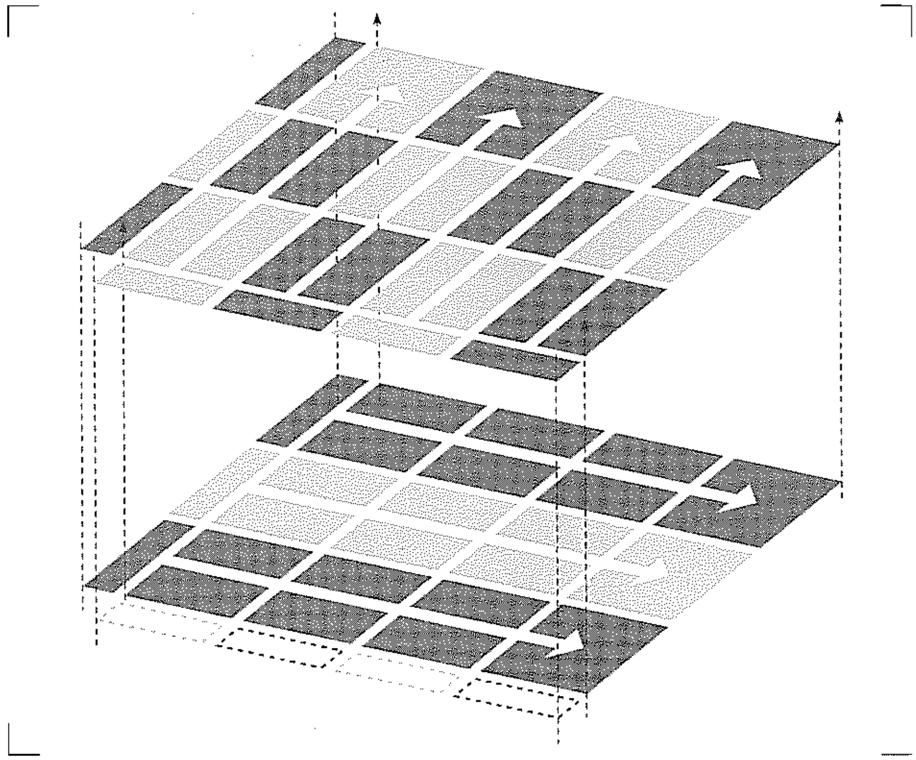
Unit VI / Math and the Mind's Eye Activities



Modeling Integers

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Modeling Integers

1 Counting Piece Collections
Bicolored counting pieces are used to introduce signed numbers and provide a model for the integers.

2 Adding and Subtracting Signed Numbers
Counting pieces are used to find sums and differences of signed numbers.

3 Counting Piece Arrays
A rectangular array of counting pieces is formed. Rows and/or columns of the array are turned over and the effect on the net value of the array is noted. Edge pieces are introduced and the relationship between the net values of an array and the net values of its edges is investigated.

4 Multiplication and Division of Signed Numbers
Counting piece arrays, with edge pieces, are used to model multiplication and division of signed numbers.

Math and the Mind's Eye materials are intended for use in grades 4-9. They are written so teachers can adapt them to fit student backgrounds and grade levels. A single activity can be extended over several days or used in part.

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Math and the Mind's Eye

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Unit VI • Activity 1

Counting Piece Collections

O V E R V I E W

Bicolored counting pieces are used to introduce signed numbers and provide a model for the integers.

Prerequisite Activity

None

Materials

Red and black counting pieces (see Comment 1); a copy of Activity Sheet VI-1 for each student

Actions

1. Draw a chart like that shown below on the overhead or chalkboard. Drop a small handful of counting pieces on a surface that all the students can see. Record the information about this collection on the first line of the chart.

Collection Number	Total No. of Pieces	No. of Red	No. of Black	Net Value
1.				
2.				
3.				
4.				
5.				
.				
.				
.				

Comments

1. Counting pieces are red on one side and black on the other. They can be made from red cardstock using the masters that appear at the end of this activity. Copy the Counting Piece Master / Front on one side of red cardstock and the Counting Piece Master / Back on the other side, then cut on the lines. One sheet of cardstock will provide enough counting pieces for four students or four groups of students.

If there is no surface that all of the students can see, counting pieces can first be dropped on a desk or table top and the resulting collection of red and black pieces replicated on the overhead. Any cardstock counting piece will appear as a black piece on the overhead, red overhead pieces can be obtained by making a copy of the Counting Piece Master / Front on red transparency paper and cutting on the lines.

Red and black pieces are said to be of *opposite* color. The *net value* of a collection of counting pieces is the number of red or black pieces in the collection that can not be matched with a piece of the opposite color. A collection in which all pieces can be matched has a net value of 0.

Continued next page.

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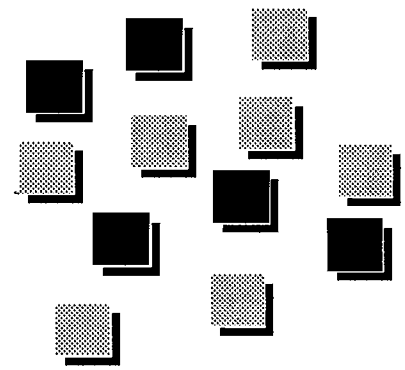
Actions

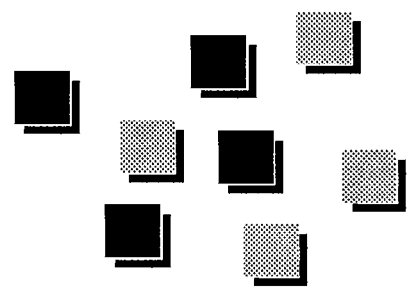
Comments

1. *(Continued.)* Collection 1 below contains 12 pieces, 5 black and 7 red. Its net value is 2 red. Collection 2 contains 8 pieces, 4 red and 4 black. Its net value is 0. This information is recorded in the table below.

Collection 1:



Collection 2:



Collection Number	Total No. of Pieces	No. of Red	No. of Black	Net Value
1.	12	7	5	2 R
2.	8	4	4	0
. . .	. . .	. . .	. . .	. . .

2. Ask a student to take a modest collection of number pieces (a dozen or so) from a container of counting pieces and drop them on their desktop. Record information about this collection on the chart. Repeat this Action with different students until there are several entries on the chart.

2. You may want to have a student record the information in the chart as you move about the room with the container of number pieces. You can have a student report the number of pieces in their collection and then ask the class for the net value of the collection.

Actions

3. Discuss the information contained in the chart. In particular, draw out the students' observations concerning net values.

Comments

3. For a collection whose net value is 2 red, possible discussion questions are: What are some other collections that have a net value of 2 red? What is the collection containing the fewest number of pieces that has this value? If a collection has a net value of 2 red and contains 10 black pieces, how many red pieces are in the collection?

Some observations concerning net values:

- Adding or removing an equal number of red and black pieces from a collection does not change its net value.

- For a given non-zero net value, the collection with the fewest pieces that has that net value contains all red pieces or all black pieces, the number and color matches the net value. For example, the collection with the fewest pieces that has a net value of 3 red is a collection of 3 red pieces.

- The collection with the fewest pieces that has net value zero is the empty collection, that is, the collection containing no pieces.

Actions

4. Explain to the students how plus and minus signs will be used to designate net values.

Comments

4. A minus sign will indicate a red net value and a plus sign will indicate a black net value. For example, a net value of 3 red will be written $^{-}3$ (read "negative three"); a net value of 2 black will be written $^{+}2$ (read "positive two"). Note that the minus and plus signs are written in superscript position.

Numbers to which a plus or minus sign are attached will be called *signed numbers.* You may want to write the appropriate signed number alongside the net values in the chart developed earlier, as shown:

Collection Number	Total No. of Pieces	No. of Red	No. of Black	Net Value	
1.	12	7	5	2 R	$^{-}2$
2.	8	4	4	0	
3.	13	4	9	5 B	$^{+}5$
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.

Actions	Comments
5. Drop a small handful of counting pieces on a surface all students can see. Ask the students for the net value of the resulting collection. Then ask the students for the net value of the collection that would be obtained if all of the counting pieces were turned over. Repeat this action for two or three other collections.	5. If not all of the students can see the collection, it can be simulated on the overhead. By the end of this Action, the students should recognize that turning over all pieces in a collection changes the sign (or, what is the same, color) of its net value. Thus, if all pieces in a collection whose net value is $^{+}3$ (or 3 black) is turned over, the resulting collection will have value $^{-}3$ (or 3 red). See collections A and B below.
6. Referring to the results of Action 5, explain *opposite collections* and *opposite net values* to the students.	6. Two collections are called *opposites* of each other if one can be obtained from the other by turning over all of its pieces. The net values of opposite collections are *opposite net values.* Collections A and B, shown below, are opposite collections. Their net values, $^{+}3$ and $^{-}3$, are opposite net values, that is, $^{+}3$ is the opposite of $^{-}3$ and $^{-}3$ is the opposite of $^{+}3$.

A. Net Value = $^{+}3$ B. Net Value = $^{-}3$

Note that a collection which has the same number of red and black pieces is its own opposite. The net value of such a collection is 0. Thus the opposite of 0 is 0.

Actions

7. Distribute a copy of Activity Sheet VI-1 to each student. Ask the students to fill in the missing numbers. Discuss with them the methods they used to arrive at their answers.

Comments

7. A master for Activity Sheet VI-1 is found at the end of this activity.

You may have to remind the students that a minus sign indicates a red net value and a plus sign indicates a black net value.

Some students may arrive at correct answers without using counting pieces. Urge students who are having difficulty to use them.

Below is a completed table for Part A.

Collections	Total No. of Pieces	No. of Red Pieces	No. of Black Pieces	Net Value
A.	15	6	9	$^{+}3$
B.	9	2	7	$^{+}5$
C.	12	7	5	$^{-}2$
D.	8	4	4	0
E.	10	5	5	0
F.	7	0	7	$^{+}7$
G.	13	5	8	$^{+}3$
H.	13	9	4	$^{-}5$

In Part B, the net values of collections X, Y and Z are $^{+}5$, $^{-}3$ and $^{-}4$ respectively. The net value of collections X and Y combined is $^{+}2$. Y and Z combined is $^{-}7$. If X and the opposite of Y are combined, the net value is $^{+}8$.

8. Explain to the students that net values are a model of the integers.

8. Net values of collections of counting pieces, when expressed as signed numbers, serve as a model of the integers. The collection of black net values, $^{+}1, ^{+}2, ^{+}3, \ldots$, represents the positive intergers and the collection of red net values, $^{-}1, ^{-}2, ^{-}3, \ldots$, represents the negative integers. A 0 net value represents the *zero* integer.

The set of positive integers, $^{+}1, ^{+}2, ^{+}3, \ldots$, can be identified with the set of whole numbers, 1, 2, 3, ... Consequently, the + sign is often omitted when referring to a positive integer, i.e., 3 is written in place of $^{+}3$.

Actions

9. (Optional) Discuss notation conventions associated with the integers.

Comments

9. As mentioned in Comment 8, the + sign is generally omitted when referring to a positive integer, i. e. $^{+}5$ is written as 5.

Also, when writing a signed number to represent a negative integer, the – sign is usually written in normal position rather than in superscript position, i.e., –3 is written instead of $^{-}3$.

The notation 'opp(n)' is sometimes used to designate the opposite of a signed number n, e.g. opp($^{-}3$) = $^{+}3$. More frequently, though, the opposite of n is denoted by '–n', e.g. –$^{+}3$ = $^{-}3$ and –$^{-}3$ = $^{+}3$. Notice that if these conventions concerning the dropping of the + sign and the location of the – sign are adopted, these two statements would appear as –3 = –3 and ––3 = 3, respectively. The first of these statements, –3 = –3, appears to be a redundancy. However, on the left, –3 is intended to represent "the opposite of +3" and, on the right, –3 is intended to represent the negative integer "$^{-}3$".

In standard practice, it is difficult to determine whether a – sign is being used as part of the symbol for a negative integer or to designate the opposite of a positive integer. As illustrated in the last paragraph, using standard practices, both the opposite of the positive integer $^{+}3$ and the negative integer $^{-}3$ are denoted symbolically as –3. Since the opposite of the positive integer $^{+}3$ is the negative integer –3, it doesn't matter, in most cases, which of these two interpretations is given to the symbol –3.

Notice that the – sign occurs in three different ways in arithmetical notation. Besides its use in denoting the opposite of a number and its use in designating a negative number, it is also used to denote the operation of subtraction. Generally, it is clear from the context what use is intended. Nonetheless, students are apt to be confused by this variety of usage.

Part A

Fill in the missing numbers:

Collections	Total No. of Pieces	No. of Red Pieces	No. of Black Pieces	Net Value
A.		6		$^{+}3$
B.		2	7	
C.	12		5	
D.		4		0
E.	10			0
F.	7			$^{+}7$
G.			8	$^{+}3$
H.	13			$^{-}5$

Part B

Collection X contains 2 red and 7 black pieces.
Collection Y contains 8 red and 5 black pieces.
Collection Z contains 7 red and 3 black pieces.

Record the net value of collection X: ______, Y: ______, Z: ______.

Record the net value if collections X and Y are combined: ______.

Record the net value if collections Y and Z are combined: ______.

Record the net value if collection X and the opposite of collection Y are combined:

______.

Counting Piece Master / Front

Counting Piece Master / Back

Adding and Subtracting Signed Numbers

OVERVIEW

Counting pieces are used to find sums and differences of signed numbers.

Prerequisite Activities

Unit II, Activity 1, *Basic Operations*; Unit VI, Activity 1, *Counting Piece Collections*.

Materials

Red and black counting pieces for each student.

Actions

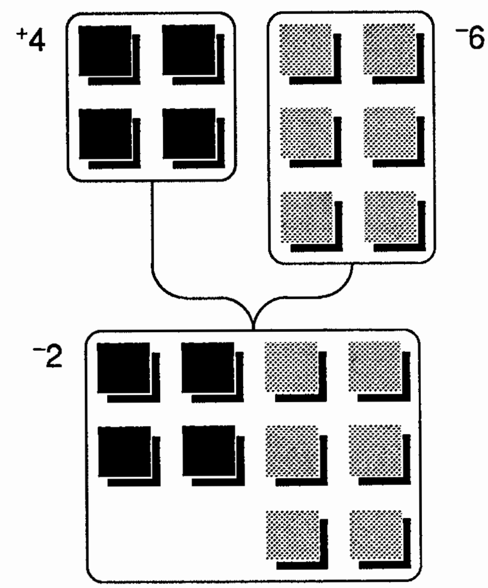
1. Distribute counting pieces to each student or group of students. Ask the students to suggest ways in which counting pieces can be used to determine the sum of two signed numbers, for example, $^{+}4 + ^{-}6$.

Comments

1. Each student, or group of students, should have about 15 counting pieces.

The students may have a variety of suggestions. One way to use counting pieces to determine the sum $^{+}4 + ^{-}6$ is to combine a collection whose net value is $^{+}4$ with a collection whose net value is $^{-}6$, and then find the net value of the combined collection.

The collections for $^{+}4$ and $^{-}6$ shown below are those that contain the fewest number of pieces. Other collections with the same net values could be used. Note that the combined collection has a net value of $^{-}2$. Hence, $^{+}4 + ^{-}6 = ^{-}2$.



$^{+}4 + ^{-}6 = ^{-}2$

In combining collections, some students may remove pairs of red and black pieces, ending up with a collection of 2 red pieces.

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Actions

2. Have the students use counting pieces to determine the following sums:

(a) $^{+}7 + ^{-}5$ (b) $^{-}4 + ^{-}5$

(c) $^{+}2 + ^{-}6$ (d) $^{+}4 + ^{-}4$

3. (Optional) Have the students compute the following sums:

(a) $^{-}25 + ^{-}40$ (b) $^{-}35 + ^{+}50$

(c) $^{-}60 + ^{+}52$

Comments

2. Some students may arrive at answers without physically manipulating counting pieces. If this happens, you can ask the students how they arrived at their answers to see if they understand the counting piece model.

3. Because of the magnitude of the numbers, finding these sums using counting pieces is impractical. However, in finding the sums, one can think in terms of counting pieces. For example, to find the sum in (b), one may think of combining a collection of 35 red pieces with a collection of 50 black pieces. The combined collection will have 15 more black pieces than red pieces. Hence, its net value is 15 black or $^{+}15$. Thus, $^{-}35 + ^{+}50 = ^{+}15$.

You may want to ask the students to find the sums of additional pairs of signed numbers.

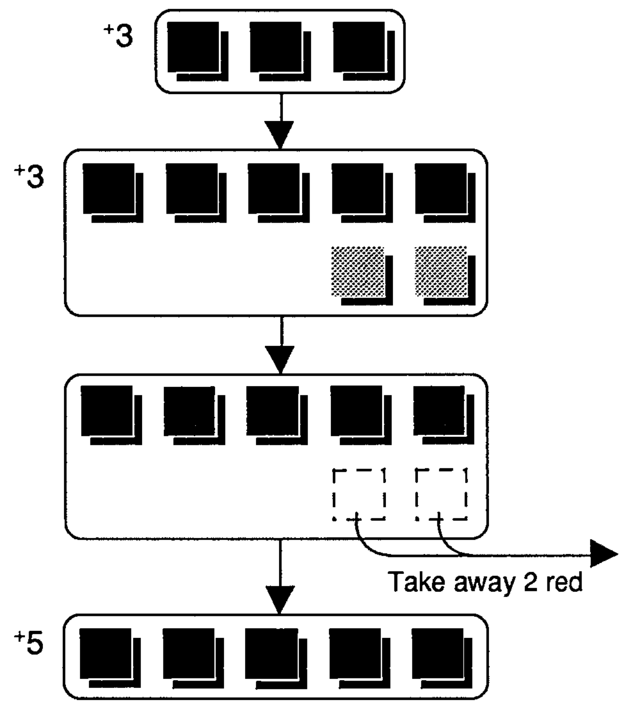
Actions

4. Write '$^{+}3 - ^{-}2$' on the overhead or chalkboard. Ask the students to suggest ways in which counting pieces can be used to compute the value of this expression.

Comments

4. There are a variety of ways in which this can be done.

One way is to take away a collection whose net value is $^{-}2$ (i.e., 2 red pieces) from a collection whose net value is $^{+}3$. In order to do this, one must build an appropriate collection for $^{+}3$. The collection with the fewest pieces for $^{+}3$ has 3 black pieces. It has no red pieces to remove. However, adding 2 black and 2 red pieces to this collection does not change its net value and results in a collection of 5 black and 2 red. Taking 2 red from this collection leaves 5 black. Hence, $^{+}3 - ^{-}2 = ^{+}5$.



$^{+}3 - ^{-}2 = ^{+}5$

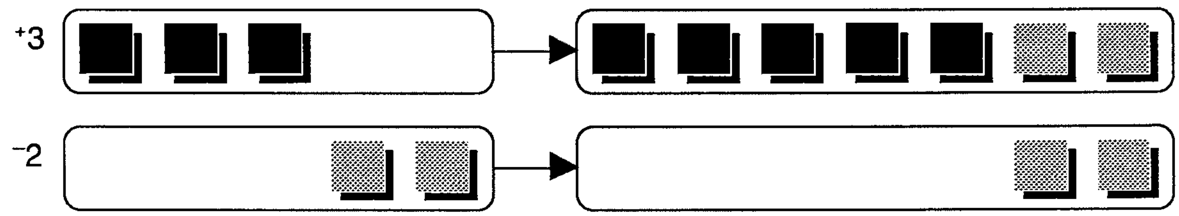
Note that adding 2 black and 2 red pieces to a collection and then removing the 2 red has the net effect of adding 2 black. Thus, subtracting $^{-}2$ from $^{+}3$ is equivelent to adding the opposite of $^{-}2$ to $^{+}3$.

Continued next page.

Actions

Comments

4. *Continued.* This computation can also be done using the difference model for subtraction. To do this, lay out collections for +3 and −2 and observe how the net value of the second differs from that of the first. If, as shown on the left below, +3 is represented by a collection of 3 black and −2 by a collection of 2 red, this difference is not immediately apparent. However, adding 2 black and 2 red to the first collection does not change its net value, and it then becomes clear that the difference of the second collection from the first is 5 black or +5.



The difference is 5 black.
Hence, $^{+}3 - {}^{-}2 = {}^{+}5$

Determining the difference of the second collection from the first is equivalent to determining what must be added to the second collection so that it has the same net value as the first. With this in mind, one might begin with a collection of 2 red pieces and directly determine that adding 5 black pieces to it results in a collection where net value is +3.

In using the difference model, it is important to determine how the second collection differs from the first, that is, what adjustment must be made to the second collection so it has the same net value as the first, and not conversely. In the above illustration, 5 black must be added to the second collection to give it the same net value as the first collection, whereas 5 red must be added to the first set to give it the same net value as the second.

Actions

5. Tell the students to use counting pieces to determine the following:

(a) $^{+}8 - {}^{+}3$ (b) $^{-}7 - {}^{-}2$

(c) $^{-}4 - {}^{+}7$ (d) $^{+}2 - {}^{-}5$

(e) $^{-}4 - {}^{-}4$

6. (Optional) Have the students compute the following:

(a) $^{-}25 - {}^{-}50$ (b) $^{+}80 - {}^{+}73$

(c) $^{+}70 - {}^{-}35$ (d) $^{-}45 - {}^{-}40$

Comments

5. Again, some students may arrive at answers without physically manipulating counting pieces. If that happens, you may want to ask these students to use counting pieces to explain to you how they arrived at their answers.

6. Because of the magnitude of the numbers, using counting pieces to perform these computations is impractical. However, it is useful to think in terms of counting pieces. For example, to compute (a), one wants a collection whose net value is $^{-}25$ from which one can take 50 red. One such collection is that containing 50 red and 25 black. Taking 50 red from this collection leaves 25 black. Hence, $^{-}25 - {}^{-}50 = {}^{+}25$.

Alternately, one could think as follows: If a first collection has 25 red and a second collection has 50 red, the second set would have the same value as the first set if 25 black were added to it. Hence the difference in value of the second set from that of the first is $^{+}25$, that is $^{-}25 - {}^{-}50 = {}^{+}25$.

Counting Piece Arrays

OVERVIEW

A rectangular array of counting pieces is formed. Rows and/or columns of the array are turned over and the effect on the net value of the array is noted. Edge pieces are introduced and the relationship between the net value of an array and the net values of its edges is investigated.

Prerequisite Activity

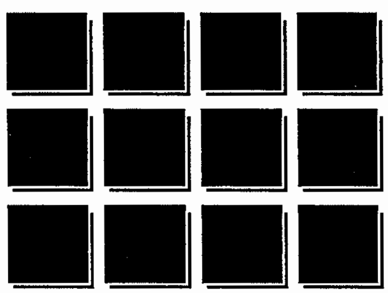
Unit VI, Activity 1, *Counting Piece Collections.*

Materials

Red and black counting pieces and red and black edge pieces for each student (see Comment 4). A copy of Activity Sheet VI-3 for each student.

Actions

1. Distribute counting pieces to each student or group of students. Have each student, or group of students, lay out an array of black counting pieces that has 3 rows and 4 columns as shown below. Ask the students what the net value of the array becomes if (a) all pieces in one column of the array, and none other, are turned over, (b) all pieces in one row, and none other, are turned over, (c) all pieces in two rows are turned over.

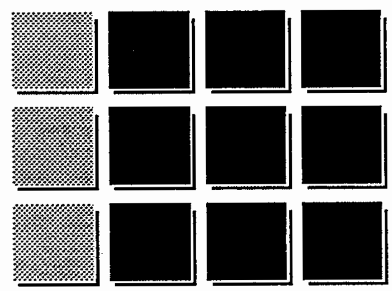


3 x 4 Array of Black Counting Pieces; Net Value: $^{+}12$

Comments

1. Each student or group of students will need no more than 20 counting pieces.

The net value of the original array is $^{+}12$. If every piece in one coulumn of the array is turned over, the net value becomes $^{+}6$:



The Array with One Column Turned Over; Net Value: $^{+}6$

Turning over every piece in one row changes the net value to $^{+}4$. The net value becomes $^{-}4$ if every piece in two rows is turned over.

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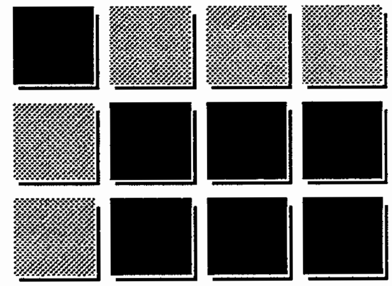
Actions

2. Ask the students what the net value of the original array becomes if first a row and then a column of the array are turned over. Discuss.

3. Ask the students what the net value of the original array becomes if two rows and then a column are turned over. Have them find other net values that can be obtained by turning over various rows and/or columns of the original array.

Comments

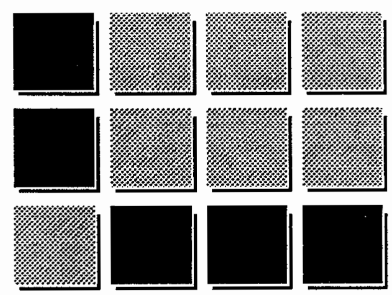
2. The net value becomes $^{+}2$:



Some points to bring out in the discussion:

- The net value is independent of which row or column is turned over.
- The net value is the same if first a column and then a row is turned over, or vice versa.
- The piece at the intersection of the row and column gets turned over twice and hence is black.

3. If two rows and a column are turned over, the net value of the resulting array is $^{-}2$:



The net values that can be obtained by turning over various combinations of rows and columns are $^{-}12, ^{-}6, ^{-}4, ^{-}2, 0, ^{+}2, ^{+}4, ^{+}6$ and $^{+}12$.

You may want to tabulate results as students report them:

Turned Over	Net Value
2 rows, 2 cols	0
0 rows, 3 cols	$^{-}6$
3 rows, 2 cols	0
3 rows, 4 cols	$^{+}12$
• •	•
• •	•
• •	•

Continued next page.

Actions

4. Distribute black and red edge pieces to each student or group of students. Show the students how edge pieces are used to indicate whether or not a row or column have been turned over.

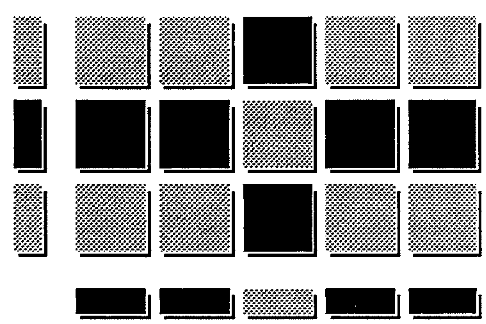
Comments

3. *Continued.* After several rows and columns of an array have been turned over, it is difficult to tell from the resulting array which rows and columns have been turned. In the next Action, red and black edge pieces are introduced. One of their functions is to provide a record of the rows and columns that are turned over.

4. Each student, or group of students, will need about a dozen edge pieces. Edge pieces are obtained by cutting counting pieces in thirds. If scissors are available, students may prepare their own.

Edge pieces are used to indicate whether or not a row or column of an array of black counting pieces has been turned over. A red edge piece at the end of a row or column indicates that that row or column has been turned over; a black edge piece indicates that it has not been turned over.

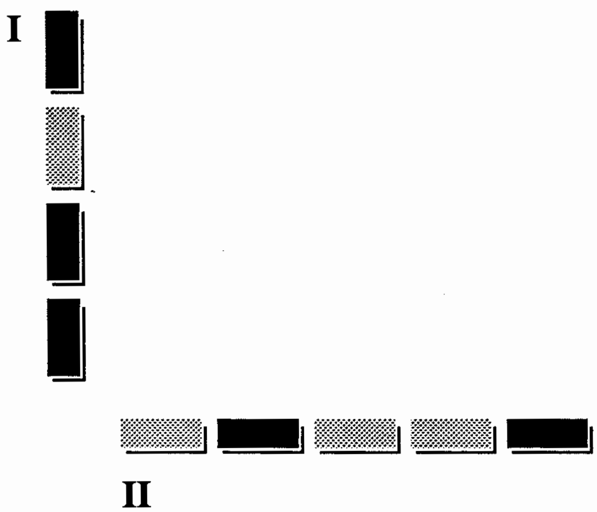
For example, the edge pieces alongside the array pictured below indicate it was obtained from an array of black pieces by successively turning over, in any order, the top and bottom rows and the middle column of the array.



Note that a piece which is the intersections of a row and column with red edge pieces has been turned over twice and hence is black. In general, if a counting piece is the intersection of a row and column which have edge pieces of the same color (both red or both black), the counting piece is black; if the row and column have edge pieces of different colors, the counting piece is red.

Actions

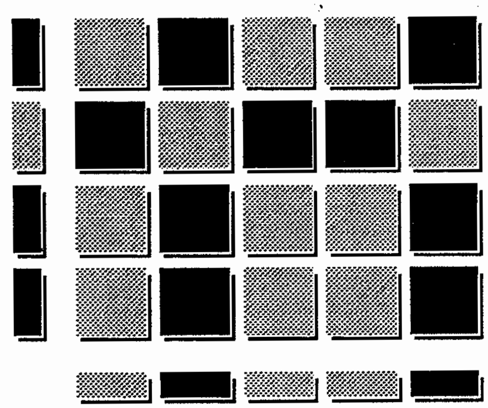
5. Ask the students to lay out edge piece collections I and II as shown below. Then ask them to lay out the associated array of counting pieces. When they have finished, give them each a copy of Activity Sheet VI-3 and, on line 1, have them record the information called for.



Comments

5. A master for Activity Sheet VI-3 is attached.

Here is a sketch of the two edge piece collections and the associated array:



The array of black and red counting pieces can be obtained by laying out a 4 x 5 array of black counting pieces and successively turning over the rows and columns which have red edge pieces. Alternately, the array can be immediately laid out by placing black pieces at the intersection of rows and columns with matching edge pieces, and placing red pieces at the intersection of rows and colums with unlike edge pieces.

Since the array has 2 more red pieces than black pieces, its net value is $^{-}2$. The net values of the edges are determined similarly. Collection I has two more black edge pieces than red and hence has a net value of $^{+}2$. Collection II contains 1 more red piece than black, so its net value is $^{-}1$.

When completed, line 1 of Activity Sheet VI-3 should appear as follows:

	Edge I		Edge II		Array		Net Values		
	R	B	R	B	R	B	Edge I	Edge II	Array
1.	1	3	3	2	11	9	$^{+}2$	$^{-}1$	$^{-}2$

Using a transparency of Activity Sheet VI-3 you can record this information at the overhead while the students are recording it at their seats.

Actions

6. Show the students an overhead transparency of Activity Sheet VI-3 in which the first line has been filled out. Ask students to suggest other collections of 5 or fewer edge pieces. Record these collections in the boxes headed Edge I and Edge II on the Activity Sheet. When this has been done for 3 or 4 lines of the activity sheet, ask the students to use their edge and counting pieces to form the associated arrays and fill in the remaining boxes on these lines. Monitor the students' work to see if they are forming arrays and determining net values correctly. Discuss any difficulties you observe.

Comments

6. Instead of asking for student suggestions, you can specify the edge collections. Here are some possibilities:

	Edge I		Edge II		Array		Net Values		
	R	B	R	B	R	B	Edge I	Edge II	Array
1.	1	3	3	2	11	9	$^+2$	$^-1$	$^-2$
2.	2	1	3	0					
3.	4	1	1	3					
4.	0	4	2	3					
5.									

For these choices, the completed table appears as follows:

	Edge I		Edge II		Array		Net Values		
	R	B	R	B	R	B	Edge I	Edge II	Array
1.	1	3	3	2	11	9	$^+2$	$^-1$	$^-2$
2.	2	1	3	0	3	6	$^-1$	$^-3$	$^+3$
3.	4	1	1	3	13	7	$^-3$	$^+2$	$^-6$
4.	0	4	2	3	8	12	$^+4$	$^+1$	$^+4$
5.									

The students' work can be monitored by observing it as you move about the room. You may also have the students work in groups or check their answers with one another. You may want to ask volunteers to record their answers on the overhead.

Actions

7. On a blank line of the transparency of Activity VI-3 used in Action 6, enter 0 in the last column (the column of Net Values headed "Array"). Ask each student, or group of students, to find an array, with edge pieces, which has this value, and record the information about this array on their activity sheet. Discuss with the students when an array will have net value 0.

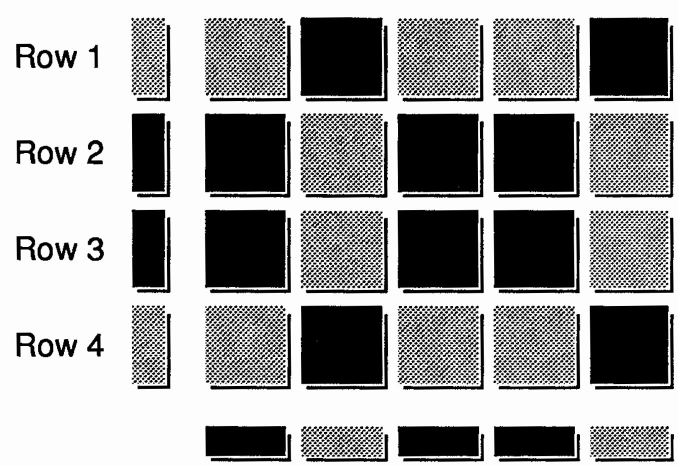
Comments

7. If the students have previously listed arrays with net value 0, ask them to find an array that is different from any of those found previously.

As students discover arrays with net value 0, the information about the array can be recorded on the overhead transparency. Listed below are some arrays whose net value is 0.

	Edge I		Edge II		Array		Net Values		
	R	B	R	B	R	B	Edge I	Edge II	Array
	⋮		⋮		⋮			⋮	
5.	2	2	1	3	8	8	0	$^+2$	0
6.	3	2	1	1	5	5	$^-1$	0	0
7.	3	3	2	2	12	12	0	0	0

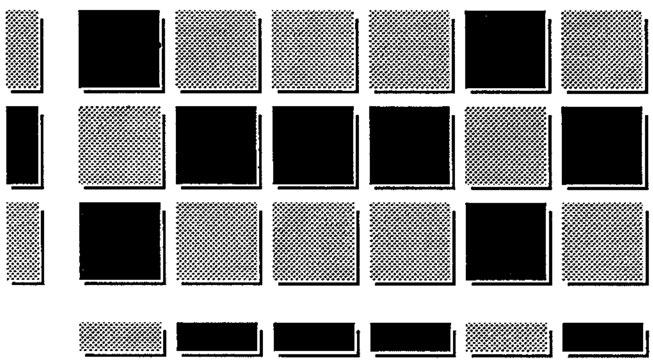
If two rows have edge pieces of opposite color, then the rows are opposites of each other and the total net value of the two rows is 0. Similarly, the total net value of two columns with edge pieces of opposite color is 0. Thus an array will have net value 0 if one of its collection of edge pieces has an equal number of red and black pieces. Note that the net value of such an edge is 0.



Rows 1 and 2 are the opposites.
So are rows 3 and 4.
The net value of the array is 0.

Actions

8. Show the students the following array of number pieces. Ask for a student volunteer to remove either two rows or two columns of the array, and the corresponding edge pieces, without changing the net value of either the array or its edges. Repeat this action until it is no longer possible to remove two rows or two columns without changing a net value. Discuss.



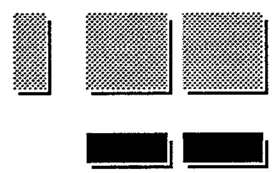
Comments

8. Two rows (or columns) are opposites of each other if they have edge pieces of opposite color. Removing them, along with their edge pieces, does not effect any net values since an equal number of black and red array pieces and black and red edge pieces are being removed.

For example, the two columns on the right of the given array can be removed without affecting any net values. Similarly the two columns on the left and the two top rows can be removed without changing any net values.

The process of removing pairs of rows and columns which have opposite colored edge pieces can be continued, without changing any net values, until all pieces in an edge have the same color. (See the exception noted in the last parargraph of this comment.) The resulting array is called a *minimal* array since no further rows or columns can be removed without changing net values.

Shown below is the minimal array that is left after pairs of rows and columns with opposite colored edge pieces have been removed from the above array.



Note that all pieces in an edge of a minimal array have the same color. Hence both sets of edge pieces will have the same color or one set of edge pieces will have one color and the other set the opposite color. If the edges have the same color, all counting pieces in the array will be black; if they have opposite colors, all counting pieces in the array will be red.

An exception occurs if an edge has net value 0, i.e., it has an equal number of red and black pieces. In this case, the above process will remove all pieces in the edge and also all counting pieces in the array.

Actions

9. Place a transparency of Activity Sheet VI-3 on the overhead. On several empty lines write $^+6$ in the last column (the column of Net Values headed "Array"). Ask the students to build minimal arrays, with edge pieces, which have this net value and fill in a line of the activity sheet for each array found. Repeat this Action using the net value $^-6$ instead of $^+6$.

Comments

9. Listed below are some minimal arrays which have net values of $^+6$ or $^-6$.

	Edge I		Edge II		Array		Net Values		
	R	B	R	B	R	B	Edge I	Edge II	Array
⋮	⋮		⋮		⋮			⋮	
8.	0	2	0	3	0	6	$^+2$	$^+3$	$^+6$
9.	2	0	3	0	0	6	$^-2$	$^-3$	$^+6$
10.	0	1	0	6	0	6	$^+1$	$^+6$	$^+6$
11.	1	0	6	0	0	6	$^-1$	$^-6$	$^+6$
12.	0	2	3	0	6	0	$^+2$	$^-3$	$^-6$
13.	2	0	0	3	6	0	$^-2$	$^+3$	$^-6$
14.	0	1	6	0	6	0	$^+1$	$^-6$	$^-6$
15.	1	0	0	6	6	0	$^-1$	$^+6$	$^-6$

Counting Piece Arrays with Edge Piece Collections

	Edge I		Edge II		Array		Net Values		
	R	B	R	B	R	B	Edge I	Edge II	Array
1.									
2.									
3.									
4.									
5.									
6.									
7.									
8.									
9.									
10.									
11.									
12.									
13.									
14.									
15									

Multiplication and Division of Signed Numbers

OVERVIEW

Counting and edge pieces are used to model multiplication and division of signed numbers.

Prerequisite Activity

Unit V, Activity 3, *Counting Piece Arrays.*

Materials

Red and black counting pieces and red and black edge pieces.

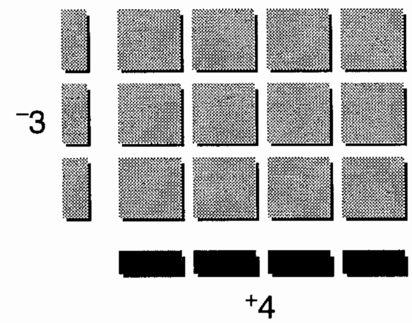
Actions

1. Distribute red and black counting pieces and edge pieces to each student or group of students. Ask the students to form an array, with edges, so that one edge has net value $^{-}3$ and the other edge has net value $^{+}4$. Have them determine the net value of the array. Discuss the multiplication statement that is being modeled.

Comments

1. Each student, or group of students, will need at least 12 counting pieces and 8 edge pieces.

A variety of arrays, with edges, can be formed which satisfy these conditions. The net value of any of these arrays will be $^{-}12$. The minimal array is shown here.



The multiplication statement being modeled is: $^{-}3 \times {}^{+}4 = {}^{-}12$.

The introduction of edge pieces allows the extension of the "rectangular array" model of multiplication to signed numbers. In Unit 3, Activity 3, *Arithmetic with Number Pieces*, and Unit 3, Activity 7, *Base 10 Multiplication*, the product of two numbers is viewed as the area of a number-piece rectangle which has those two numbers as dimensions.

Similarly, the product of two signed numbers can be viewed as the net value of an array which has these two numbers as net values of its edges. Note that if both signed numbers are positive, the net value of a minimal array is the same as its area and the net values of its edges are the same as its dimensions. Hence, the concept of net value extends the notions of area and dimension to signed numbers.

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Actions

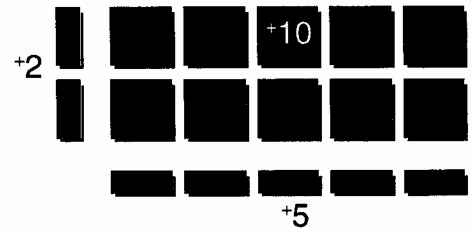
2. For each of the following, ask the students to form a minimal array, with edges, which models the product and then write the multiplication statement that is being portrayed:

(a) $^{+}2 \times {}^{+}5$ (b) $^{-}3 \times {}^{-}3$

(c) $^{+}2 \times {}^{-}4$ (d) $^{-}1 \times {}^{+}4$

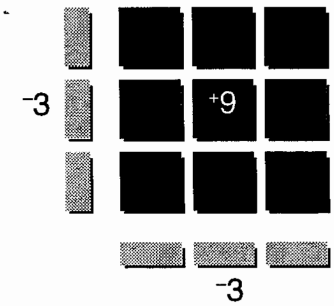
Comments

2. (a)



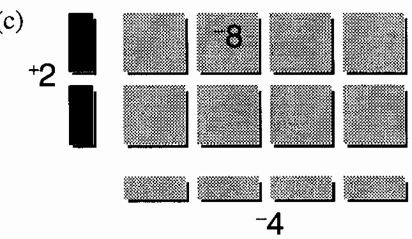
$^{+}2 \times {}^{+}5 = {}^{+}10$

(b)



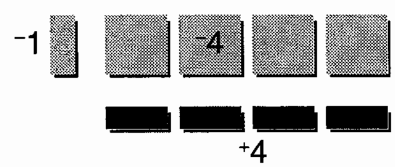
$^{-}3 \times {}^{-}3 = {}^{+}9$

(c)



$^{+}2 \times {}^{-}4 = {}^{-}8$

(d)



$^{-}1 \times {}^{+}4 = {}^{-}4$

Actions

3. Discuss with the students the relationship between the signs of two signed numbers and the sign of their product.

4. Ask the students to form an array, with edges, so that one edge has net value 0 and the other edge has net value $^{+}3$. Have them find the net value of the array. Discuss multiplication by 0.

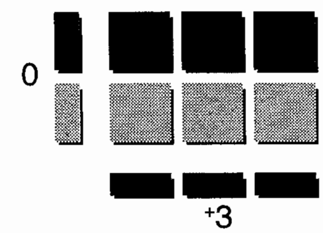
Comments

3. If both edges of a minimal array have the same color, all pieces in the array will be black. Put in another way: if both edges have positive net values or both have negative net values, the array will have a positive net value. This means that the product of two positive numbers is a positive number and the product of two negative numbers is also a positive number.

If the edges of a minimal array have different colors, all pieces in the array will be red. Put in another way: if one edge has a positive net value and the other has a negative net value, the array will have a negative net value. This means that the product of a positive number and a negative number is a negative number.

The above can be summarized as follows: The product of two numbers with like signs is positive; the product of two numbers with unlike signs is negative.

4. Below is one array. The net value of the array is 0.



$$0 \times {}^{+}3 = 0$$

If an edge has net value 0, the edge contains an equal number of red and black pieces. If an edge contains an equal number of red and black pieces, the set of counting pieces associated with red edge pieces is the opposite of the set associated with black edge pieces. Hence the net value of the array is 0. Since this is independent of the other edge, the product of 0 and any signed number is 0.

Continued next page.

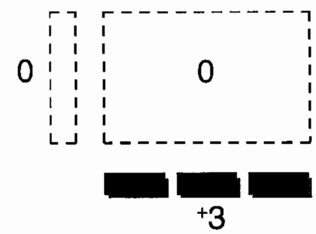
Actions

5. Tell the students that a certain array, with edges, has net value $^{-}15$ and has one edge whose net value is $^{-}3$. Ask them to determine the net value of the other edge. Discuss the division statement that is being modeled.

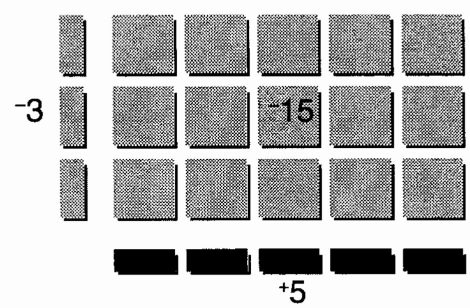
Comments

4. *Continued.* It may be interesting to discuss with your students whether or not the array pictured above is a minimal array. The question is: Can one remove the two rows without affecting any net values? The answer is "Yes," provided there is agreement that an empty set of pieces has net value 0.

Making this agreement and removing the two rows results in the following "minimal" array, in which the array and one edge are empty sets.



5. Below is a minimal array that satisfies the conditions. Since the array has a negative net value, the two edges must have unlike colors. Hence, the other edge is black. Its net value is $^{+}5$. The division statement being modeled is: $^{-}15 \div {}^{-}3 = {}^{+}5$.



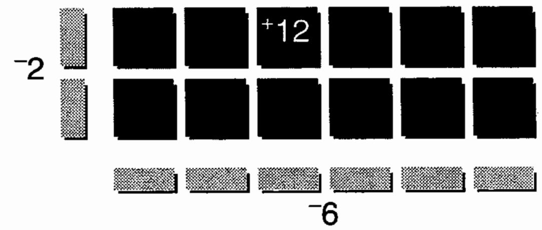
The quotient of two signed numbers can be found by forming an array, with edges, such that (1) the net value of the array is the dividend and (2) the net value of one edge is the divisor. The net value of the other edge is the quotient.

6. For each of the following, ask the students to form a minimal array, with edges, which models the quotient and then write the division statement that is being portrayed:

(a) $^{+}12 \div {}^{-}2$ (b) $^{-}8 \div {}^{-}4$

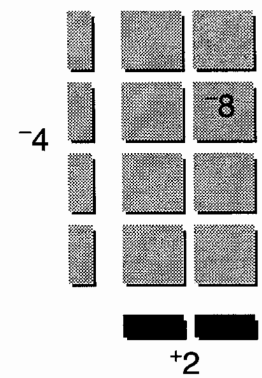
(c) $^{-}10 \div {}^{+}5$ (d) $^{+}3 \div {}^{-}1$

6. (a)



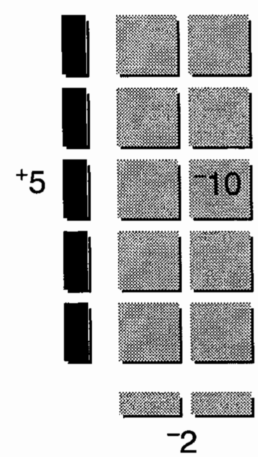
$^{+}12 \div {}^{-}2 = {}^{-}6$

(b)



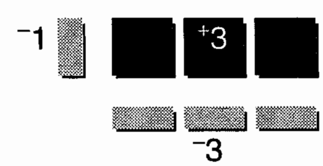
$^{-}8 \div {}^{-}4 = {}^{+}2$

(c)



$^{-}10 \div {}^{+}5 = {}^{-}2$

(d)



$^{+}3 \div {}^{-}1 = {}^{-}3$

Actions

7. Discuss with the students the relationship between the signs of the dividend, the divisor and the quotient in a division statement.

Comments

7. When a division statement is portrayed by a minimal array, with edges, the dividend corresponds to the net value of the array, the divisor corresponds to the net value of one edge and the quotient corresponds to the net value of the other edge.

If a positive number is being divided, the corresponding minimal array has a positive net value. Hence its edges' net values are both positive or both negative, that is, the quotient has the same sign as the divisor. So, if a positive number is divided by a positive number, the quotient is positive and if a positive number is divided by a negative number, the quotient is negative.

If a negative number is being divided, the corresponding minimal array has a negative net value. Hence one edge has a positive net value and one has a negative net value, that is, the quotient and divisor have opposite signs. So, if a negative number is divided by a positive number, the quotient is negative and if a negative number is divided by a negative number, the quotient is positive.

This information can be put in tabular form:

dividend	÷ divisor	= quotient
+	+	+
+	−	−
−	+	−
−	−	+

Note that if the dividend and divisor have like signs, the quotient is positive. If the dividend and divisor have unlike signs, the quotient is negative.

Actions

8. Ask the students to explore the possibility of the following:

(a) forming an array, with edges, so that the array has net value 0 and one edge has net value $^{+}4$,

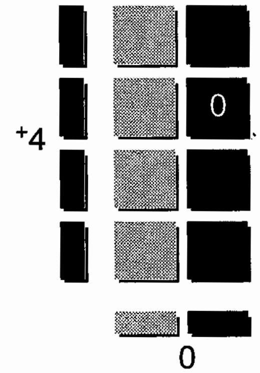
(b) forming an array, with edges, so that the array has net value $^{+}4$ and one edge has net value 0,

(c) forming an array, with edges, so that the array has net value 0 and one edge also has net value 0.

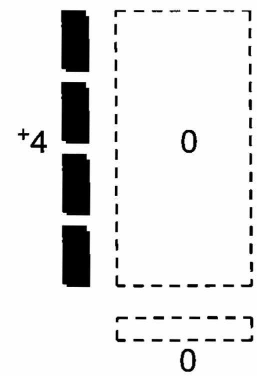
Discuss divisions involving 0 with the students.

Comments

8. (a) Here is one possibility:

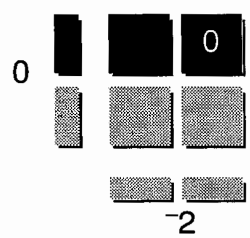


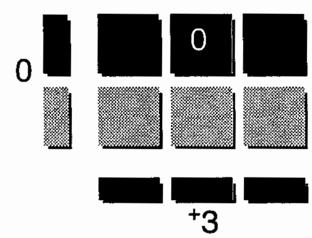
Some students may form a minimal array in which the array and one edge are empty sets:



(b) No such array exists. If an edge has net value 0, the array will have net value 0.

(c) Many such arrays exist. Two are pictured in (a) above. Here are two others:





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Unit VI / Math and the Mind's Eye

Transparencies

for Modeling Integers

The Math Learning Center
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Salem, Oregon 97302

Catalog #MET6

Part A

Fill in the missing numbers:

Collections	Total No. of Pieces	No. of Red Pieces	No. of Black Pieces	Net Value
A.		6		$^{+}3$
B.		2	7	
C.	12		5	
D.		4		0
E.	10			0
F.	7			$^{+}7$
G.			8	$^{+}3$
H.	13			$^{-}5$

Part B

Collection X contains 2 red and 7 black pieces.
Collection Y contains 8 red and 5 black pieces.
Collection Z contains 7 red and 3 black pieces.

Record the net value of collection X: ______, Y: ______, Z: ______.

Record the net value if collections X and Y are combined: ______.

Record the net value if collections Y and Z are combined: ______.

Record the net value if collection X and the opposite of collection Y are combined:

______.

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Counting Piece Arrays with Edge Piece Collections

	Edge I		Edge II		Array		Net Values		
	R	B	R	B	R	B	Edge I	Edge II	Array
1.									
2.									
3.									
4.									
5.									
6.									
7.									
8.									
9.									
10.									
11.									
12.									
13.									
14.									

Activity Sheet VI-3

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