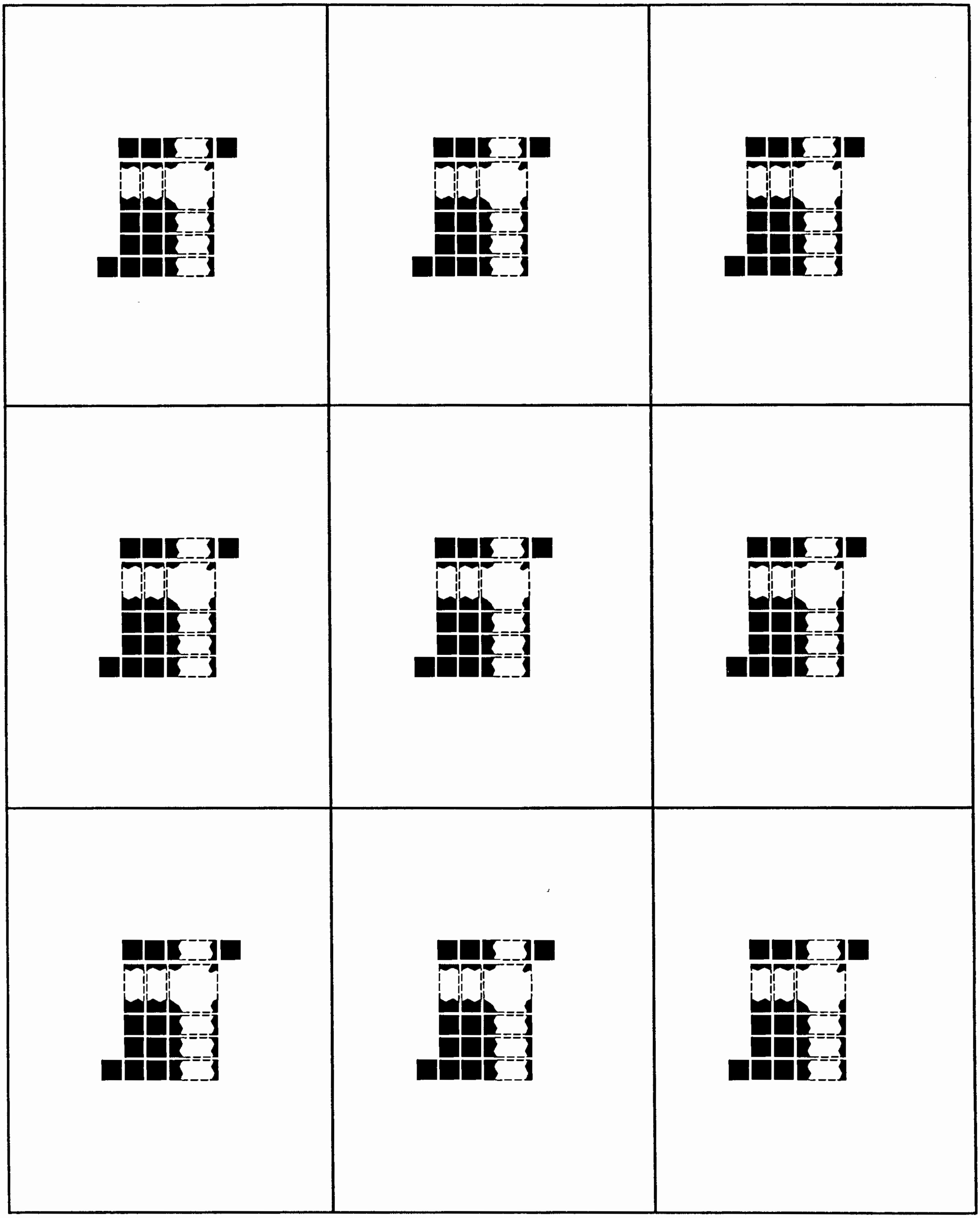
1st 2nd 3rd 4th 5th 20th

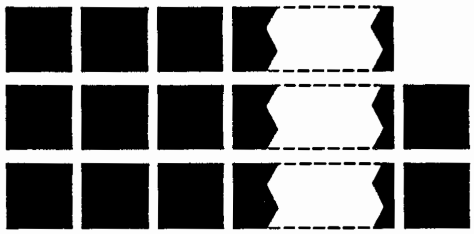


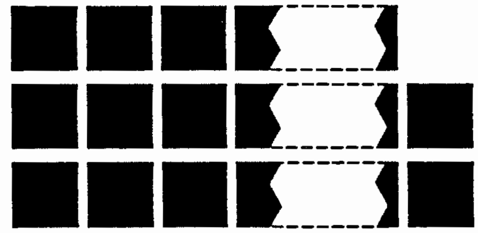
Arrangement

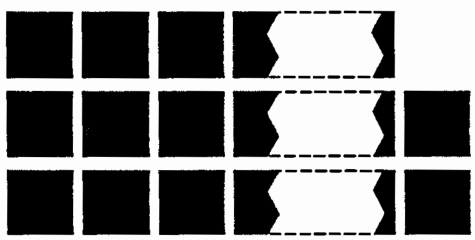
1st 2nd 3rd 4th 5th 20th

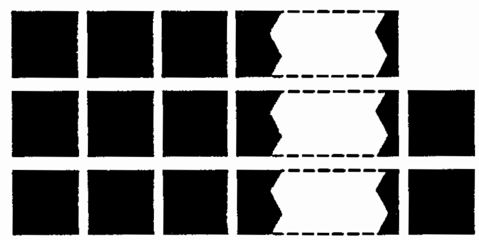


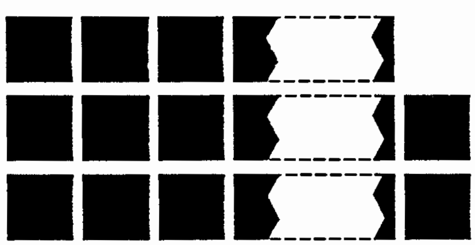


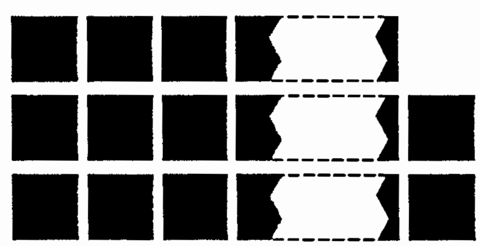


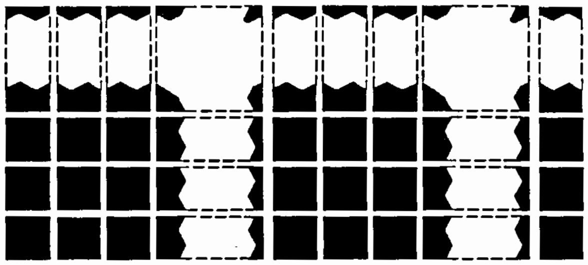
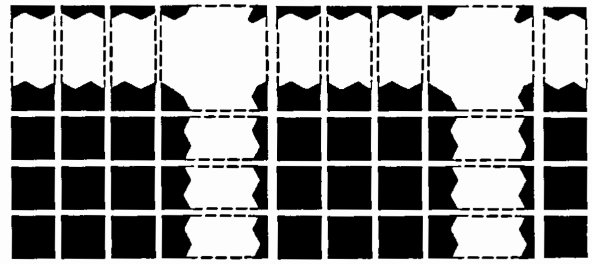
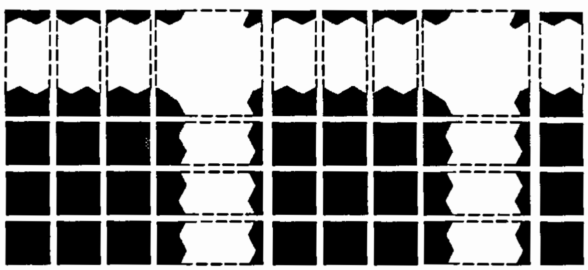
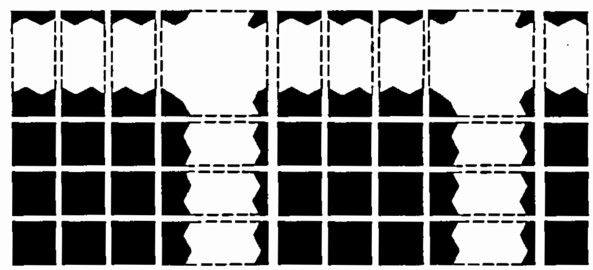
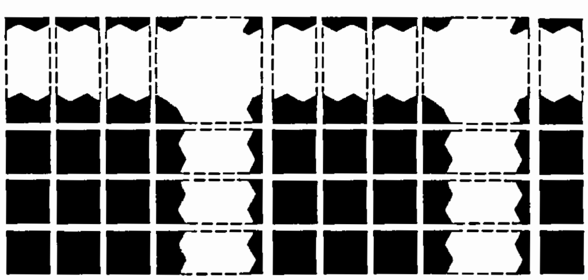
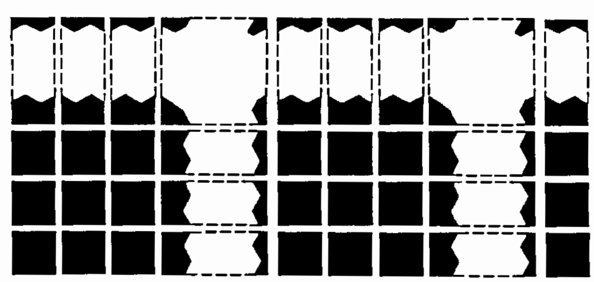










Activity Sheet IX-3-A

Name ______________________

1. Form the first three tile arrangements in a sequence for which the number of tile in the nth arrangement is $2(n + 1) + 3$. Sketch these three arrangements below.

2. Use tile pieces to form a representation of the nth arrangement. Sketch your representation here:

3. Which arrangement contains 225 tile?

4. Which is the largest arrangement that contains 500 or fewer tile?

5. A total of 400 tile is required to build two successive arrangements. Which arrangements are these?

Name ______________________

1. Form the first three tile arrangements in a sequence for which the number of tile in the nth arrangement is $(n + 1)(2n + 1)$. Sketch these three arrangements below.

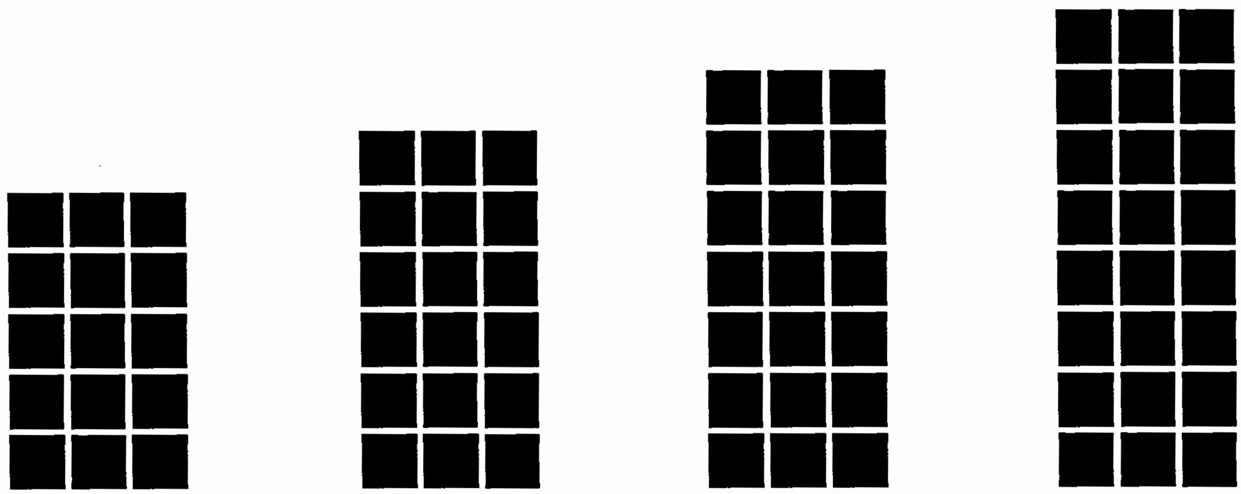
2. Use tile pieces to form a representation of the nth arrangement. Sketch your representation here:

3. Which arrangement contains 2145 tile?

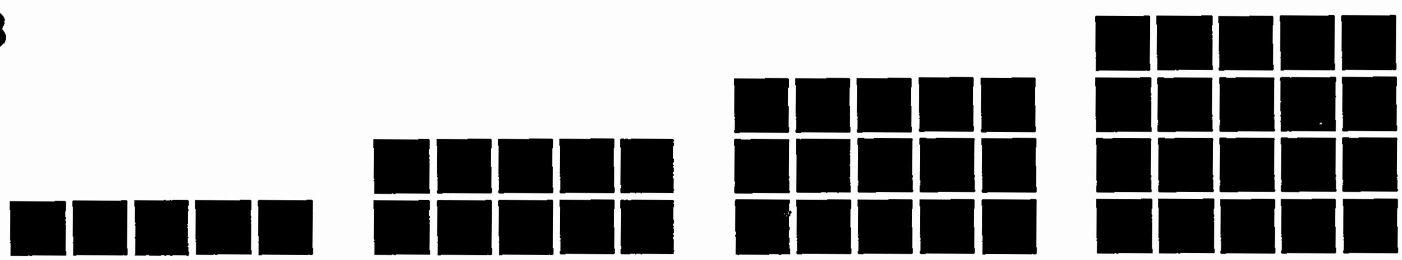
4. If the number of tile in a certain arrangement is doubled, 50 more tile are needed to form a 50 x 50 square. Which arrangement is this?

5. The larger of two successive arrangements contains 125 more tile than the smaller. Which two arrangements are these?

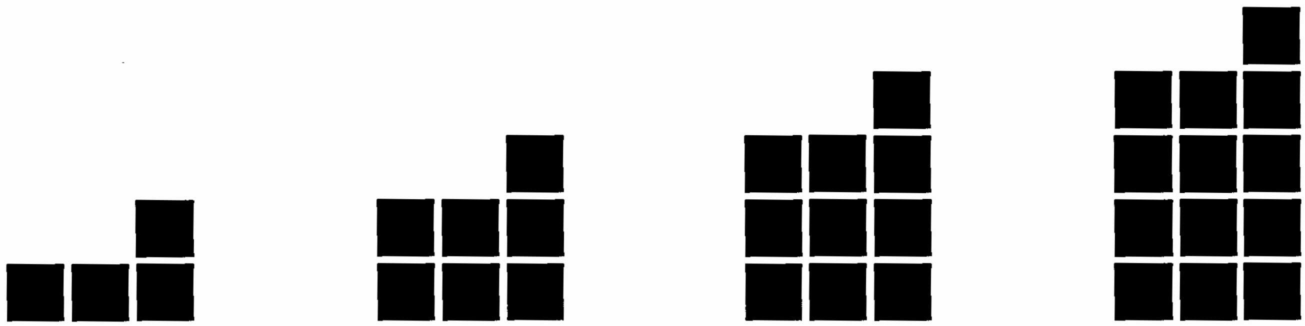
A



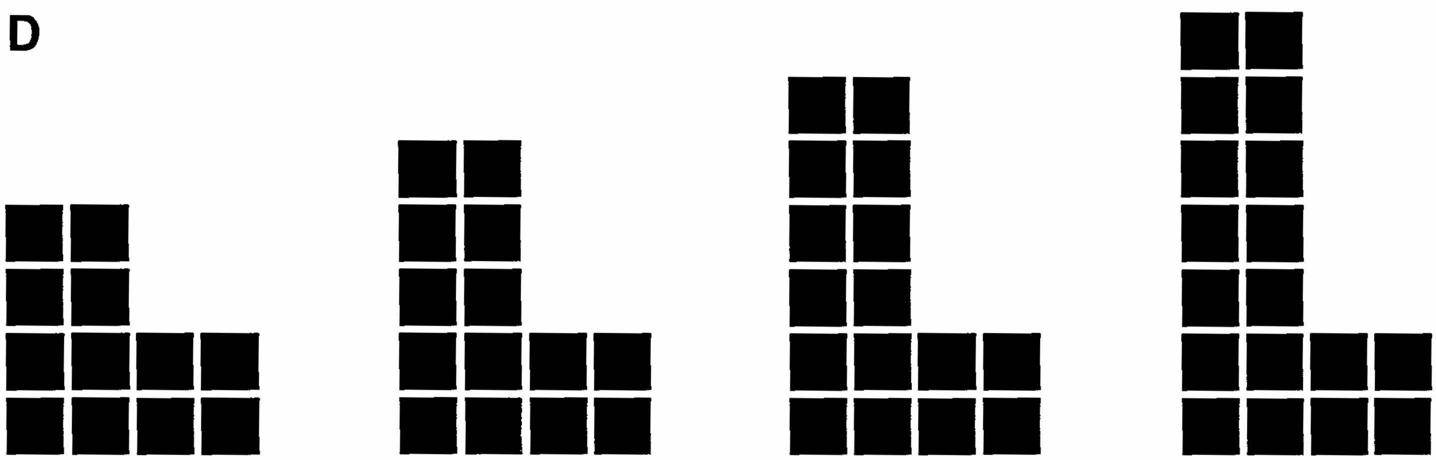
B



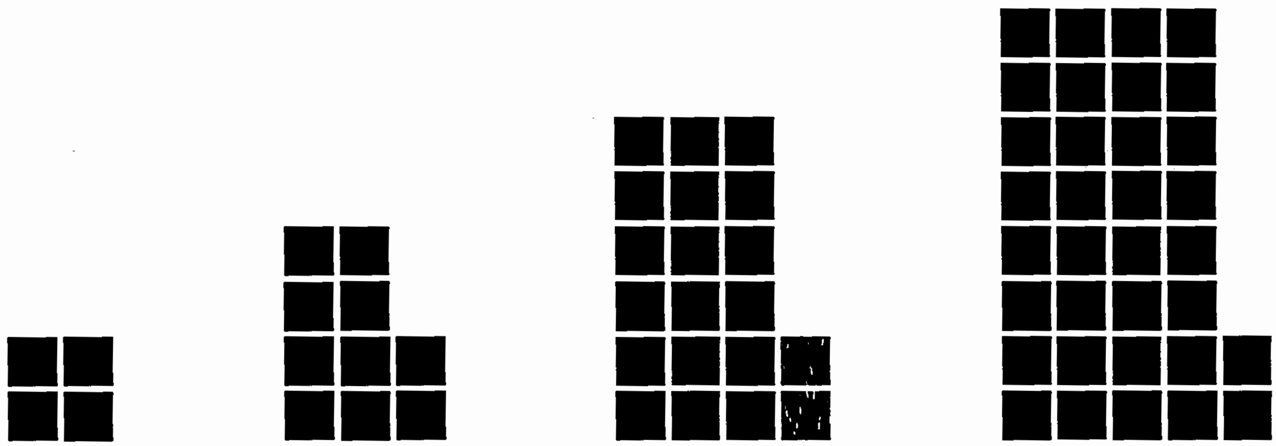
C



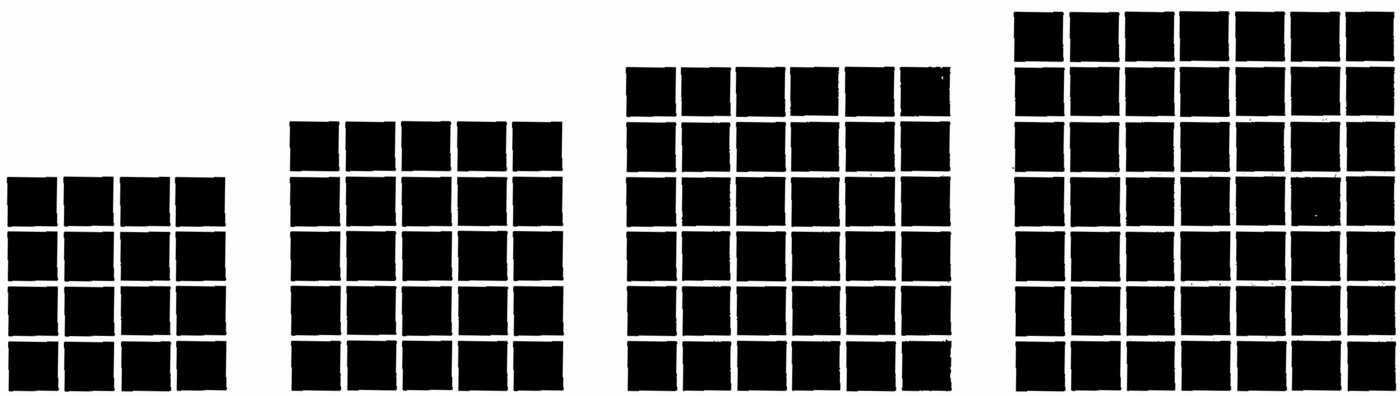
D



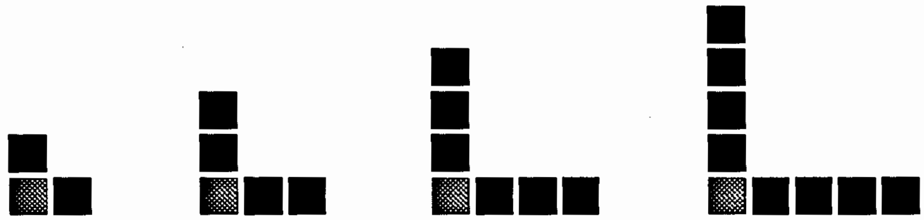
E



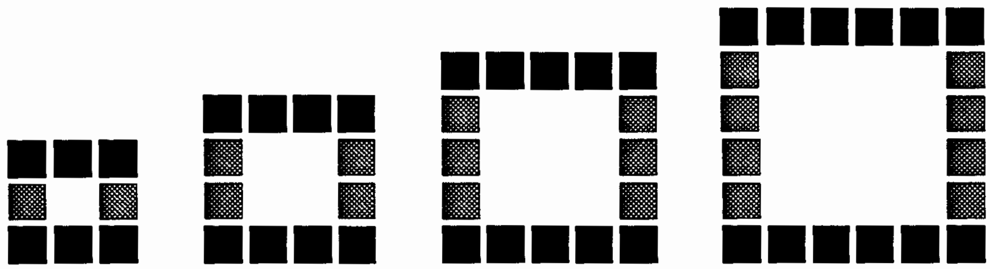
F



(a)



(b)



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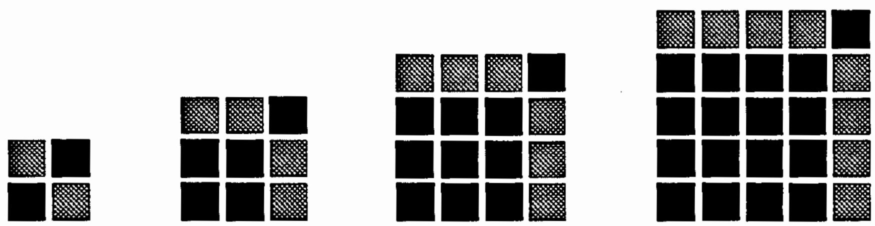
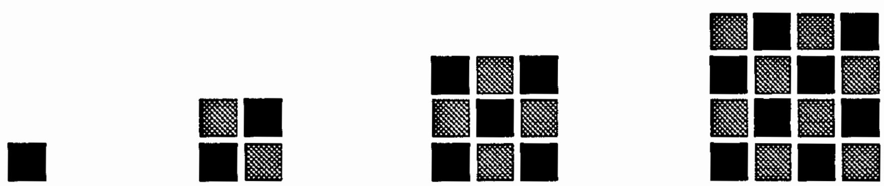
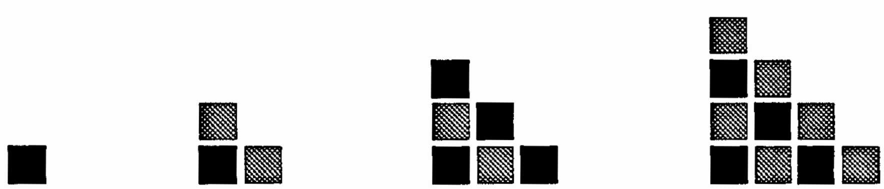


(d)



(e)



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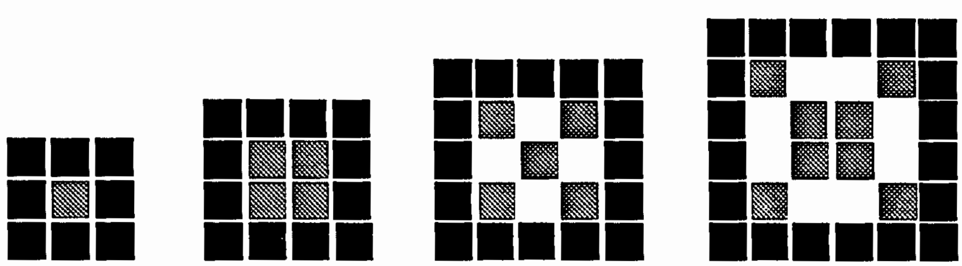
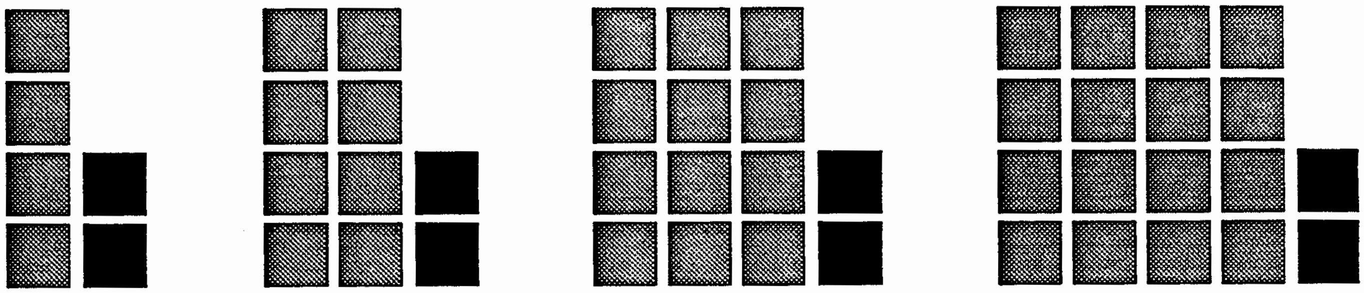


Name ______________________________

A.



Shown above are the first 4 arrangements in a sequence of counting piece arrangements. List some equivalent expressions for $v(n)$.

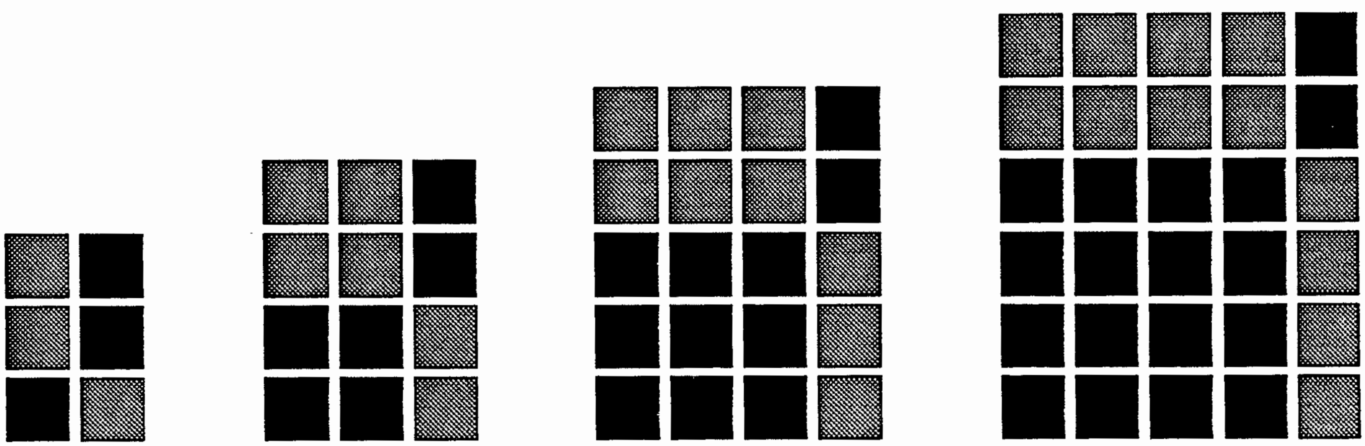
Complete the following statements:

(a) $v(15) =$ __________.

(b) If $v(n) = -250$, then $n =$ __________.

(c) If $v(n) = -98$, then $v(n-2) =$ __________.

B.



Shown above are the first 4 arrangements in a sequence of counting piece arrangements. List some equivalent expressions for $v(n)$.

Complete the following statements:

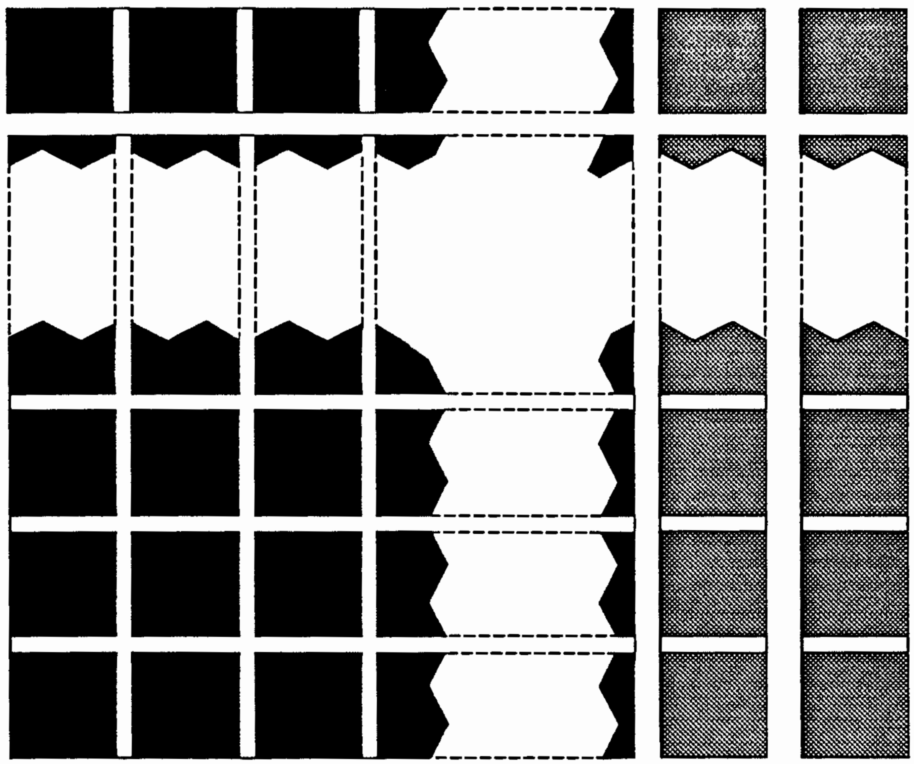
(a) $v(20) =$ __________.

(b) If $v(n) = 90$, then $n =$ __________.

(c) If $v(n+1) - v(n) = 50$, then $n =$ __________.

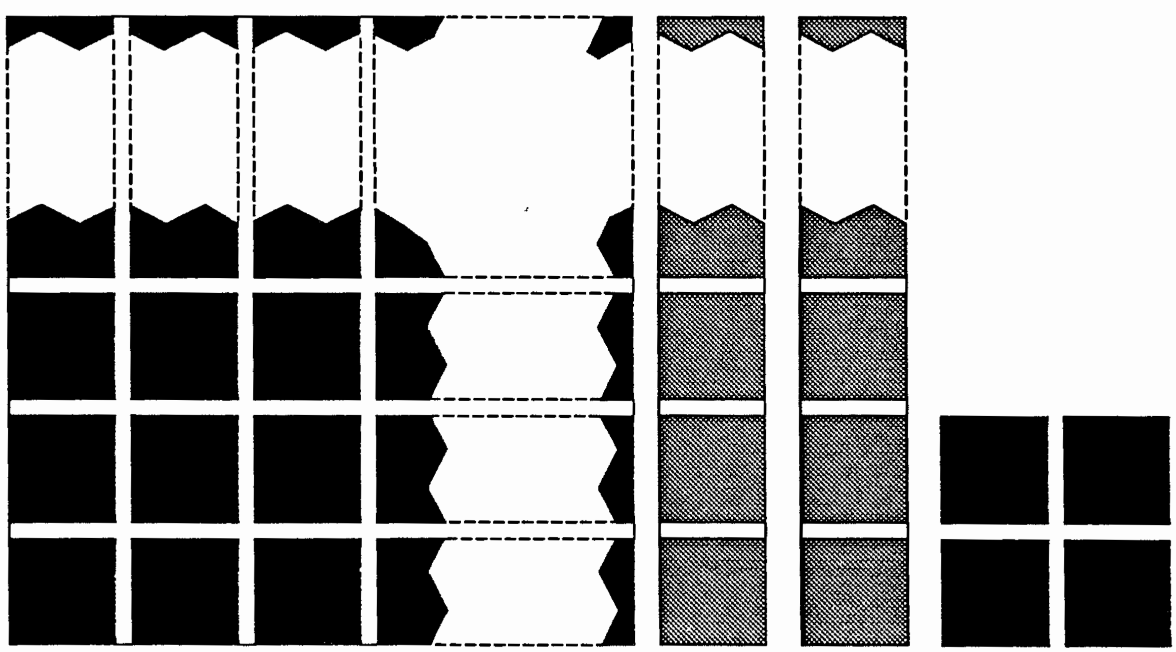
*n*th Arrangements

Sequence 1

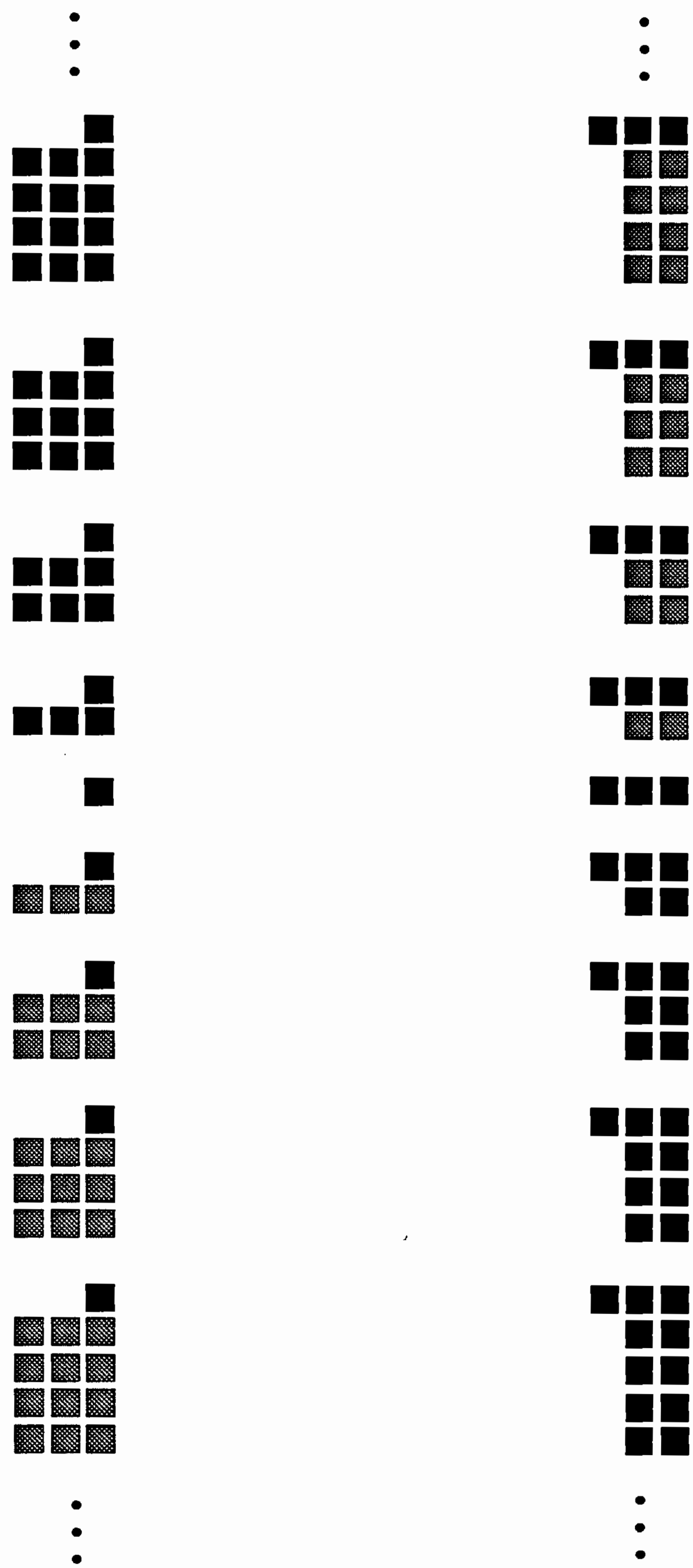


$v_1(n) = ?$

Sequence 2



$v_2(n) = ?$

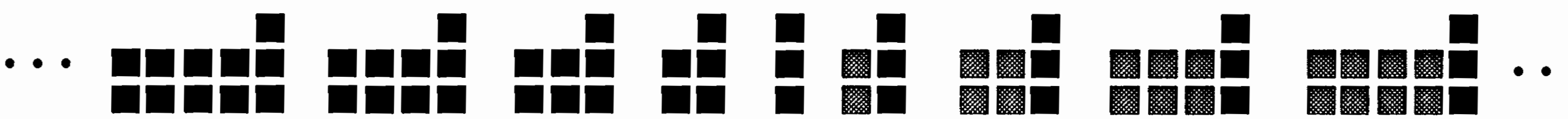


Arrangement number, n

Value, $v(n)$	–11	–8	–5	–2	1	4	7	10	13

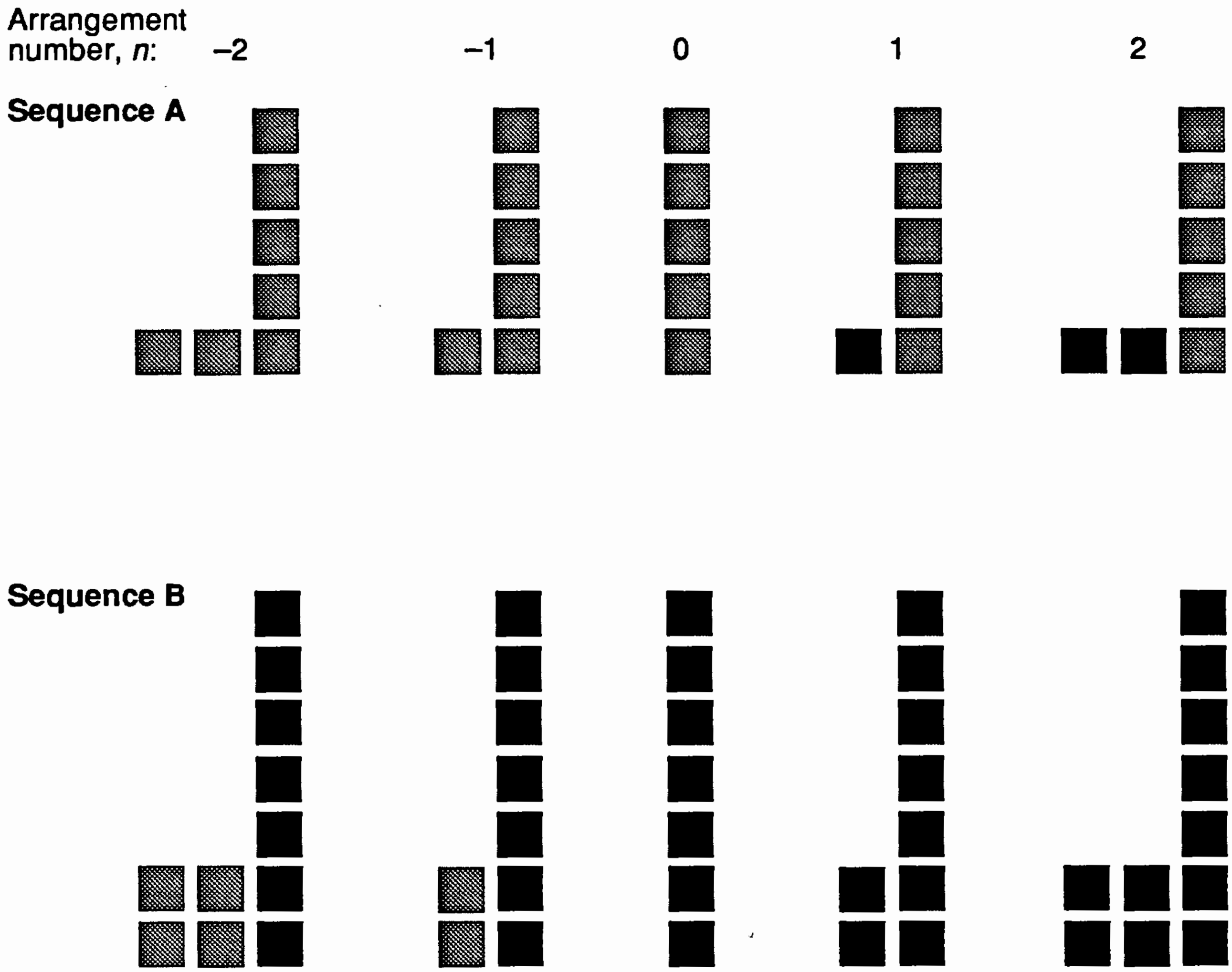
$v(n) =$

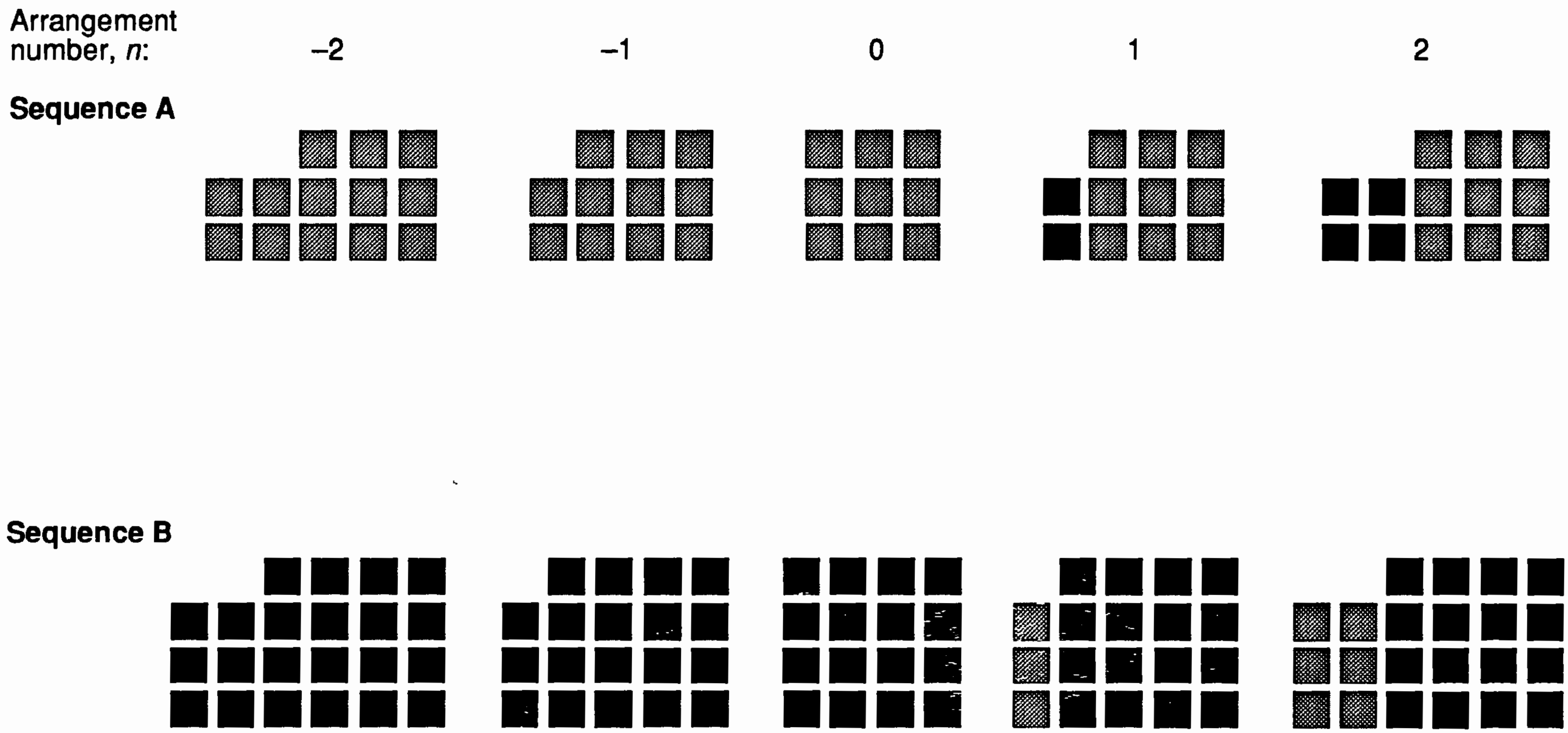
Arrangement number, n



Value, $v(n)$	11	9	7	5	3	1	–1	–3	–5

$v(n) =$


Arrangement number, n:
–2
–1
0
1
2
Sequence A
Sequence B


Arrangement number, n:
–2
–1
0
1
2
Sequence A
Sequence B

Arrangement number