

Modeling Rationals

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Modeling Rationals



Egg Carton Fractions

Egg carton diagrams are used as visual models to introduce fractions and fraction equivalence.



Fractions on A Line

Line segments are divided into equal parts as a means of introducing the division model for fractions.



Fraction Bars

The Fraction Bar[®] model for fractions is introduced and used to discuss fraction equality and inequality.



Addition and Subtraction with Fraction Bars

Fraction bars are used to illustrate processes for adding and subtracting fractions.



Multiplication and Division with Fraction Bars

Fraction bars are used to illustrate processes for multiplying and dividing fractions.



Introduction to Decimals

With the aid of base 10 number pieces, the concept of a decimal is introduced and decimal notation is discussed.



Decimal Addition and Subtraction

Base 10 number pieces are used to develop processes for adding and subtracting decimals.



Decimal Length and Area

The dimensions and areas of rectangles are found and the distinction between linear measure and area measure is discussed.



Decimal Multiplication and Division

Base 10 number piece rectangles are used to find the product and quotient of decimals.



Fraction Operations via Area: Addition and Subtraction

Fractions are represented by areas of rectangular regions, and fraction sums and differences found by finding the sums and differences of areas.



Fraction Operations via Area: Multiplication

Two fractions are multiplied by viewing them as the dimensions of a rectangle and their product as the rectangle's area.



Fraction Operations via Area: Division

The quotient of two fractions is found by constructing a rectangle for which the area and one dimension are given.



Math and the Mind's Eye materials are intended for use in grades 4-9.

They are written so teachers can adapt them to fit student backgrounds and grade levels. A single activity can be extended over several days or used in part.

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Egg Carton Fractions

O V E R V I E W

Egg cartons and diagrams of egg cartons are used as visual models to introduce fractions and fraction equivalence.

Prerequisite Activity

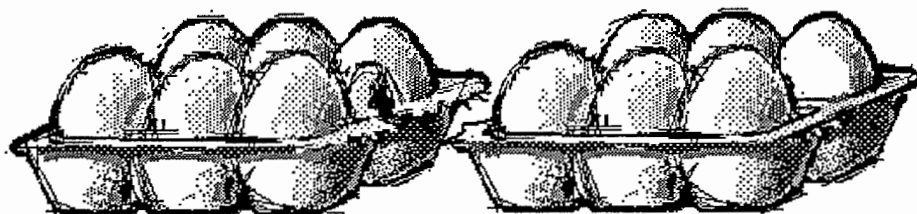
None

Materials

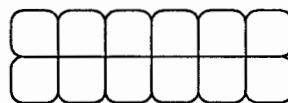
A few one-dozen egg cartons, a copy of Activity Sheet A for each student, an overhead transparency of Activity Sheet A; index cards for optional Action 6; and, for optional Action 7, an overhead transparency and student copies of Activity Sheet B

Actions

1. Show the students an egg carton cut in half as an example of how some people buy one-half dozen eggs.



Ask the students how they could cut the carton differently to get one-half dozen eggs. Discuss.

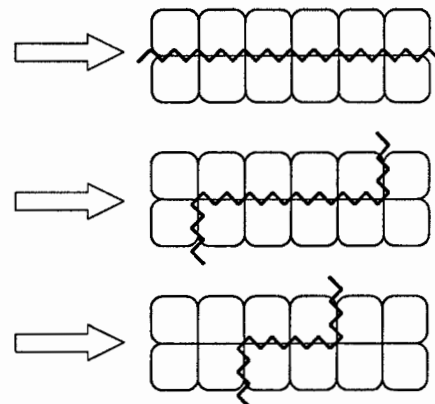


Comments

1. It is common to see egg cartons that have been cut in half in supermarkets.

Using an overhead transparency made from Activity Sheet A, discuss different ways to cut an egg carton to get one-half dozen eggs. Encourage volunteers to come forth to draw their methods.

Here are three ways:



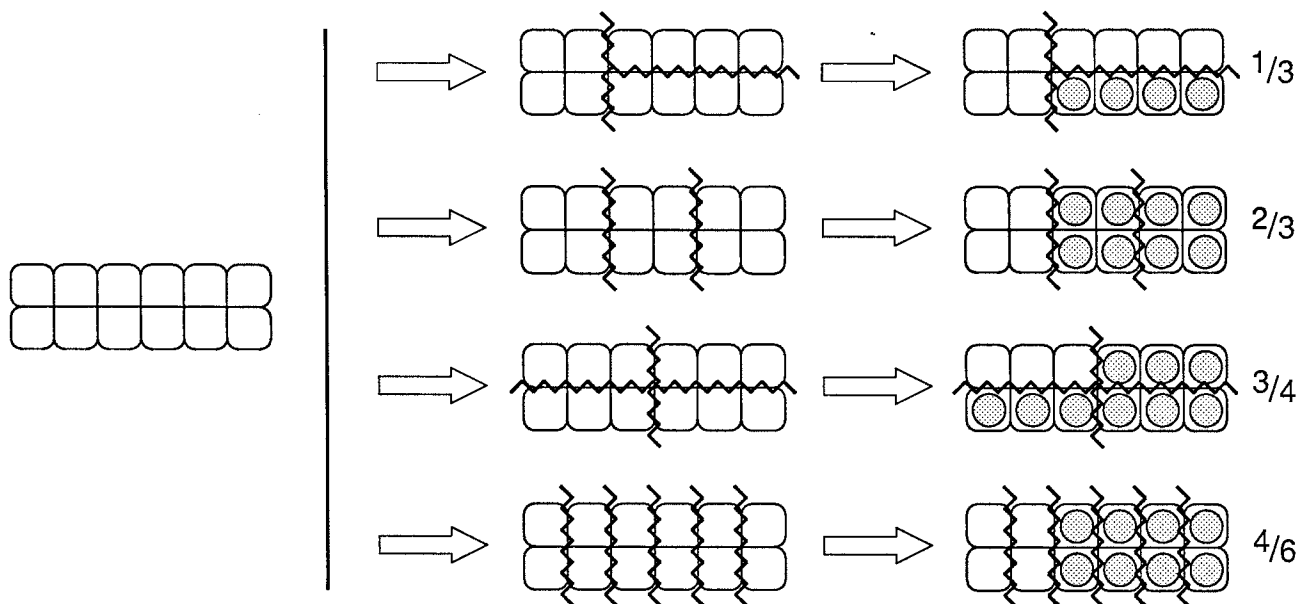
Actions

2. Distribute copies of the egg carton diagrams (Activity Sheet A). Ask the students to devise ways to subdivide the carton to show one-third of a dozen eggs. Extend this question to include two-thirds, three-fourths and four-sixths of a dozen.

Comments

2. You may wish to begin by doing one or two examples on the overhead. To diagram the egg carton fraction $\frac{2}{3}$, for example, subdivide the carton into three equal parts, fill two of those parts with eggs and then write $\frac{2}{3}$ under the diagram. As the students draw their diagrams have them write the fraction symbol next to their drawing.

There are many ways to do each of these, but here are some typical responses:



The essential idea for this model of the fraction $\frac{a}{b}$ is to subdivide the whole carton into b equal parts (equal in the sense that each part holds the same number of eggs) and then fill a of these parts with eggs. Discuss enough examples so that students can relate fractions to egg cartons.

Actions

3. Ask the students to find other egg carton fractions of a dozen, recording each fraction and its diagram. When the students finish, display and discuss the results.

Comments

3. This activity works well in small groups. Here are the 32 egg carton fractions that students usually discover:

$0/12, 1/12, 2/12, \dots, 11/12, 12/12$

$0/6, 1/6, 2/6, \dots, 5/6, 6/6$

$0/4, 1/4, 2/4, \dots, 3/4, 4/4$

$0/3, 1/3, 2/3, 3/3$

$0/2, 1/2, 2/2$

Sometimes students want to add $0/1$ and $1/1$ to the list of egg carton fractions. $0/1$ and $1/1$ are obtained by not subdividing the carton and filling all or none of it. You may wish to add these two fractions to the list if your students discover them.

The zeroes, like $0/4$, come from dividing an egg carton into 4 equal parts but not filling any of them.

Because the question is open-ended there may be some ingenious answers. Some students have obtained up to 90 egg carton fractions. For example, they get elevenths by removing one egg from the carton and then dividing the carton into eleven parts. Of course, they are no longer getting fractions of a dozen. These interesting approaches should be acknowledged.

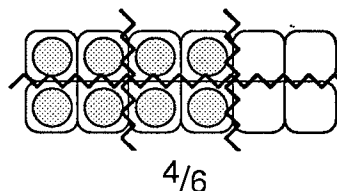
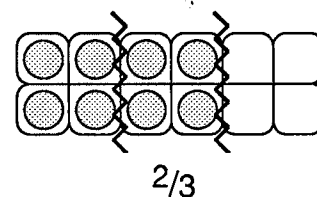
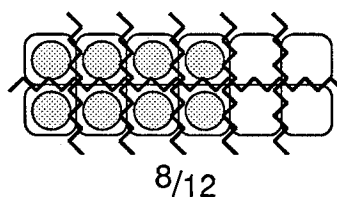
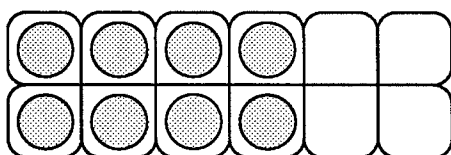
Seldom do students ask if $6/12$ of a dozen is equal to $2/4$ of a dozen. As symbols they are different, but they do represent the same number of eggs. The fact that some fractions are *equivalent* to others will be addressed in Action 5.

Actions

Comments

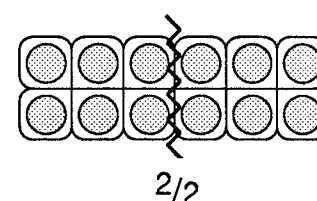
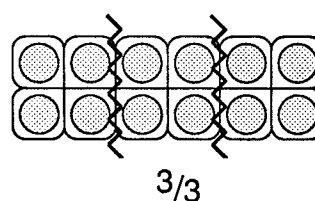
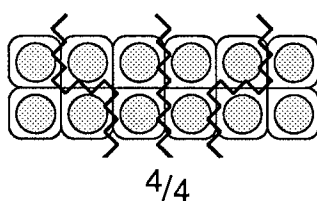
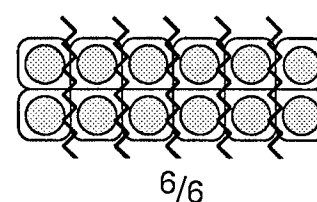
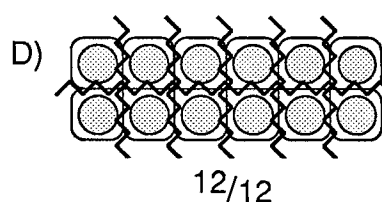
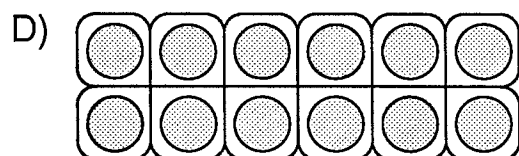
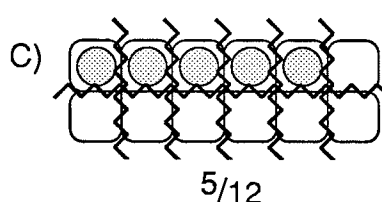
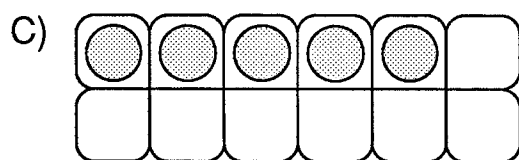
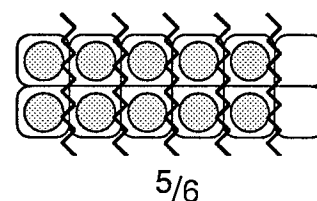
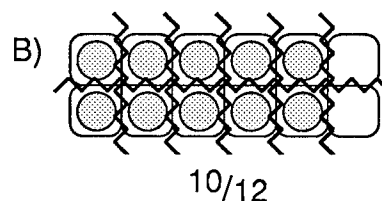
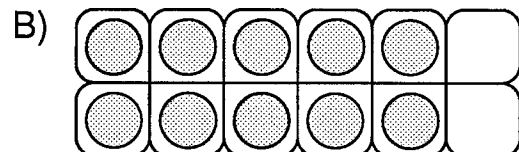
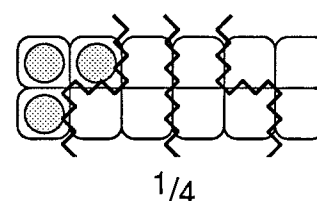
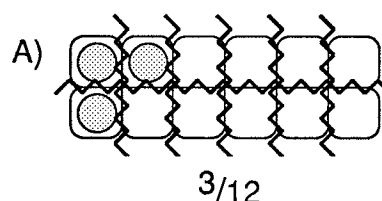
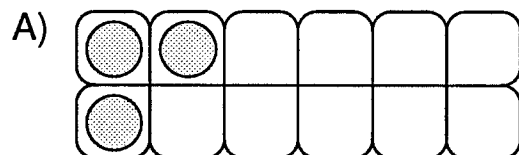
4. Using the overhead transparency of the egg cartons, show the students the following diagram and ask them to determine what fraction of a dozen eggs is in the carton.

4. The carton with eight eggs can be described as eight-twelfths of a dozen, two-thirds of a dozen, or four-sixth of a dozen depending upon how the subdivision is visualized.

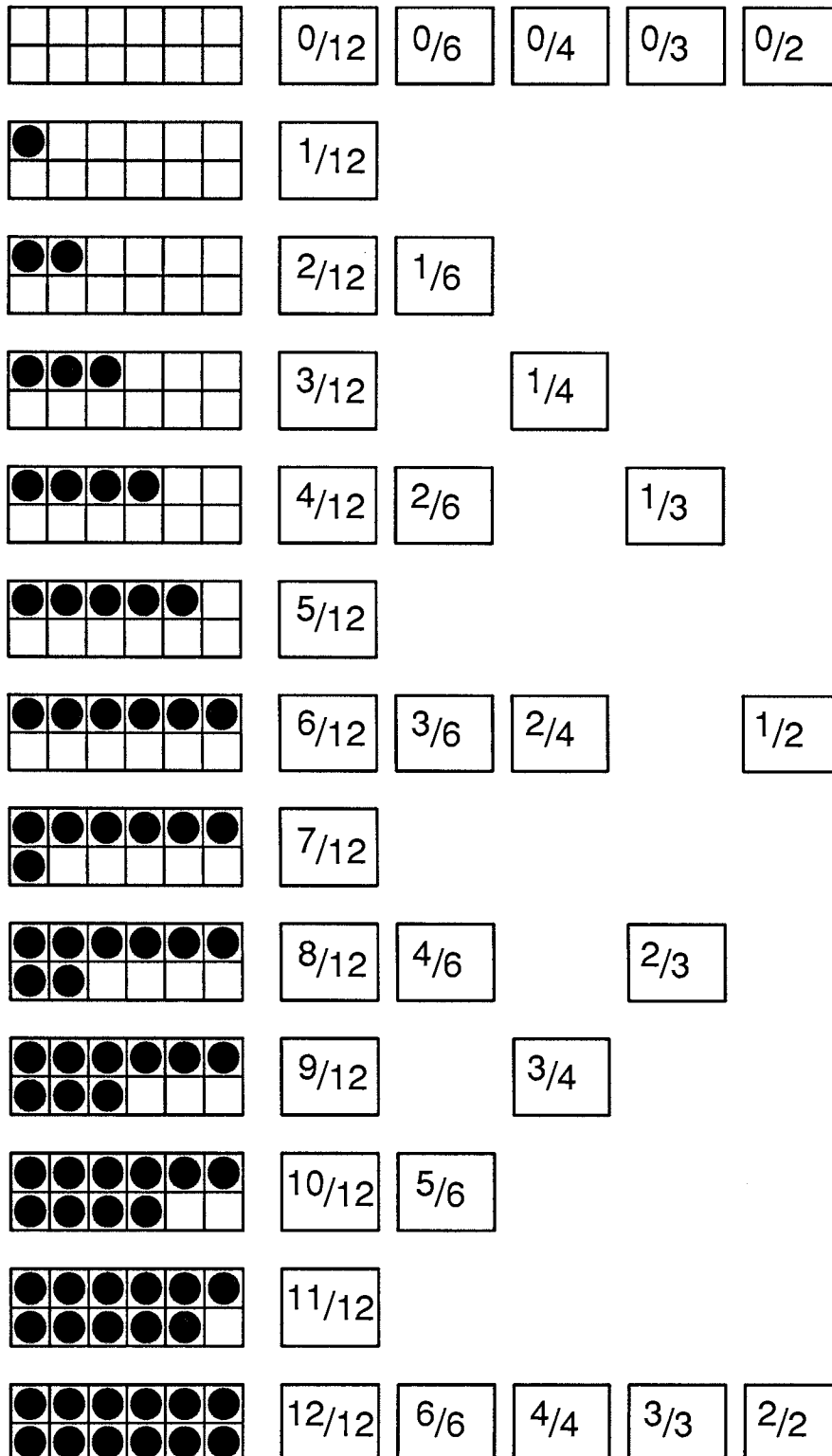


Have them determine the fraction of a dozen in each of the following (you may wish to add more examples). Discuss each case.

The others can be represented as follows:



6. (Optional — Egg Carton Fraction Wall Chart) Print each of the 32 egg carton fractions on an index card. Make 13 large egg carton diagrams with a different number of eggs in each. Have the students help you arrange a wall chart similar to the following:



6. Leaving this chart on the wall for several days is helpful for many students. It can be used for warm-up activities at the beginning of class. Here are two ideas for warm ups:

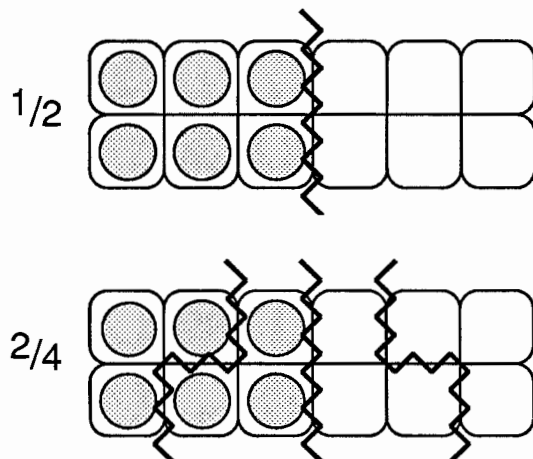
a. Before students arrive in class, scramble a few fraction cards to see if students can spot and correct the errors.

b. Take a few fraction cards off the wall and select students to correctly replace them.

In both warm-ups the students can be asked to "prove" the correct replacement by sketching an egg carton diagram corresponding to their fraction.

Actions

5. Use a diagram to illustrate that $\frac{1}{2}$ and $\frac{2}{4}$, of a dozen, are both egg carton fractions that represent 6 eggs. Ask them what other egg carton fractions represent 6 eggs?



Ask them to separate the remaining egg carton fractions into groups, so that each fraction in the same group represents the same number of eggs. You may wish to point out that fractions in the same groups are called *equivalent* fractions.

Comments

5. The egg carton fractions $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$ and $\frac{6}{12}$ all represent six eggs.

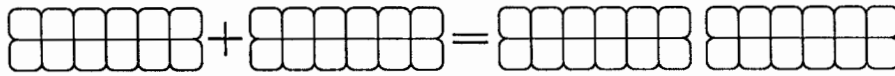
The 32 egg carton fractions form twelve groups of equivalent fractions:

$\frac{0}{12}$	$\frac{0}{6}$	$\frac{0}{4}$	$\frac{0}{3}$	$\frac{0}{2}$
$\frac{1}{12}$				
$\frac{2}{12}$	$\frac{1}{6}$			
$\frac{3}{12}$		$\frac{1}{4}$		
$\frac{4}{12}$	$\frac{2}{6}$		$\frac{1}{3}$	
$\frac{5}{12}$				
$\frac{6}{12}$	$\frac{3}{6}$	$\frac{2}{4}$		$\frac{1}{2}$
$\frac{7}{12}$				
$\frac{8}{12}$	$\frac{4}{6}$		$\frac{2}{3}$	
$\frac{9}{12}$		$\frac{3}{4}$		
$\frac{10}{12}$	$\frac{5}{6}$			
$\frac{11}{12}$				
$\frac{12}{12}$	$\frac{6}{6}$	$\frac{4}{4}$	$\frac{3}{3}$	$\frac{2}{2}$

Actions

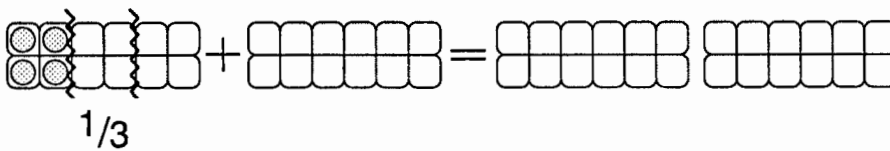
Comments

7. (Optional — Adding Egg Carton Fractions) Put this diagram on the overhead and go through steps like the following:

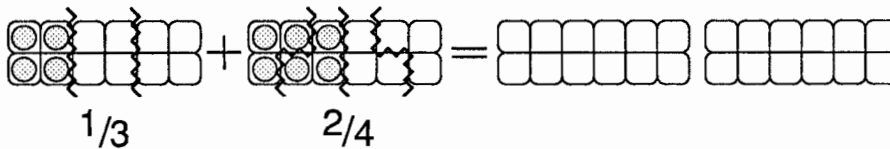


a. Shuffle a deck of 32 fraction cards like the ones made for the wall chart in *Action 6*.

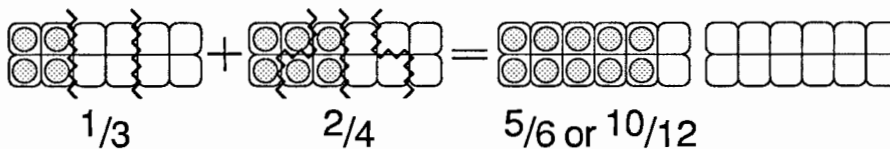
b. Select one card and ask for a volunteer to sketch that fraction of a dozen in the first egg carton and write the fraction below the egg carton (suppose it is $\frac{1}{3}$).



c. Select a second card and repeat the previous directions for the second egg carton (suppose it is $\frac{2}{4}$).



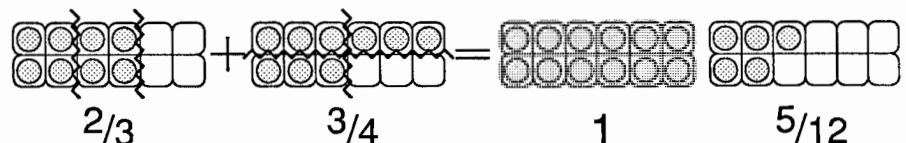
d. Ask a volunteer to combine the eggs into one carton and write a number for the fraction of a dozen eggs that result.



Give each student a copy of activity sheet **B** and then repeat this activity by drawing cards and having students record the fractions and adding individually.

7. (See below.)

Two egg cartons have been placed to the right of the equal sign on Activity Sheet B. This is because it is possible that some fraction addition exercises generated by this activity will produce a full dozen and a fraction of a dozen as in this example:



After doing this addition activity you may wish to repeat the directions, but this time, instead of using the egg carton addition diagrams, focus attention on the wall chart of Action 6. See if students can discover an easier way to add using equivalent fractions from the chart.

Egg Carton Recording Paper

Egg Carton Addition Paper

A visual representation of the addition $4 + 4 = 8$ using ten-frames. Two ten-frames, each with 4 dots, are added to form a single ten-frame with 8 dots.

A visual representation of the addition $20 + 20 = 40$ using tens blocks. On the left, there are two groups of two tens blocks (each a 2x10 grid). In the middle is a plus sign. To the right of the plus sign is another group of two tens blocks, followed by an equals sign. To the right of the equals sign are four tens blocks, arranged in two groups of two. This shows that two groups of 20 are equal to one group of 40.

A visual representation of the addition $8 + 8 = 16$ using ten-frames. Two ten-frames, each containing 8 dots, are added together to equal two ten-frames, each containing 10 dots.

A visual representation of the addition $20 + 20 = 40$ using base ten blocks. Two tens rods (each composed of 10 unit cubes) are added to two more tens rods, resulting in four tens rods.

A visual representation of the addition $4 + 4 = 8$ using ten frames. The first part shows two ten frames, each containing 4 dots (2 in the top row, 2 in the bottom row). These are followed by a plus sign and an equals sign. The final part shows two ten frames, each containing 8 dots (4 in the top row, 4 in the bottom row).

Two ten-frames, each containing 10 dots, are added together to equal two ten-frames, each containing 10 dots.

40 + 40 = 80

Fractions on a Line

O V E R V I E W

One way to view fractions is in terms of division. That is, $\frac{3}{5}$ can be visualized as the length of one part when a segment of 3 units is divided into 5 equal parts. This activity introduces the division model for fractions. Part I introduces a practical method for dividing segments into equal parts. Part II uses this line-dividing technique to explore the division model for fractions.

Prerequisite Activity

Unit IV • Activity 1 or some previous work with fractions

Materials

A Parallel-line sheet and Activity Sheets A and B for each student. An overhead transparency of each of these sheets.

Actions

Part I Dividing Line Segments into Equal Parts

1. Demonstrate the parallel line method of dividing a segment into equal parts.

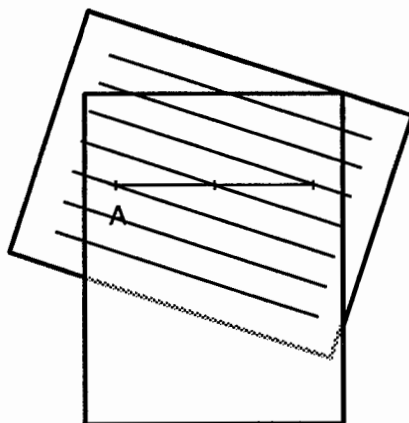


Figure 1

Comments

1. Place a transparency of equally spaced parallel lines beneath another transparency which has a line segment on it. A master for a parallel line sheet is attached.

Figure 1 shows the segment divided into 2 equal parts. Figure 2 shows the segment divided into 3 equal parts.

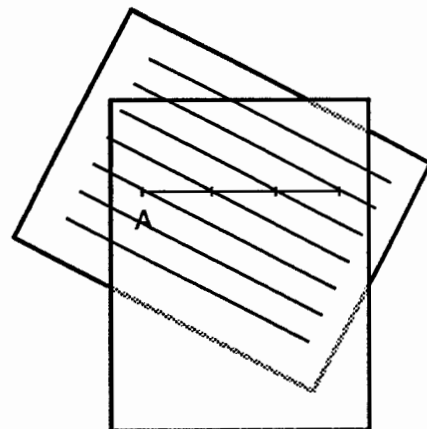


Figure 2

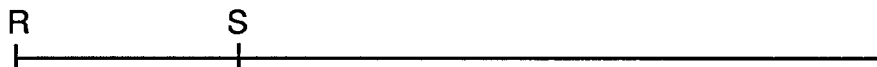
2. Distribute Activity Sheet A and a parallel line sheet to each student. Have the students subdivide the segments in number 1 as indicated.

2. This can be done by placing a parallel line sheet with black lines under their activity sheet and proceeding as you did on the overhead projector.

Actions

3. Draw the following diagram on the overhead.

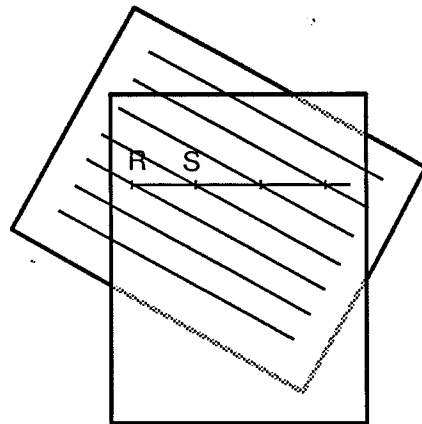
Tell the class you want to mark points to the right of S so that each space (or interval) between points is the same length as interval RS. Ask how this can be done using equally spaced parallel lines.



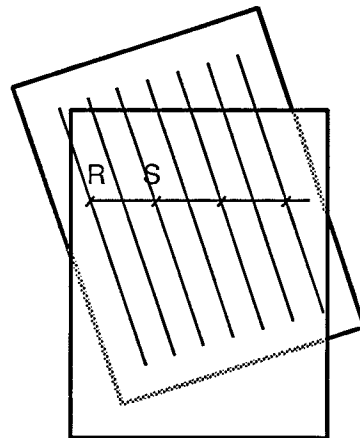
4. Ask the students to use their parallel lines to solve the remaining problems on the Activity Sheet A. Have the class discuss their results.

Comments

3. Students may enjoy demonstrating their methods on the overhead. Some may use 1 space for each interval.



Others may use 2 spaces for each interval.

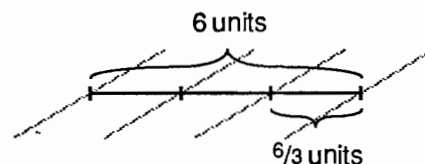


4. There may be more than one way to solve a problem.

Part II The Division Model for Fractions

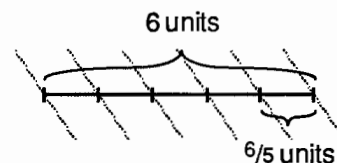
5. Draw a segment representing 6 units on the overhead projector. Use parallel lines to divide it into 3 equal parts. Ask the students how long each part is. Tell them that we can also speak of the length as six-thirds, $\frac{6}{3}$.

5. This is called the division model for fractions. A fraction like $\frac{6}{3}$ can be represented as the length of one part when 6 units are split into 3 equal parts.



6. Duplicate the above 6 unit line and divide it into 5 equal parts. Ask the students to determine the length of each equal part.

6. Some may answer $1\frac{1}{5}$ units. Explain that six-fifths, $\frac{6}{5}$, is another acceptable name and conveys the action of dividing 6 units into 5 equal parts.

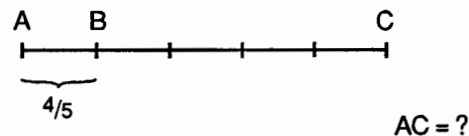
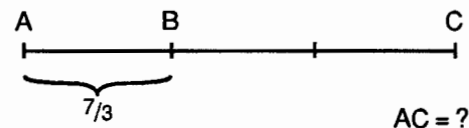


7. Have students work problems 1 and 2 on Activity Sheet B.

7. You may wish to do the first part of each problem on the overhead with class discussion.

8. Put the following sketch on your overhead. Ask the students: If AB is $\frac{2}{3}$ units, how long is AC? Discuss.

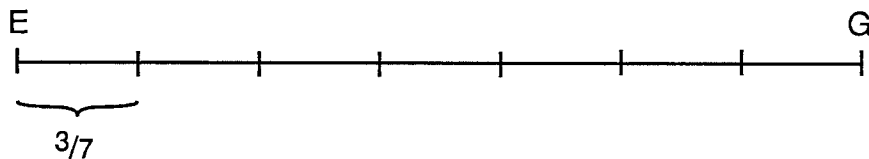
8. Since $\frac{2}{3}$ is the length obtained when 2 units are divided into 3 equal parts, then three lengths of $\frac{2}{3}$ must be 2 units. You may wish to include a few more examples here:



9. Put this drawing on the overhead projector and ask the students to help you

(a) determine the length of segment EG

(b) locate a point F so that EF is 1 unit long.



10. Have the students complete problems 3 to 5 on Activity Sheet B.

9. EG must be 3 units long. Because $\frac{3}{7}$ results from dividing 3 into 7 equal parts, 7 lengths of $\frac{3}{7}$ must be 3 units.

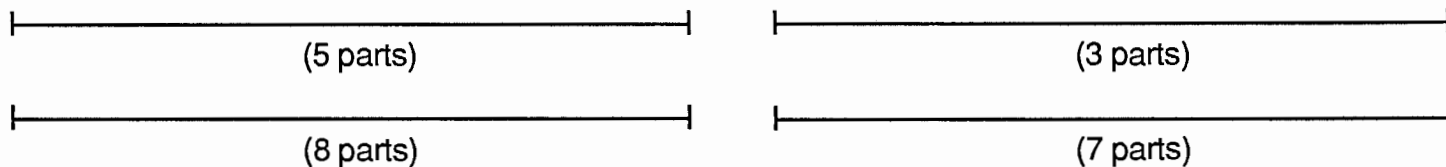
To find point F, EG must be divided into 3 equal parts.

10. Again, there may be more than one way to solve a problem.

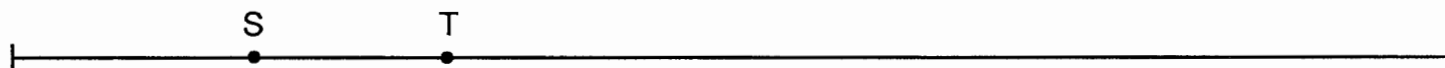
Name _____

Fractions on a Line

1. Use the parallel line sheet to divide each segment into the indicated number of parts.



2. Locate points to the right of T and to the left of S so that distance between adjacent points is the same as ST.



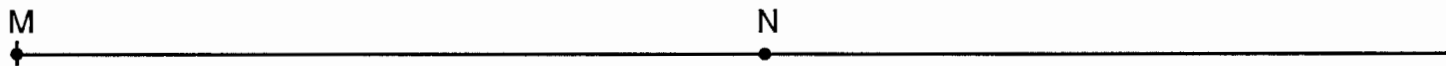
3. If the distance from X to Y is 1 unit, what is the distance from X to Z?



4. If the distance from A to B is 7 units, locate a point P which is 5 units from A.



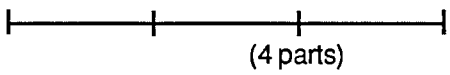
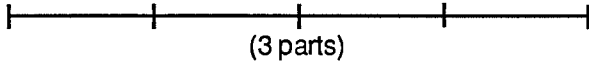
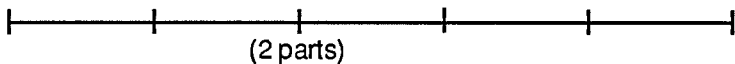
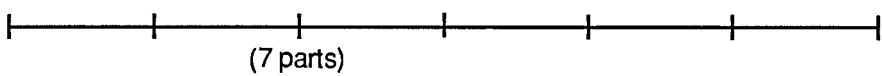
5. If MN is 3 units, find point Q so that MQ is 5 units.




Name _____


Fractions on a Line


1. Use the parallel line sheet to divide each segment into the indicated number of parts. Then write a fraction name for each part

		Length of One Part
3 units		a. <input type="text"/>
4 units		b. <input type="text"/>
5 units		c. <input type="text"/>
6 units		d. <input type="text"/>

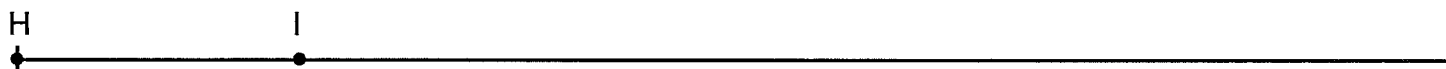
2. Use the parallel lines to locate the indicated fraction on the given number line.

3 units  $\frac{3}{5}$

5 units  $\frac{5}{2}$

7 units  $\frac{7}{10}$

3. HI is $\frac{1}{4}$ of a unit. Find point J so that HJ is 1 unit.

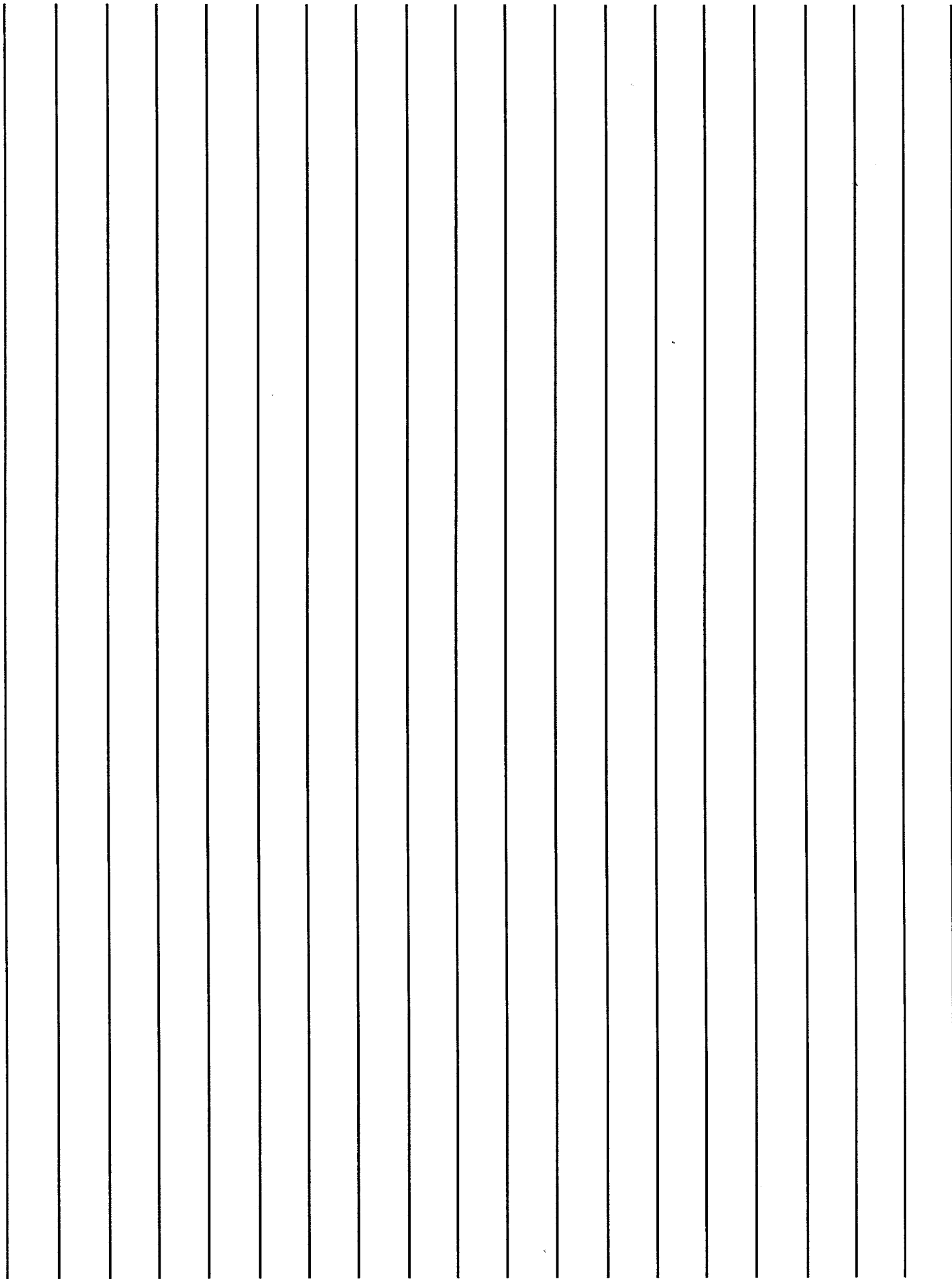


4. UV is $\frac{3}{5}$ of a unit. Find point M so that UM is 3 units.



5. UV is $\frac{3}{5}$ units. Find point W so that UW is 1 unit.

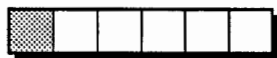




Fraction Bars

O V E R V I E W

This activity uses FRACTION BARS® as one model for fractions. In the fraction bar model, the denominator of a fraction is represented by the number of parts in a bar and the numerator by the number of shaded parts. In addition to dealing with the basic concept of fractions, the activity introduces fraction equality and inequality.


 $\frac{2}{3}$

 $\frac{1}{6}$

 $\frac{3}{4}$

Prerequisite Activity

None

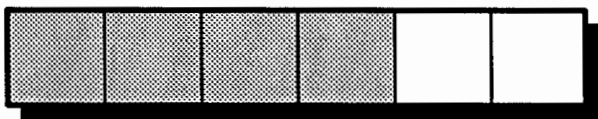
Materials

Fraction bars or transparencies of fraction bars (see Comment 1)

Actions

1. Distribute fraction bars to each student or group of students.

2. Show the students a sixth bar (red) with 4 parts shaded.



Tell the students this is one model for "four-sixths". Write the fraction $\frac{4}{6}$ on the board and discuss the meaning of the "top number" and the "bottom number".

3. Show the students, or have them select, several different bars. Have them describe each bar and then give its fraction.

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Comments

1. If fraction bars are not available, this activity can be done by using fraction bar transparencies on the overhead: make transparencies of the fraction bar master sheets included with this activity and cut out the bars. Coloring the bars (halves – green, thirds – yellow, fourths – blue, sixths – red, twelfths – orange) helps in differentiating bars with different numbers of parts.

2. The top number, or *numerator*, tells the number of shaded parts. The bottom number, or *denominator*, tells the number of equal parts in the bar. You may wish to defer the introduction of the terms "numerator" and "denominator" until students are more familiar with fractions.

3. Some students may describe a bar by giving its color and the number of shaded parts. Others may tell the total number of parts and the number of shaded parts.

Actions

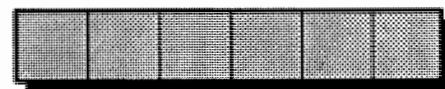
4. Show the students a bar with all parts shaded, and have them write the fraction for the bar. Do the same for a bar with no parts shaded. Note that the fraction for the former equals 1 and for the latter equals 0.

5. Write a few fractions on the board (or overhead) and ask students to visualize and then describe the bars for these fractions. For example, the bar for $\frac{3}{8}$ has 8 equal parts and 3 of them are shaded.

6. Place a $\frac{3}{12}$ bar under a $\frac{1}{4}$ bar. Describe what this shows. Ask the students for other pairs of bars which have the same amount of shading. If the students have fraction bars, have them sort the bars into piles according to their shaded amounts.

Comments

4. A fraction bar with all parts shaded is called a *whole bar*, and one with no shaded parts is called a *zero bar*.



$$\frac{6}{6} = 1$$



$$\frac{0}{4} = 0$$

5. You may want to use fractions with relatively small denominators, say, less than 20. However, it may be instructive to see if the students can describe bars for fractions such as $\frac{3}{50}$, $\frac{99}{100}$, or $\frac{1}{1000}$.

6. One part out of four has the same shading as three parts out of twelve. If two bars have the same amount of shading, we say their fractions are *equal*. Thus $\frac{1}{4} = \frac{3}{12}$.



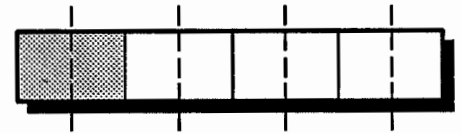
$$\frac{1}{4} = \frac{3}{12}$$

Actions

7. Show the students a $\frac{1}{4}$ bar and ask them to describe bars with more parts but the same amount of shading. Repeat this activity for bars for $\frac{1}{2}$, $\frac{3}{4}$, $\frac{2}{3}$, or other bars of your choice. Discuss with students the methods they use.

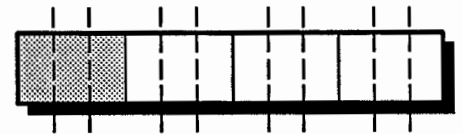
Comments

7. Some other bars that would have the same amount of shading as a $\frac{1}{4}$ bar are bars for $\frac{2}{8}$, $\frac{3}{12}$, and $\frac{4}{16}$. One method of forming these bars is to divide each part of a $\frac{1}{4}$ bar into an equal number of parts. For example, dividing each part of a $\frac{1}{4}$ bar into two equal parts doubles both the number of parts and the number of shaded parts on a $\frac{1}{4}$ bar. The result is a bar for $\frac{2}{8}$.



$$\frac{1}{4} = 2 \times \frac{1}{2} \times 4 = \frac{2}{8}$$

Similarly, dividing each part of a $\frac{1}{4}$ bar into 3 equal parts triples both the number of parts and the number of shaded parts, resulting in a bar for $\frac{3}{12}$.

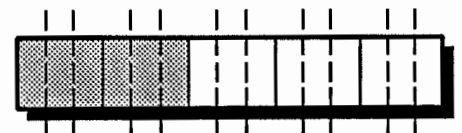


$$\frac{1}{4} = 3 \times \frac{1}{3} \times 4 = \frac{3}{12}$$

If transparency fractions bars are being used, a transparency sheet can be placed over a bar and dotted lines for dividing the parts of the bar can be drawn on the sheet.

8. Ask the students to sketch a bar for $\frac{2}{5}$. Then ask them to divide the parts of this bar to show that $\frac{2}{5} = \frac{6}{15}$. Discuss the relationship between numerators and denominators of these two fractions.

8. Since each part is divided into 3 equal parts, both the number of shaded parts (the numerator) and the total number of parts (the denominator) are increased by a factor of 3. The "new" numerator is 3 times the "old" numerator and the "new" denominator is 3 times the "old" denominator.



$$\frac{2}{5} = \frac{6}{15}$$

Actions

9. Discuss a general method for forming equal fractions. Give several examples.

10. Write the fractions $\frac{1}{3}$ and $\frac{1}{4}$ on the board (or overhead) and ask the students to determine which is the greater fraction. Discuss their reasons. Show the students a symbolic way of writing their conclusion.

11. Ask the students to determine the greater fraction for each of the following pairs: $\frac{1}{2}$ and $\frac{4}{6}$, $\frac{5}{12}$ and $\frac{2}{3}$, $\frac{5}{6}$ and $\frac{7}{12}$, $\frac{2}{3}$ and $\frac{3}{4}$. Discuss with them how they arrived at their conclusions.

Comments

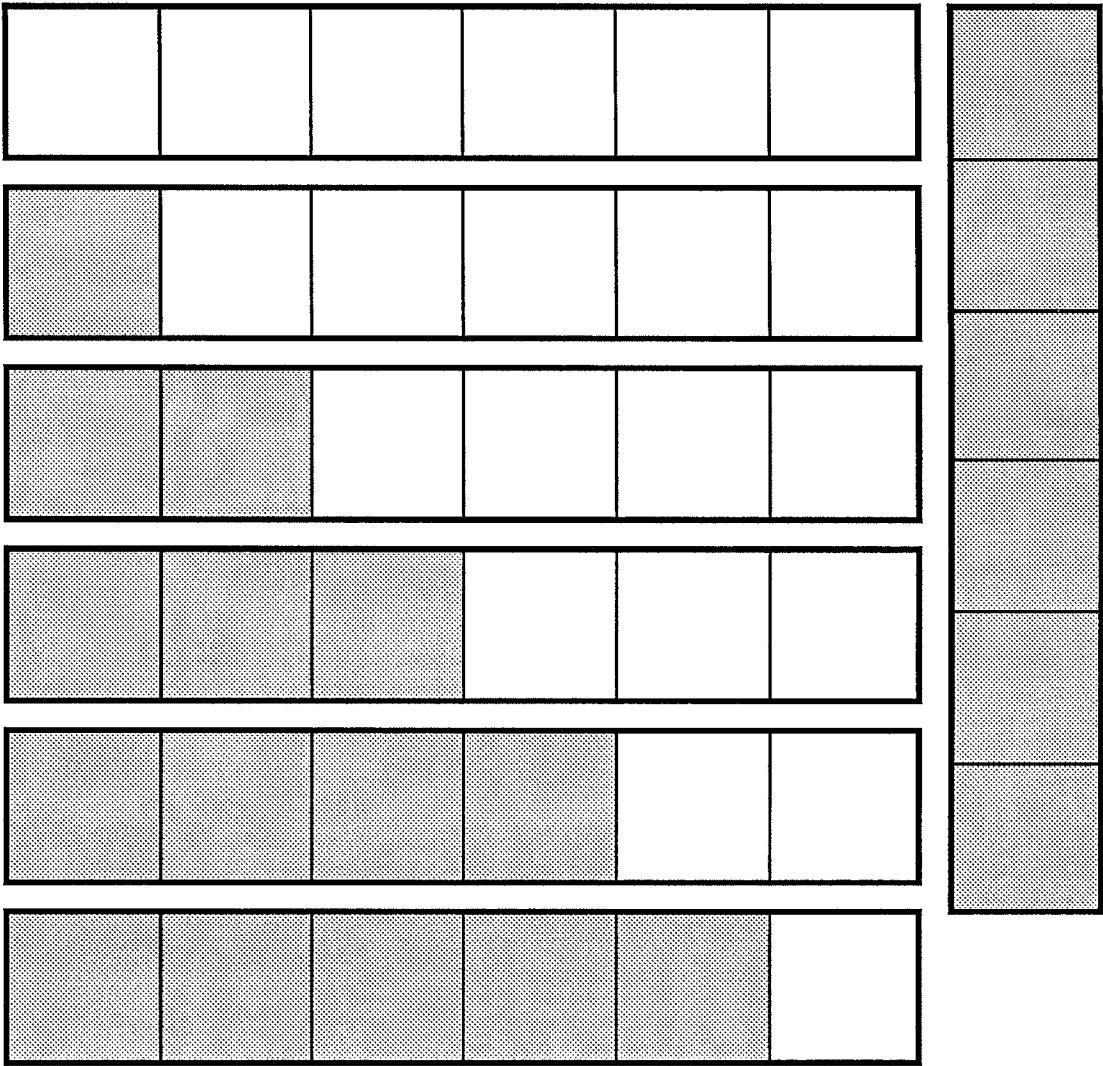
9. Dividing each part of a fraction bar into the same number of equal parts increases both the number of shaded parts and the total number of parts by the same factor, i.e. multiplying the numerator and the denominator of a fraction by the same whole number produces an equal fraction.

10. One explanation that $\frac{1}{3}$ is greater than $\frac{1}{4}$ is that the shaded portion of a $\frac{1}{3}$ bar is larger than the shaded portion of a $\frac{1}{4}$ bar. Another explanation is that dividing a bar into 3 equal parts produces larger parts than dividing a bar into 4 equal parts.

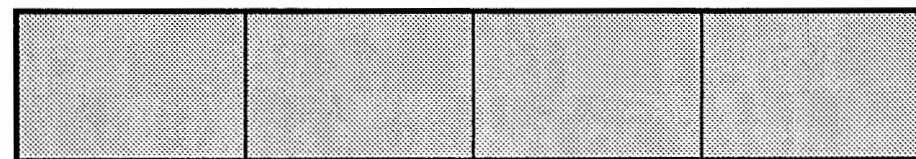
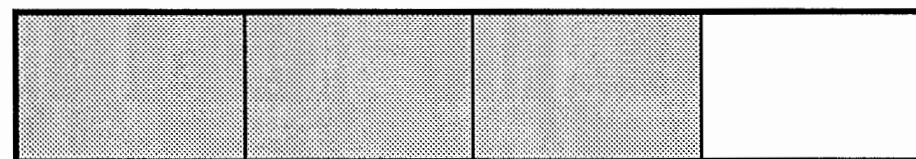
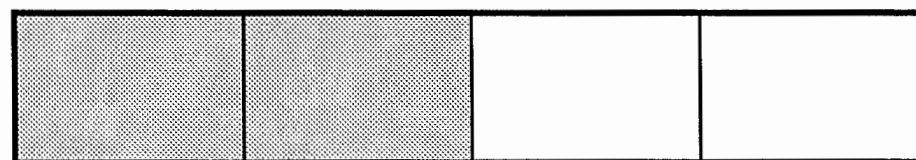
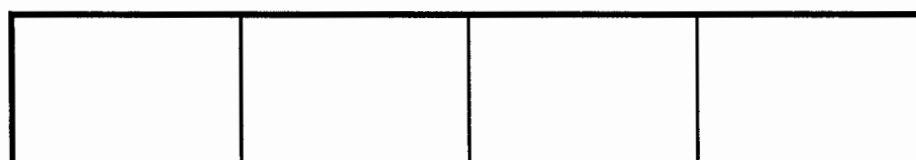
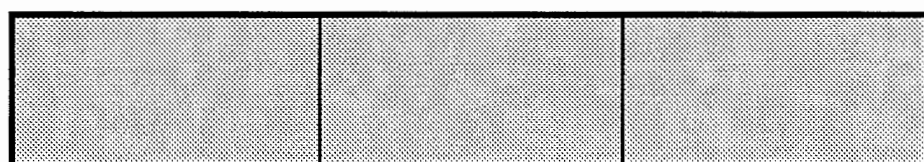
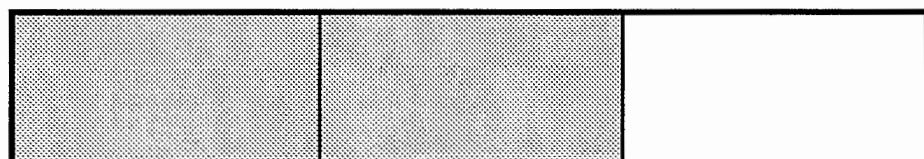
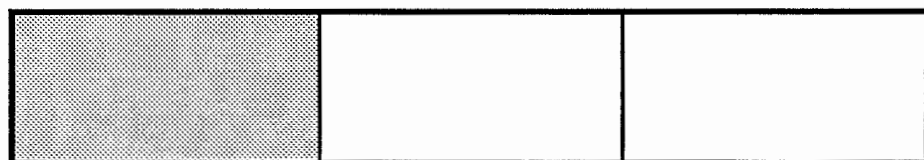
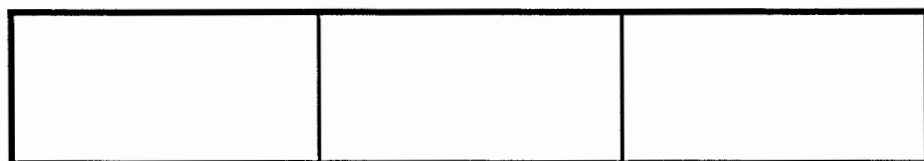
The statement " $\frac{1}{3}$ is greater than $\frac{1}{4}$ " can be written as " $\frac{1}{3} > \frac{1}{4}$ ". This is equivalent to the statement " $\frac{1}{4}$ is less than $\frac{1}{3}$ ", which is written " $\frac{1}{4} < \frac{1}{3}$ ". The inequality symbols, $<$ and $>$, can be thought of as arrows which always point to the smaller number.

11. If students are working with fraction bars, many of them will compare the shaded areas of the appropriate bars to reach their conclusions. Others may arrive at conclusions without physically comparing bars. For example, $\frac{4}{6} > \frac{1}{2}$ because a $\frac{4}{6}$ bar is more than half shaded; $\frac{5}{12} < \frac{2}{3}$ because a $\frac{5}{12}$ bar is less than half shaded and a $\frac{2}{3}$ bar is more than half shaded; $\frac{7}{12} < \frac{5}{6}$ because $\frac{5}{6} = \frac{10}{12}$ and $\frac{7}{12} < \frac{10}{12}$; $\frac{2}{3} < \frac{3}{4}$ because $\frac{2}{3} = \frac{8}{12}$ and $\frac{3}{4} = \frac{9}{12}$. Encourage students to find a variety of reasons to support their conclusions.

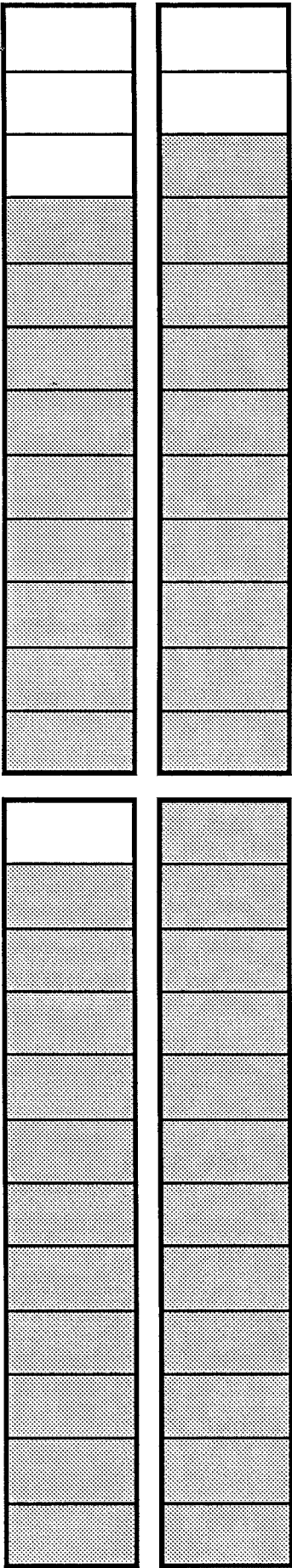
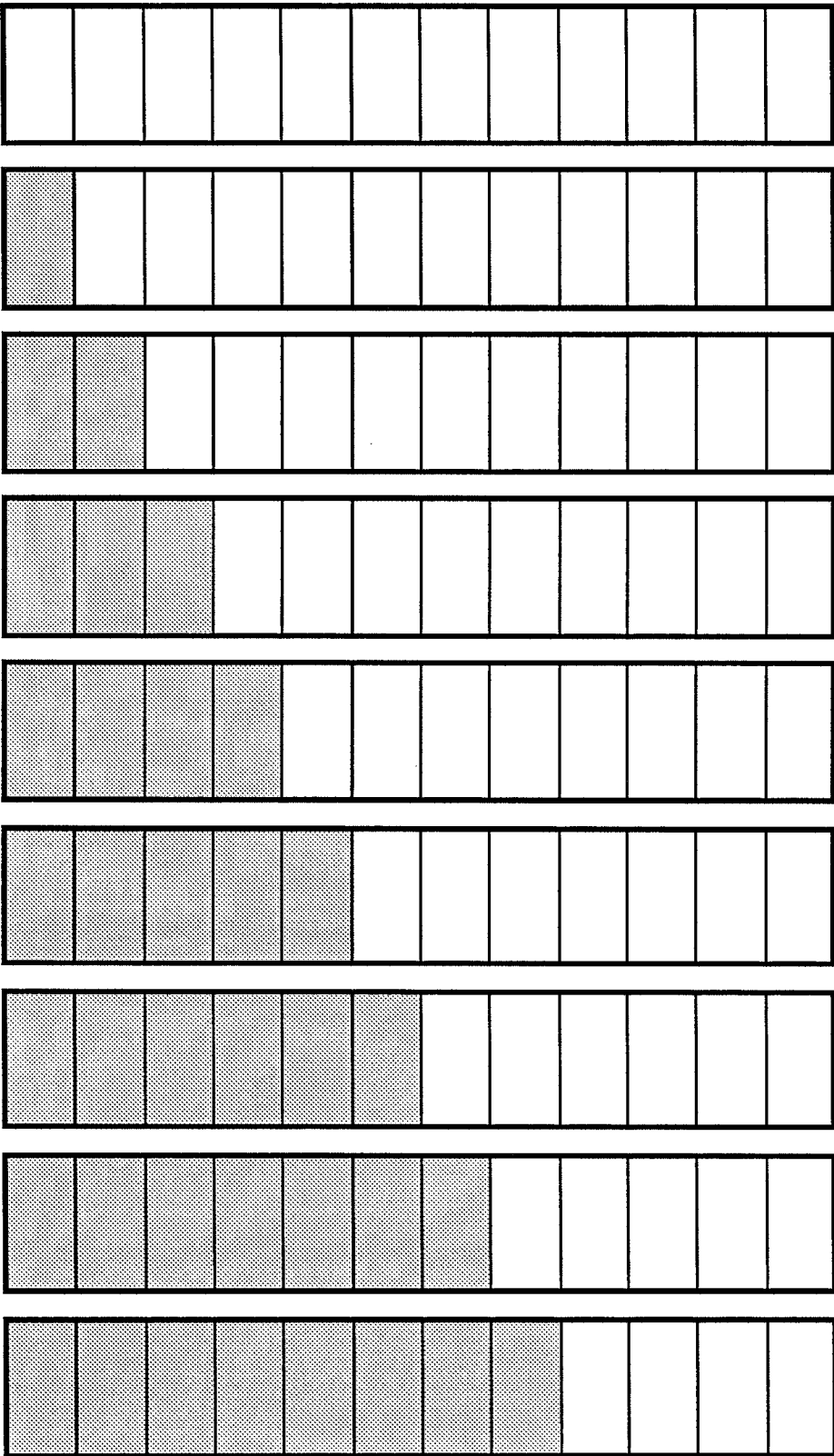
Fraction Bars for Halves and Sixths



Fraction Bars for Thirds and Fourths



Fraction Bars for Twelfths



Addition and Subtraction with Fraction Bars

O V E R V I E W

Fraction Bars™ are used to illustrate the processes of adding and subtracting fractions.

Prerequisite Activity

Unit IV, Activity 3, *Fraction Bars*

Materials

Fraction bars or transparencies of fraction bars (see Comment 1)

Actions

1. Distribute fraction bars to each student or group of students. Write the following sums on the chalkboard or overhead:

(a) $\frac{1}{4} + \frac{2}{3}$

(b) $\frac{2}{3} + \frac{5}{6}$

Ask the students to devise ways to use fraction bars to find these sums. Discuss the methods the students use.

Comments

1. If fraction bars are not available, this activity can be done as a class discussion using fraction bar transparencies on the overhead. Directions for making transparencies are continued in Comment 1, Unit IV, Activity 3, *Fraction Bars*.

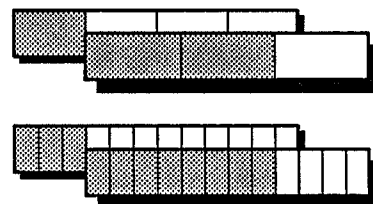
The students may use a variety of methods to find the sums. You may want to ask some of them to demonstrate their methods.

(a) One way to find this sum is to find a bar whose shaded amount is the total of the shaded amounts on the $\frac{1}{4}$ and $\frac{2}{3}$ bars:



$$\frac{1}{4} + \frac{2}{3} = \frac{11}{12}$$

Some students may replace the bars for $\frac{1}{4}$ and $\frac{2}{3}$ by equivalent bars which have the same number of parts:



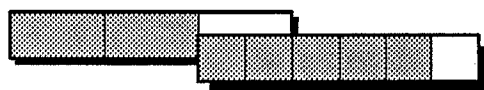
$$\frac{1}{4} + \frac{2}{3} = \frac{3}{12} + \frac{8}{12} = \frac{11}{12}$$

This illustrates the process of adding fractions by expressing them as fractions with a common denominator.

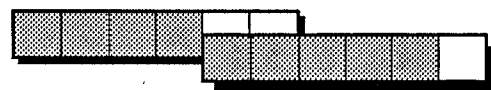
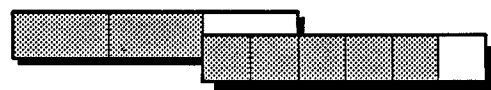
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1. *Continued.*

(b) Here are two ways to find the sum:



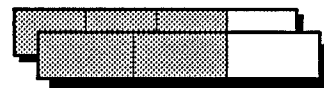
$$\frac{2}{3} + \frac{5}{6} = 1 \frac{1}{2}$$



$$\frac{2}{3} + \frac{5}{6} = \frac{4}{6} + \frac{5}{6} = \frac{9}{6}$$

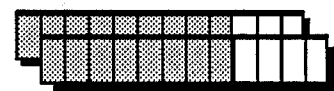
Other students may report that the sum is $\frac{3}{2}$ or $1 \frac{3}{6}$. Accept any correct form of the answer.

2. The subtraction can be done by finding a bar whose shaded amount is the difference of the shaded amounts on the $\frac{3}{4}$ and $\frac{2}{3}$ bars:



$$\frac{3}{4} - \frac{2}{3} = \frac{1}{12}$$

Another approach is to replace the $\frac{3}{4}$ and $\frac{2}{3}$ bars by equivalent bars with the same number of parts, and then find the difference:



$$\frac{3}{4} - \frac{2}{3} = \frac{9}{12} - \frac{8}{12} = \frac{1}{12}$$

2. Ask the students to use their fraction bars to find $\frac{3}{4} - \frac{2}{3}$. Discuss.

Actions

3. Have the students use the methods discussed to find additional fraction bar sums and differences.

4. (Optional.) Ask the students to draw a sketch of a $\frac{4}{9}$ bar. Then ask them to explore ways of using fraction bar sketches to find $\frac{1}{3} + \frac{4}{9}$. Discuss.

Comments

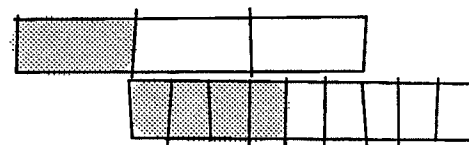
3. Here are some possibilities:

$$\begin{array}{lll} \frac{1}{2} + \frac{1}{3} & \frac{5}{6} + \frac{1}{4} & \frac{5}{12} - \frac{1}{6} \\ \frac{2}{3} - \frac{1}{4} & \frac{7}{12} + \frac{3}{4} & 1\frac{1}{2} - \frac{5}{6} \end{array}$$

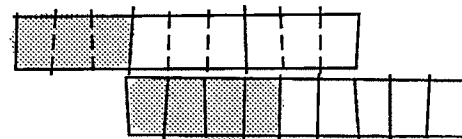
4. One way to obtain a sketch of a $\frac{4}{9}$ bar is to trace around a fraction bar to obtain a blank bar. This blank bar can be divided into 9 equal parts and then 4 of these parts can be shaded. You may want to discuss with the students how the blank bar can be divided into 9 roughly equal parts by sight. A fairly accurate subdivision can be obtained by first dividing the bar into thirds and then dividing each third into thirds.

The students should recognize that sketches are an aid to thinking. Even though their sketch of a $\frac{4}{9}$ bar may not have 9 precisely equal parts, they may use their sketches to help them visualize an ideal bar.

Two sketches for finding $\frac{1}{3} + \frac{4}{9}$ are shown here. In the second sketch, the dotted lines indicate that a $\frac{1}{3}$ bar has been converted to a $\frac{3}{9}$ bar.



$$\frac{1}{3} + \frac{4}{9} = \frac{7}{9}$$



$$\frac{1}{3} + \frac{4}{9} = \frac{3}{9} + \frac{4}{9} = \frac{7}{9}$$

You may want to ask some of the students to show their sketches on the overhead.

Actions

5. (Optional.) Ask the students to sketch a $\frac{1}{5}$ bar and a $\frac{1}{2}$ bar. Then ask them to subdivide these bars so they have the same number of parts, and provide the names of the resulting bars. Discuss.

6. (Optional.) Have the students draw fraction bar sketches to find:

(a) $\frac{5}{8} - \frac{1}{4}$ (b) $\frac{3}{5} + \frac{1}{2}$ (c) $\frac{1}{4} + \frac{3}{10}$

Comments

5. Dividing each part of the $\frac{1}{5}$ bar into 2 equal parts and each part of the $\frac{1}{2}$ bar into 5 equal parts results in tenth bars:



$$\frac{1}{5} = \frac{2}{10}$$

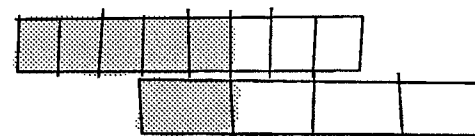


$$\frac{1}{2} = \frac{5}{10}$$

Subdividing bars to obtain bars with the same denominator is useful in the next action. You may want to discuss with the students how other pairs of bars can be subdivided to obtain the same number of parts, e.g. a $\frac{1}{3}$ bar and a $\frac{1}{5}$ bar, a $\frac{1}{6}$ bar and a $\frac{1}{8}$ bar.

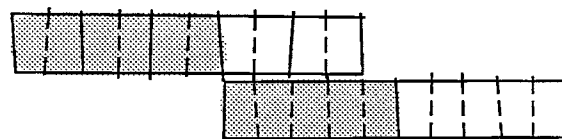
6. Here are some possible sketches:

(a)



$$\frac{5}{8} - \frac{1}{4} = \frac{3}{8}$$

(b) The $\frac{3}{5}$ bar and the $\frac{1}{2}$ bar can be converted into tenth bars:



$$\frac{3}{5} + \frac{1}{2} = \frac{6}{10} + \frac{5}{10} = \frac{11}{10}$$

(c) This sum can be found by converting both bars to twentieth bars:



$$\frac{1}{4} + \frac{3}{10} = \frac{6}{20} + \frac{5}{20} = \frac{11}{20}$$

Multiplication and Division with Fraction Bars

O	V	E	R	V	I	E	W
<p>Fraction Bars are used to illustrate the processes of multiplying and dividing fractions.</p>							
<p>Prerequisite Activity Unit IV, Activity 3, <i>Fraction Bars</i>; Unit IV, Activity 4, <i>Addition and Subtraction with Fraction Bars</i></p> <p>Materials Fraction bars or transparencies of fraction bars (see Unit IV, Activity 3, Comment 1)</p>							

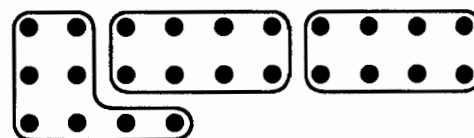
Actions

1. Distribute fraction bars to each student or group of students.
2. Discuss with the students how fractions bars may be used to find $1/4 \times 8/12$ and $3/4 \times 8/12$.

Comments

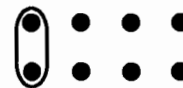
1. If fraction bars are not available, this activity can be done as a class discussion using fraction bar transparencies on the overhead.

2. You may want to begin the discussion by recalling the "repeated addition" model of multiplication for whole numbers, e. g. 3×8 may be thought of as $8 + 8 + 8$ or, what is the same, combining 3 groups of 8:



$$3 \times 8 = 3 \text{ groups of } 8 = 24$$

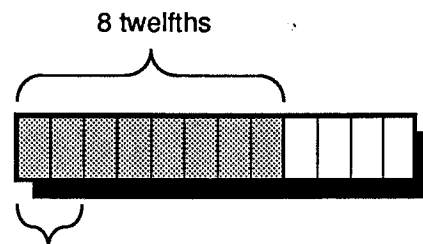
Similarly, 2×8 may be thought of as 2 groups of 8, $1/2 \times 8$ may be interpreted as $1/2$ of a group of 8 and $1/4 \times 8$ as $1/4$ of a group of 8. One often reads " $1/4 \times 8$ " as "one-fourth of eight."



$$1/4 \times 8 = 1/4 \text{ of a group of } 8 = 2$$

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2. (Continued) Turning to fraction bars, an $\frac{8}{12}$ bar can be thought of as a group of 8 twelfths. Thus $\frac{1}{4} \times \frac{8}{12}$ is $\frac{1}{4}$ of a group of 8 twelfths, or 2 twelfths:



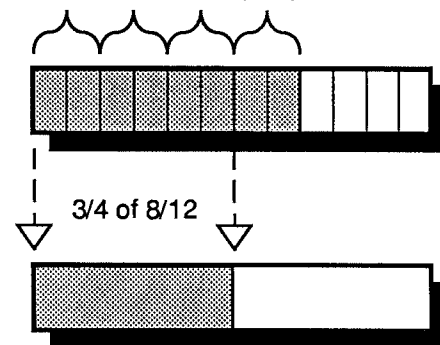
$\frac{1}{4}$ of 8 twelfths

$$\frac{1}{4} \times \frac{8}{12} = \frac{2}{12}$$

Note that the multiplier $\frac{1}{4}$ does not represent the shaded area on a $\frac{1}{4}$ bar. Rather it indicates the portion to be taken of the shaded area of the $\frac{8}{12}$ bar.

Similarly, $\frac{3}{4} \times \frac{8}{12}$ can be thought of as $\frac{3}{4}$ of a group of 8 twelfths. This can be obtained by dividing 8 twelfths into four equal parts and taking 3 of them. The result is $\frac{6}{12}$ or, alternatively, $\frac{1}{2}$:

$\frac{8}{12}$ divided into 4 equal parts



$$\frac{3}{4} \times \frac{8}{12} = \frac{6}{12} = \frac{1}{2}$$

Actions

3. Ask the students to use fraction bars to find:

(a) $\frac{1}{2} \times \frac{4}{6}$

(b) $\frac{1}{2} \times \frac{1}{3}$

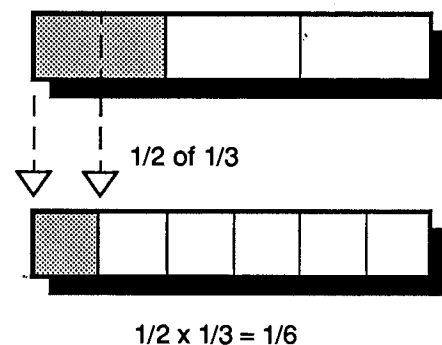
(c) $\frac{2}{3} \times \frac{3}{4}$

(d) $\frac{1}{4} \times \frac{1}{3}$

Comments

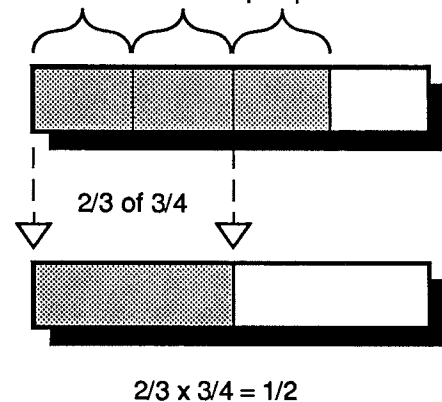
3. (a) Some students may report the answer is $\frac{2}{6}$, others $\frac{1}{3}$. Either is appropriate.

(b)

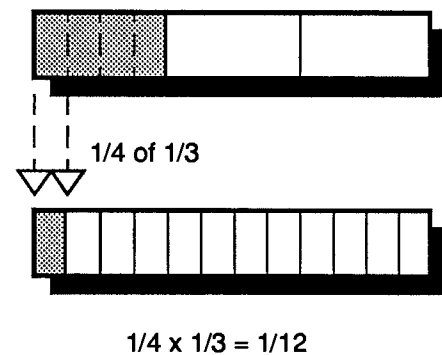


(c)

$\frac{3}{4}$ divided into 3 equal parts



(d)



Actions

4. (Optional) Have the students imagine or sketch fraction bars to help them find the following:

(a) $\frac{1}{2} \times \frac{2}{5}$

(b) $\frac{1}{2} \times \frac{3}{5}$

(c) $\frac{2}{3} \times \frac{6}{7}$

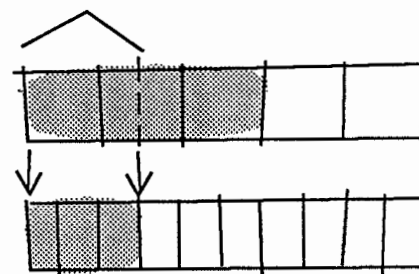
(d) $\frac{2}{3} \times \frac{4}{5}$

Comments

4. (a) Some students may see, without a sketch, that half the shaded area of a $\frac{2}{5}$ bar is the same as the amount shaded on a $\frac{1}{5}$ bar. Other students may draw a sketch.

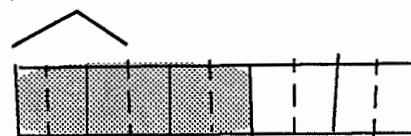
(b) Here are two possible sketches. In the second sketch, a $\frac{3}{5}$ bar has been converted into a $\frac{6}{10}$ bar.

(i) $\frac{1}{2}$ of $\frac{3}{5}$



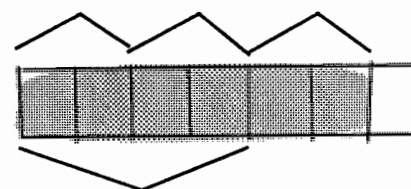
$$\frac{1}{2} \times \frac{3}{5} = \frac{3}{10}$$

(ii) $\frac{1}{2}$ of $\frac{3}{5} = \frac{1}{2} \times \frac{6}{10}$



$$\frac{1}{2} \times \frac{3}{5} = \frac{3}{10}$$

(c) $\frac{6}{7}$ divided into 3 equal parts



$$\frac{2}{3} \times \frac{6}{7} = \frac{4}{7}$$

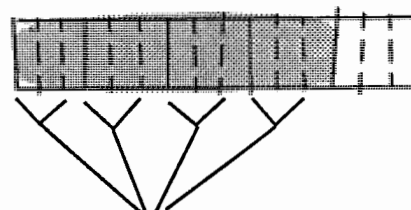
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4. (Continued)

(d) One way to determine $\frac{2}{3}$ of $\frac{4}{5}$ is to first divide each part of a $\frac{4}{5}$ bar into 3 equal parts. This converts the bar to a fifteenths bar:



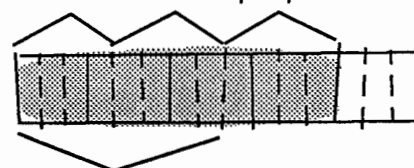
Then $\frac{2}{3}$ of $\frac{4}{5}$ can either be determined by (1) taking $\frac{2}{3}$ of each of the original shaded parts:



$$\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$$

Or (2) dividing the entire shaded area into 3 parts and taking 2 of them:

$\frac{4}{5}$ divided into 3 equal parts



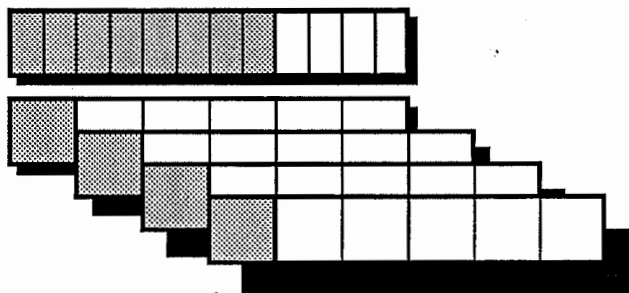
$$\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$$

Actions

5. Ask the students for their ideas about using fraction bars to find $8/12 \div 1/6$. Discuss, then ask for ideas to find $5/6 \div 1/3$.

Comments

5. Some students may say the shaded area of the $1/6$ bar fits into the shaded area of the $8/12$ bar 4 times. Thus, $8/12 \div 1/6 = 4$.



$$8/12 \div 1/6 = 4$$

This way of viewing the division of fractions is an extension of the *grouping* method of dividing whole numbers (see Comment 4, Unit II, Activity 1, *Basic Operations*). In this method, $8 \div 2$ is the number of groups of 2 in a collection of 8.

There are 4 groups of 2 in 8:



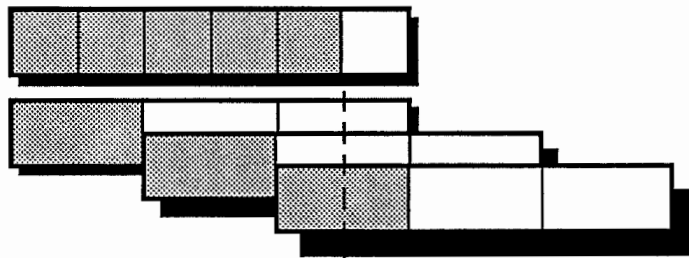
$$8 \div 2 = 4.$$

Similarly, there are $2 \frac{2}{3}$ groups of 3 in 8. (In the illustration below, the grouping on the right contains 2 parts of the 3 needed for a group. Hence it is $2/3$ of a group.)



$$8 \div 3 = 2 \frac{2}{3}$$

Since there are $2 \frac{1}{2}$ of the shaded regions of a $1/3$ bar in the shaded region of a $5/6$ bar, $5/6 \div 1/3 = 2 \frac{1}{2}$:



$$5/6 \div 1/3 = 2 \frac{1}{2}$$

6. Ask the students to use fraction bars to find:

(a) $\frac{3}{4} \div \frac{3}{12}$

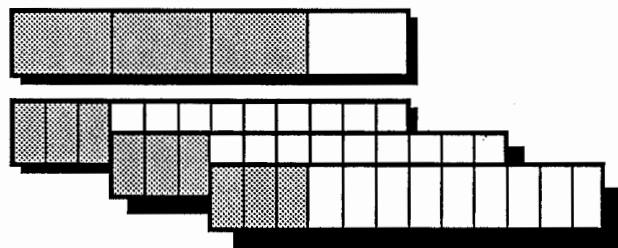
(b) $1 \frac{1}{2} \div \frac{3}{4}$

(c) $\frac{5}{12} \div \frac{1}{6}$

(d) $\frac{11}{12} \div \frac{1}{3}$

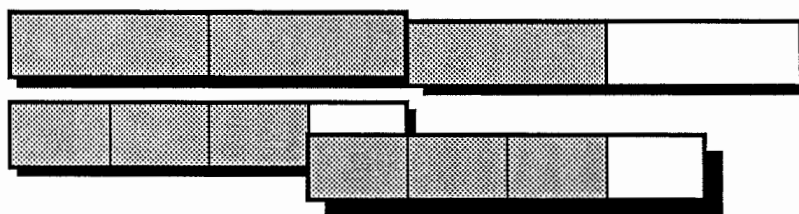
(e) $\frac{1}{2} \div \frac{3}{4}$

6. (a)



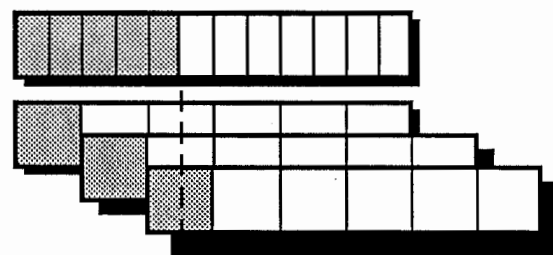
$$\frac{3}{4} \div \frac{3}{12} = 3$$

(b)



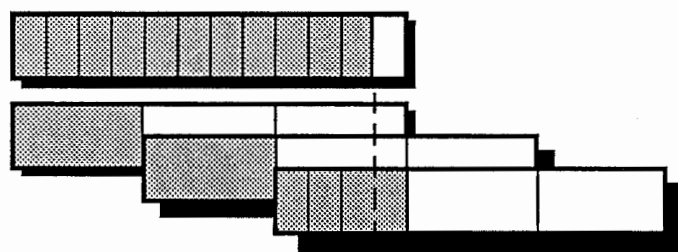
$$1 \frac{1}{2} \div \frac{3}{4} = 2$$

(c)



$$\frac{5}{12} \div \frac{1}{6} = 2 \frac{1}{2}$$

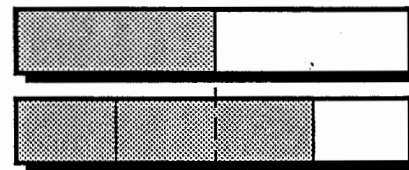
(d)



$$\frac{11}{12} \div \frac{1}{3} = 2 \frac{3}{4}$$

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6. (Continued) (e) Only $\frac{2}{3}$ of the shaded area of a $\frac{3}{4}$ bar will fit into the shaded area of a $\frac{1}{2}$ bar:

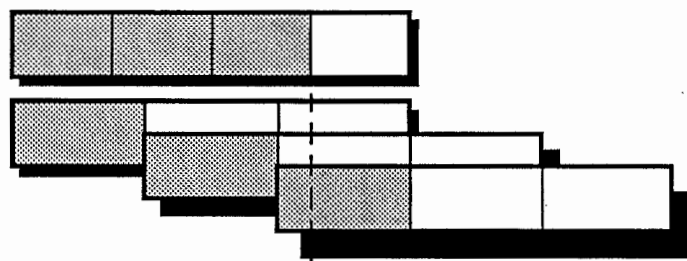


$$\frac{1}{2} + \frac{3}{4} = \frac{2}{3}$$

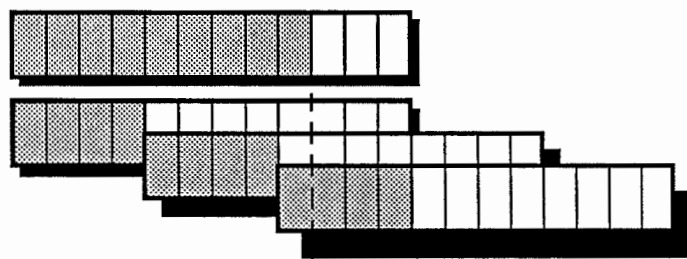
Note that in this statement, the quotient $\frac{2}{3}$ does not represent the shaded area on a $\frac{2}{3}$ bar. Rather $\frac{2}{3}$ indicates the portion of the shaded area of the $\frac{3}{4}$ bar that fits into the shaded area of the $\frac{1}{2}$ bar.

7. Discuss with the students how $\frac{3}{4} \div \frac{1}{3}$ may be found by replacing the $\frac{3}{4}$ and $\frac{1}{3}$ bars with bars which have a common number of parts. Then ask the students to use this method to find $\frac{1}{3} \div \frac{1}{2}$ and $\frac{3}{4} \div \frac{5}{6}$.

7. Comparing the $\frac{3}{4}$ and $\frac{1}{3}$ bars, it is apparent that the shaded area of the $\frac{1}{3}$ bar fits into the shaded area of the $\frac{3}{4}$ bar 2 and a fraction times. It may not be clear what this fraction is. By replacing the bars by equivalent twelfths bars, one sees that this fraction is $\frac{1}{4}$.



$$\frac{3}{4} + \frac{1}{3} = 2 ?$$

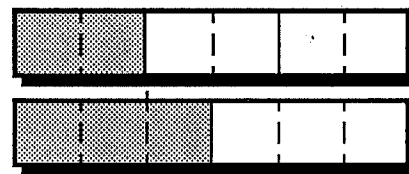


$$\frac{3}{4} + \frac{1}{3} = \frac{9}{12} + \frac{4}{12} = 2 \frac{1}{4}$$

Dividing one fraction by another is frequently simplified by replacing the fractions by equivalent fractions with a common denominator.

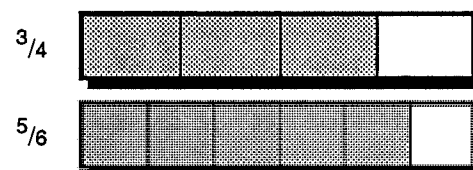
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7. (Continued) Only $\frac{2}{3}$ of the shaded area of a $\frac{1}{2}$ bar fits into the shaded area of a $\frac{1}{3}$ bar. This can be seen by converting the bars into sixth bars.

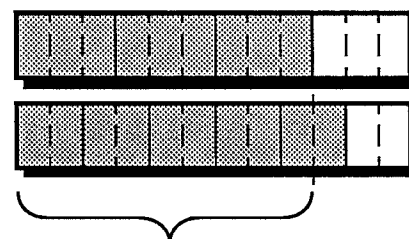


$$\frac{1}{3} + \frac{1}{2} = \frac{2}{6} + \frac{3}{6} = \frac{2}{3}$$

As seen below, the shaded area of a $\frac{5}{6}$ bar is greater than the shaded area of a $\frac{3}{4}$ bar. Hence, only a portion of the shaded area of a $\frac{5}{6}$ bar will fit into the shaded area of a $\frac{3}{4}$ bar. Thus $\frac{3}{4} + \frac{5}{6}$ is less than 1.



To find what portion of the shaded area of a $\frac{5}{6}$ bar fits into the shaded area of a $\frac{3}{4}$ bar, the bars may be converted to twelfth bars. This divides the shaded area of a $\frac{5}{6}$ bar into 10 equal parts — 9 of these 10 parts equals the shaded area of a $\frac{3}{4}$ bar. Hence, $\frac{3}{4} + \frac{5}{6} = \frac{9}{10}$.



$\frac{9}{10}$ of $\frac{5}{6}$

$$\frac{3}{4} + \frac{5}{6} = \frac{9}{12} + \frac{10}{12} = \frac{9}{10}$$

Actions

8. (Optional) Ask the students to use sketches of fraction bars to find:

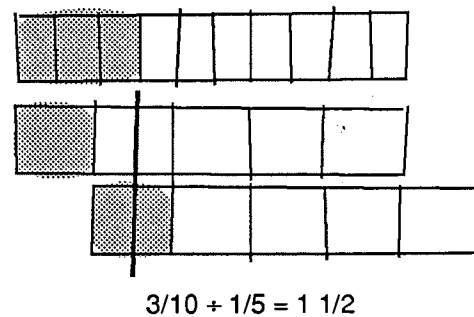
(a) $3/10 \div 1/5$

(b) $1/2 \div 2/5$

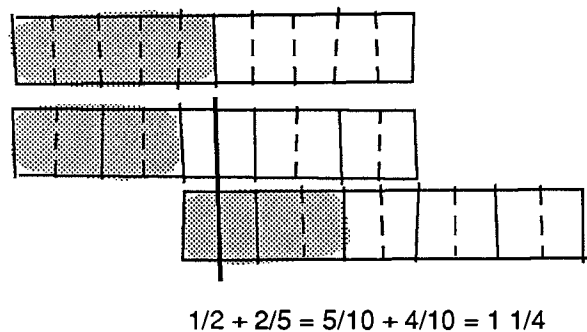
(c) $4/5 \div 1/3$

Comments

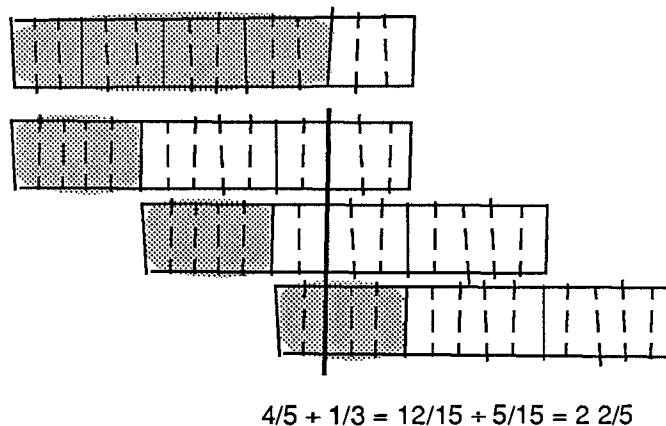
8. (a)



(b)



(c)



Introduction to Decimals

O V E R V I E W

Base 10 number pieces are used to introduce the concept of decimal, writing decimals and reading decimals.

Prerequisite Activity

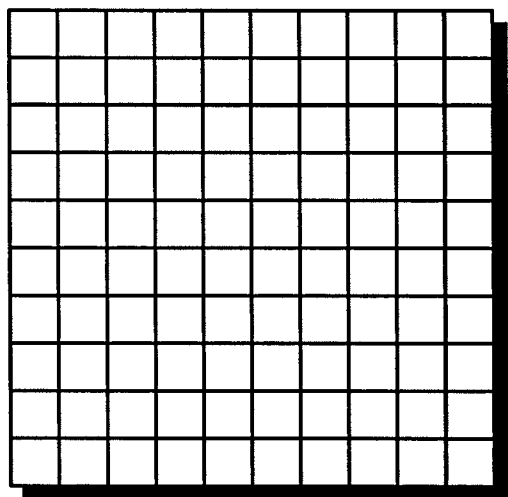
Grouping and Numeration, Unit III, Activity 1; *Fraction Bars*, Unit IV, Activity 3

Materials

Base 10 number pieces (see Comment 1)

Actions

1. Distribute the base 10 number pieces to each student. Tell them that in this activity the largest piece represents the unit, 1. Ask them to determine the value of the other number pieces. Discuss.



1
(unit)



?

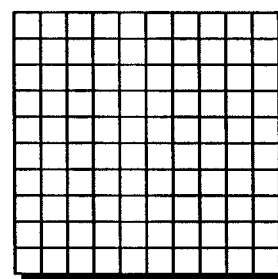


?

Comments

1. Base 10 number pieces can be made by copying the last page of this activity on tagboard and cutting along the indicated lines. You may wish to have your students do the cutting. Each student, or group of students, should have at least 11 unit, 15 tenth and 21 hundredth pieces.

If the largest piece represents the unit, then the pieces have the following values:



1

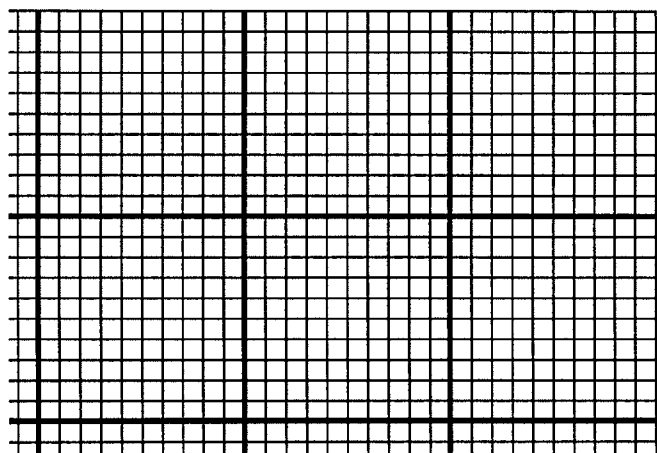


$\frac{1}{10}$
(one tenth)

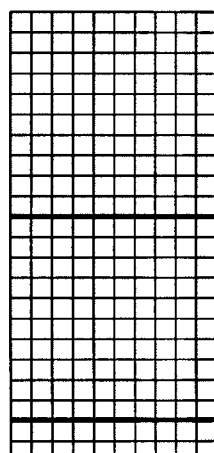


$\frac{1}{100}$
(one hundredth)

2. One way would be to make a strip of 10 units to represent 10, and then join 10 of these strips side-by-side to represent 100.



← 100
(hundred)

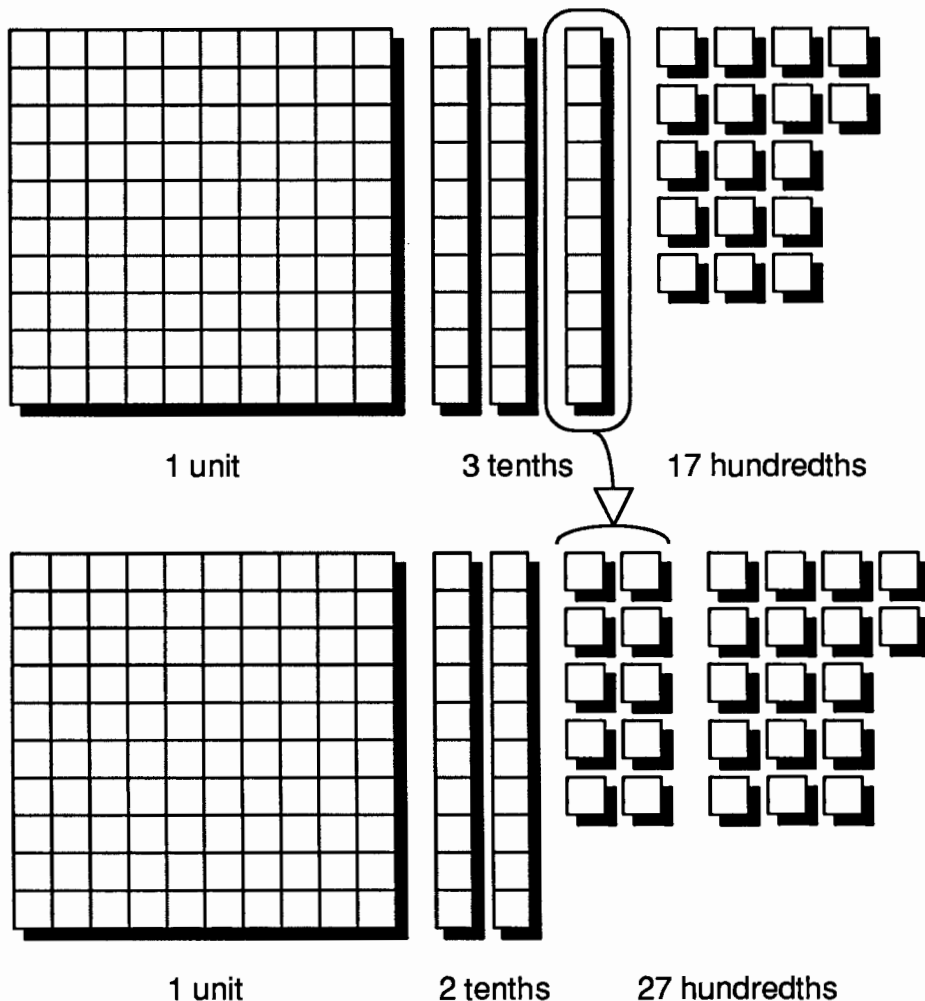


← 10
(ten)

It may aid the students visualization to construct number pieces for 10 and 100 and post them in view. The hundreds piece will be a square meter and the tens piece will be 10 centimeters by 1 meter.

Actions

3. Put the following collection of number pieces on the overhead projector: 1 unit, 3 tenths, and 17 hundredths. Record these pieces, and the total number of base 10 number pieces on a chart like the one below. Trade one tenth for 10 hundredths and record the resulting collection on the second line of the chart.



Ask the students to make more equal exchanges recording each result on the chart.

(100) hundreds	(10) tens	(1) units	(1/10) tenths	(1/100) hundredths	Total Number of Pieces
0	0	1	3	17	21
0	0	1	2	27	30

Emphasize the headings of the chart and read them aloud to help students distinguish between “hundreds” and “hundredths,” and “tens” and “tenths.”

Comments

3. For this activity, you may want students to work in small groups.

Collect and display the students results on a chart on the overhead. There are 20 possible collections that can be listed in the chart. Because some exchanges require a large number of hundredth pieces, you may want the students to imagine making trades for those cases without physically doing it. The asterisk marks the collection with the fewest number of base 10 pieces.

100	10	1	1/10	1/100	Total # of Pieces
0	0	1	3	17	21
0	0	1	2	27	30
0	0	1	1	37	39
0	0	1	0	47	48
0	0	1	4	7*	12
0	0	0	14	7	21
0	0	0	13	17	30
0	0	0	12	27	39
0	0	0	11	37	48
0	0	0	10	47	57
0	0	0	9	57	63
0	0	0	8	67	75
0	0	0	7	77	84
0	0	0	6	87	93
0	0	0	5	97	104
0	0	0	4	107	111
0	0	0	3	117	120
0	0	0	2	127	129
0	0	0	1	137	138
0	0	0	0	147	147

Actions

4. Discuss this question with the students: what do the collections in the chart have in common?

5. Have each student, or group of students, form a collection of 11 tenths and 13 hundredths. Ask them to make the minimal collection for this set of number pieces.

Have the students form minimal collections for the following sets of number pieces:

- a) 9 tenths, 21 hundredths
- b) 1 unit, 4 tenths, 10 hundredths
- c) 15 tenths
- d) 15 hundredths
- e) 10 units, 10 tenths, 10 hundredths
- f) 10 units, 9 tenths, 11 hundredths

Comments

4. There are many correct responses to this question. The main objective is to discover that each collection has the same value. This can be seen in different ways: each collection contains the same number of hundredths, each collection of pieces covers the same area, or, while making exchanges the number of pieces changes but the amount of material remains the same.

Point out that the collection that uses the fewest number of base pieces is the *minimal collection*. So out of all 20 equal collections above, the collection with 1 unit, 4 tenths, and 7 hundredths is the minimal collection.

100	10	1	1/10	1/100
0	0	1	4	7

5. The minimal collection for 11 tenths and 13 hundredths is:

100	10	1	1/10	1/100
0	0	1	2	3

Here are the minimal collections for the other sets of number pieces:

	100	10	1	1/10	1/100
a)			1	1	1
b)			1	5	
c)			1	5	
d)				1	5
e)		1	1	1	
f)		1	1		1

Actions

6. Point to the entries **a**, **e** and **f** in the chart of *Comment 5*, write the numeral 111, and observe that usually 111 means one hundred eleven. Ask the students to think of ways to represent entries **a**, **e** and **f** so there is no confusion. Then discuss decimal notation.

7. Write the following decimal numbers on a chalkboard or overhead and ask the students to represent these numbers with their base 10 pieces:

1.21 2.07 .15 .02

Comments

6. Some students may suggest a decimal point immediately because of their past experiences. Accept it, but encourage students to invent other systems of their own. Historically, many systems were used. For the collection in *Action 5*, we write "1.23", but in early times this number was written in forms such as:

1 | 23 1(23) 1|23 1 23

In England, Europe and the Scandinavian countries the same decimal is written, respectively, as:

1.23 1,23 1,²³

In our system, the period in 1.23 is called a *decimal point* and is always located between the units digit and the tenths digit.

In entry **f** more than a decimal point is needed to resolve all ambiguity. A place-holder is needed to show that there are no tenths. While we normally use a zero, 11.01, students may enjoy creating their own methods during the discussion.

You may wish to introduce more collections of base 10 pieces at this point to give the students additional practice in finding minimal collections and writing decimal names.

7. If you have made base 10 pieces for 10 and 100, you may wish to represent a few numbers like 10.01, 121.1 or 101.01. Some students may wish to investigate questions like: imagine trading all base 10 pieces for 121.1 into tenths. How many tenths would there be? Or, imagine trading all base pieces for 101.01 into hundredths. How many hundredths will there be altogether?

Actions

8. Ask the students how they would read the names of the numbers listed in *Action 7* if they were talking to someone over the telephone. Discuss ways of reading decimals.

Comments

8. There are several ways that a number like 1.21 can be read.

- a) One way that is easy to say and easy to understand is, "one point two one."
- b) Another possibility is, "one and two tenths and one hundredth."
- c) A common way is, "one and twenty-one hundredths."

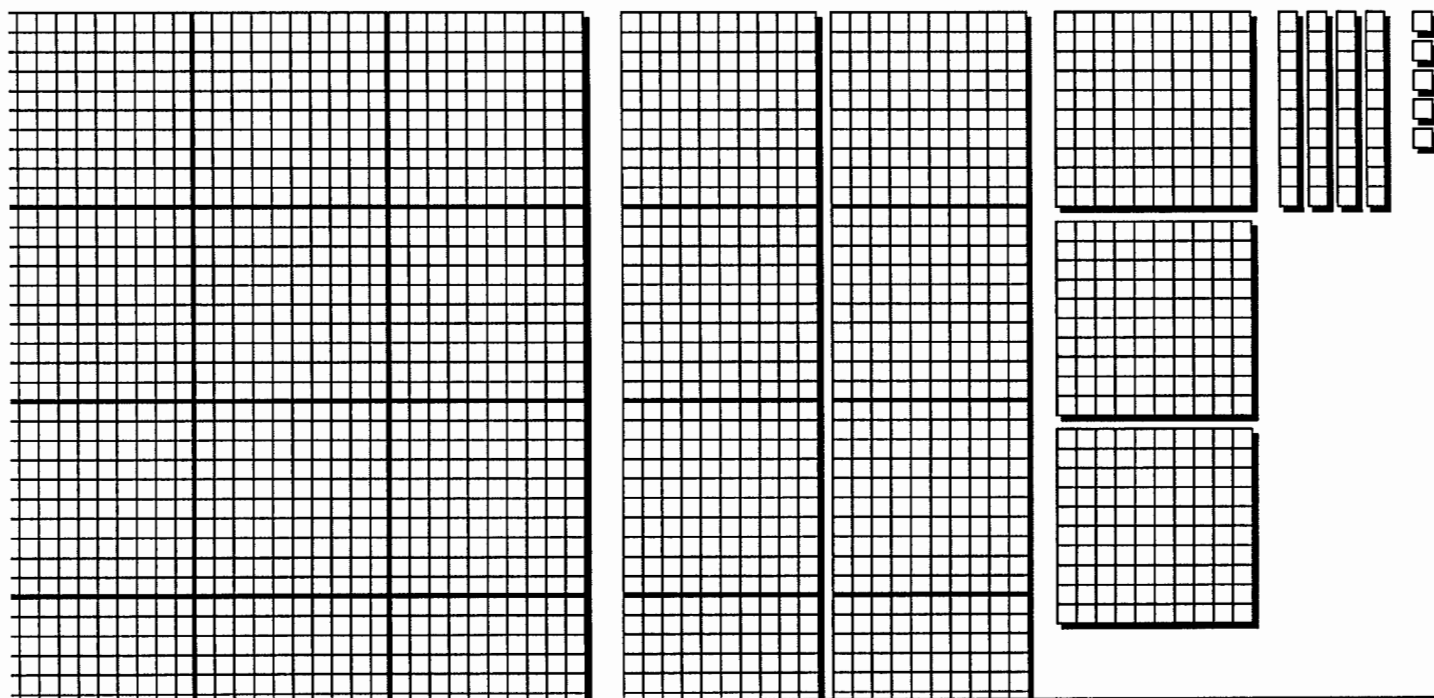
Method (c) is the usual textbook version. The whole-number part of the expression (to the left of the decimal point) is read as a whole number, the word "and" is used for the decimal point, and the decimal part (to the right of the decimal point) is also read like a whole number followed by the name of the smallest base 10 piece used. So, for example, 123.45 is read:

"one hundred twenty-three and forty-five hundredths."

Visualizing the number piece collection for 123.45 may help one understand this way of reading the number.

The collection consists of 1 hundred piece, 2 ten pieces, 3 unit pieces, 4 tenth pieces and 5 hundredth pieces. "One hundred twenty-three" describes the whole number part of this collection (the word "twenty" means "two groups of ten"). For the fractional part, the 4 tenths can be traded for 40 hundredths. This gives a total of 45 hundredths.

Continued on next page.

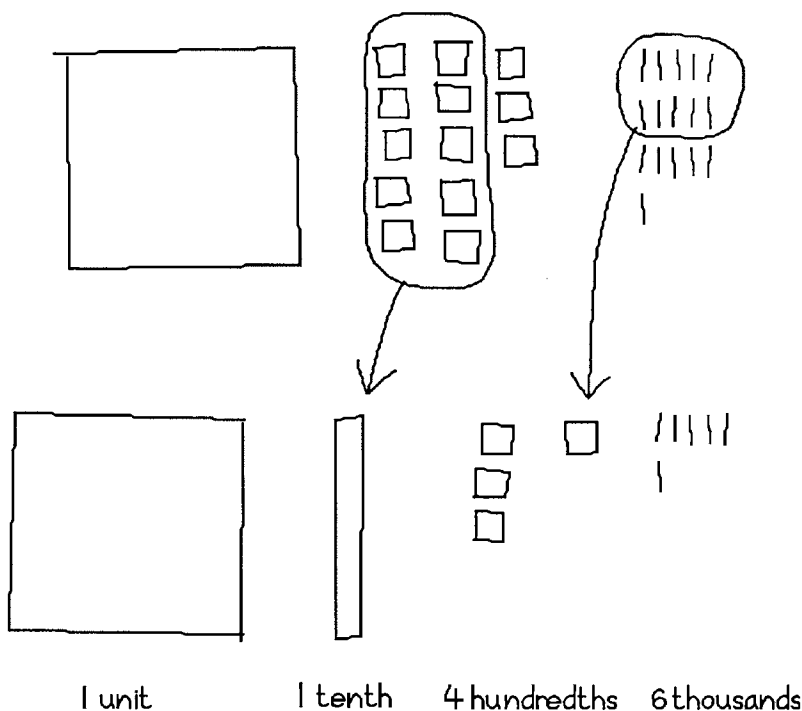
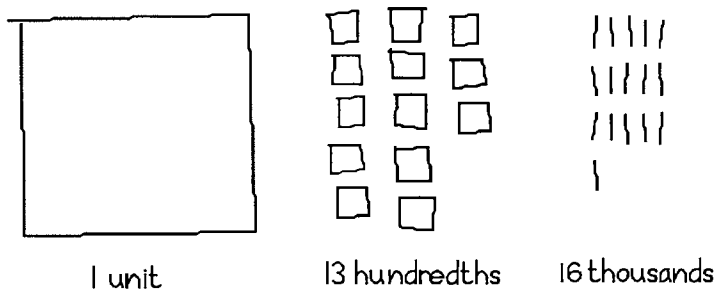


8. *Continued.* You may wish to have students represent numbers like .12, .29 and .08 with their pieces and then exchange the tenths for hundredths to see that they yield 12 hundredths, 29 hundredths, and 8 hundredths, respectively.

9. The new little strips are thousandths. It is difficult to cut a hundredth piece into ten equal strips so you may want to represent the number pieces with diagrams. For example, 1 unit, 13 hundredths, and 16 thousandths can be represented as follows:

9. (Optional) a) Tell the students to imagine a hundredth number piece divided into 10 strips of equal size. Ask them what fraction of the unit piece each of the little strips represents.

b) Ask the students to sketch a collection that has 1 unit, 13 hundredths and 16 thousandths. Then ask them to sketch the minimal collection that has the same value and write the decimal number which represents it. Discuss ways of reading this number.

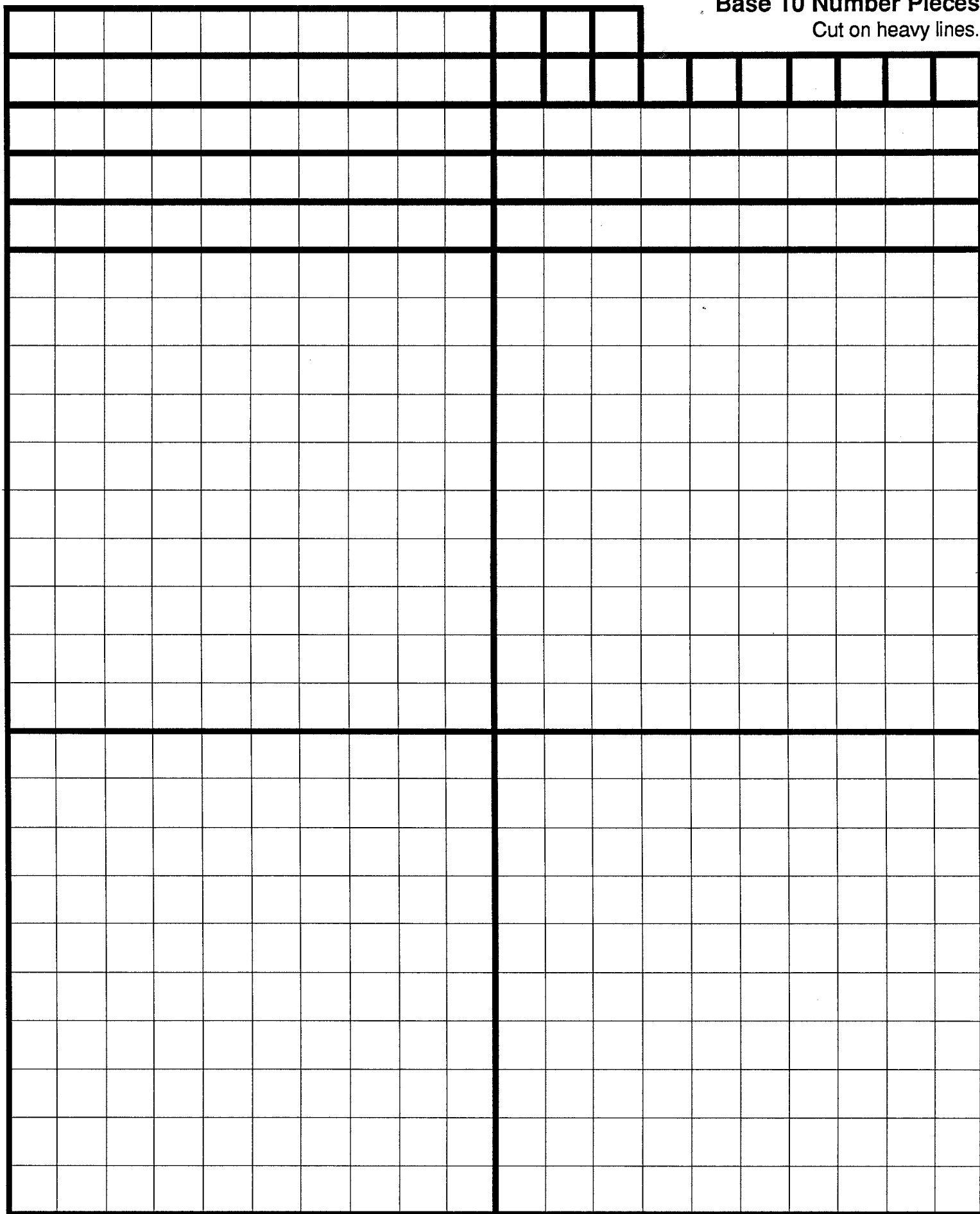


The trading needed to represent the pieces in base 10 can also be done with a diagram.

This collection represents the decimal 1.146 and is read as, "one and one hundred forty-six thousandths."

Base 10 Number Pieces

Cut on heavy lines.



Decimal Addition and Subtraction

O V E R V I E W

Base 10 number pieces are used to provide a model for understanding addition and subtraction of decimals.

Prerequisite Activity

Unit IV, Activity 6, *Introduction to Decimals*

Materials

Base 10 number pieces

Actions

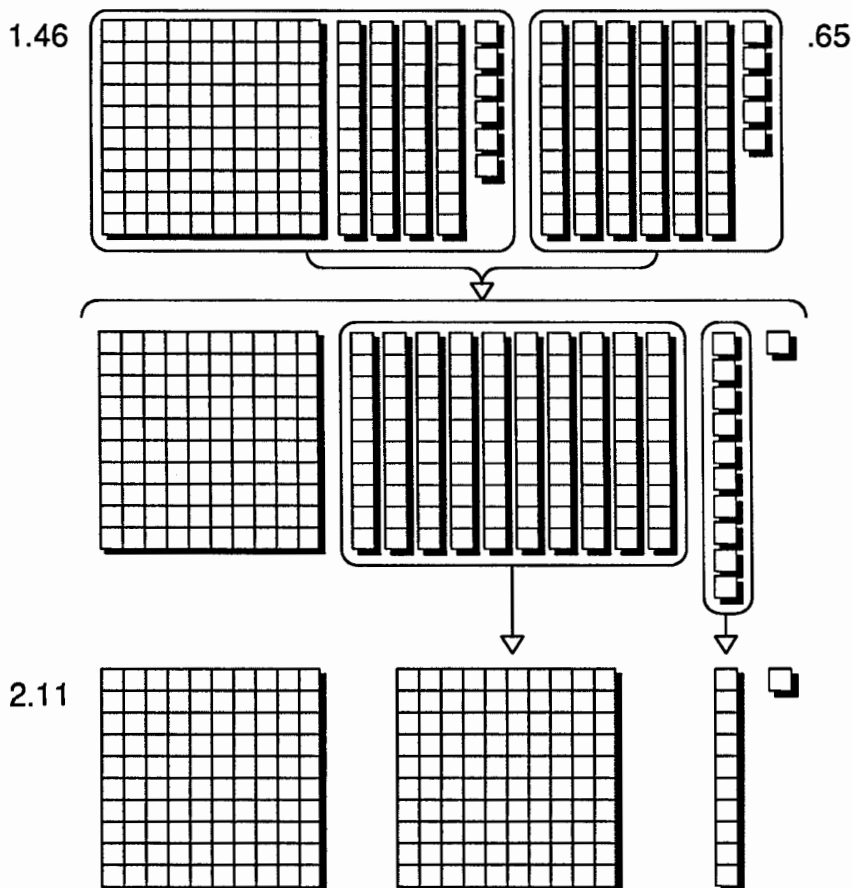
1. Distribute the base 10 pieces to each student or group of students.
2. Ask the students to use their number pieces to find the sum of 1.46 and .65. Discuss.

Comments

1. Each student or group of students should have 8 units, 16 tenths, and 26 hundredths.

2. Combining number piece collections for 1.46 and .65, and then making exchanges, results in the minimal collection for 2.11 shown below. You may wish to have students show their procedures on the over-

head. Different approaches will show that the order in which exchanges of number pieces are made does not affect the sum.



$$1.46 + .65 = 2.11$$

Actions

3. Ask the students to use their base 10 pieces to find these sums:

$$.46 + .39 \quad .09 + .56 \quad 2.67 + 3.33 \quad 1.06 + 1.6 + 1.34$$









Observe methods the students are using and discuss some of them.

4. Show the students how a table can be used to record their methods for finding sums with number base pieces. Make tables which reflect methods you observed the students using.

Comments

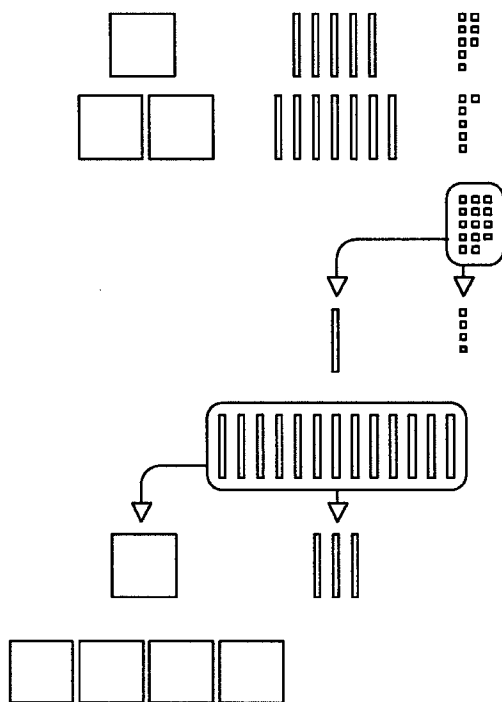
3. The different methods can be presented to the class by their student creators. Sums which involve few number pieces can be shown on the overhead. Sums involving many number pieces are often better demonstrated on a table or the floor.

4. One effective way to present a recording process is to have a student demonstrate a method while you record each move in a table. Here is an example of a record for one student's method of finding $1.58 + 2.76$:

	Student Moves	Record			
		units	10ths	100ths	
	form collection for 1.58	1	5	8	
		form collection for 2.76	+	2	7
	combine units	3			
		combine tenths		12	
	exchange 10 tenths for 1 unit	1	2		
		combine hundredths			14
	exchange 10 hundredths for 1 tenth		1	4	
		minimal collection for sum	4	3	4

Actions

5. (Optional) Explain how the usual paper and pencil algorithm can be explained in terms of number base pieces.



Comments

5. As the following illustration shows, the usual paper and pencil procedure is an abbreviated table for recording number piece moves, and exchanges, done in a certain order.

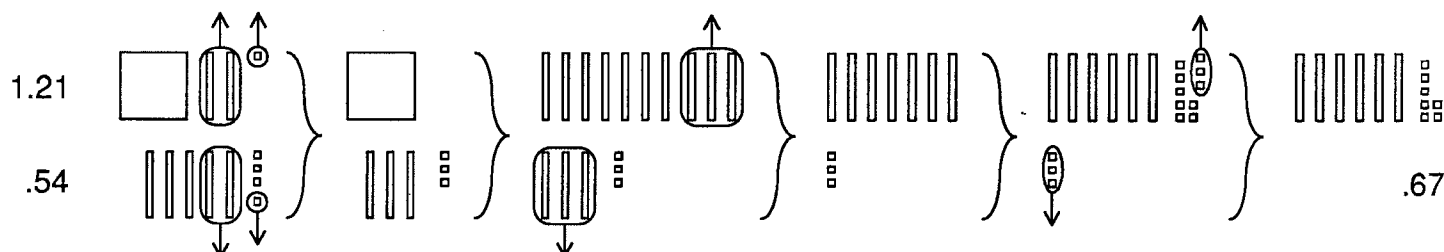
Units	10ths	100ths	
1	5	8	
+	2	7	6
		14	1.58
	1	4	2.76
			④
		13	1.58
	1	3	2.76
			③4
			1.58
			2.76
			④.34

You may wish to have students compute additional decimal sums using this algorithm, but ask them to imagine number pieces and describe each move they make in terms of those number pieces.

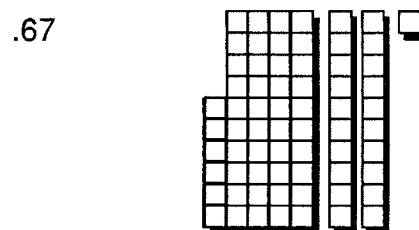
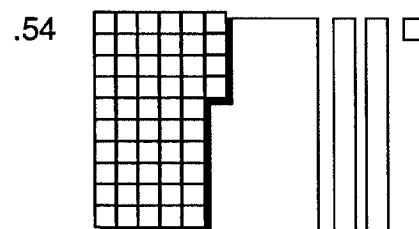
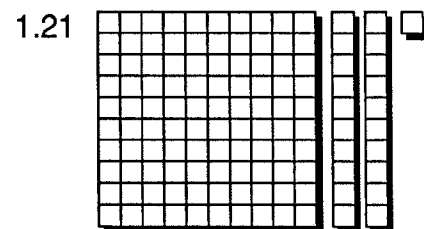
6. Write the arithmetic expression $1.21 - .54$ on the chalkboard or overhead. Ask the students to use their number base pieces to perform this subtraction. Observe the methods being used and then discuss.

6. Ask students who have used different methods to share them with the class. Here are two common methods. You may see others. (see Comment 6, Unit III, Activity 5, *Base 10 Addition and Subtraction*)

a. Here the same amount is taken from both sets of number pieces until one set is gone, making exchanges as necessary:



b. In this method a collection of number pieces is found which makes up the difference between the two sets of number pieces:



$$1.21 - .54 = .67$$

Actions

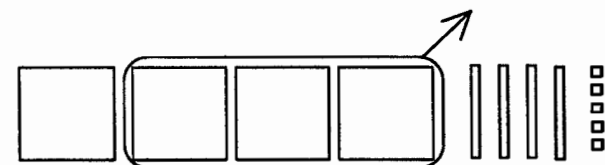
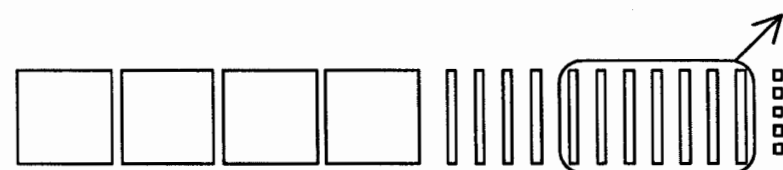
Comments

7. Ask the students to use their number base pieces to do these subtractions:

$$2.17 - 1.63 \quad 3.42 - 1.83 \quad 1.02 - .7 \quad 2.3 - 1.39$$

Observe the methods the students use.

8. (Optional) Show how paper and pencil algorithms for subtraction can be described in terms of number pieces.



$$\begin{array}{r} 5.20 \\ - 3.75 \\ \hline \end{array}$$

$$\begin{array}{r} 1 \quad 1 \\ 5.\cancel{2}0 \\ - 3.75 \\ \hline 5 \end{array}$$

$$\begin{array}{r} 4 \quad 11 \quad 1 \\ \cancel{5}.\cancel{2}0 \\ - 3.75 \\ \hline .45 \end{array}$$

$$\begin{array}{r} 4 \quad 11 \quad 1 \\ \cancel{5}.\cancel{2}0 \\ - 3.75 \\ \hline 1.45 \end{array}$$

7. You may want to suggest that the students try different methods of subtracting with their number pieces.

8. Here is one way to model the usual algorithm to compute $5.2 - 3.75$:

a. Arranging the numbers vertically, with the decimal points aligned, shows how much the collection must be reduced, number piece by number piece. In this case, 3 units, 7 tenths and 5 hundredths are to be removed from a collection of 5 units, 2 tenths and 0 hundredths.

b. 5 hundredths cannot be taken from the collection of number pieces until 1 tenth is exchanged for 10 hundredths. This exchange is recorded by reducing the number of tenths from 2 to 1 and increasing the number of hundredths from 0 to 10. Now 5 hundredths are removed, leaving 5 hundredths.

c. 7 tenths cannot be taken from the collection until 1 unit is exchanged for 10 tenths. This transaction is recorded by reducing the number of units from 5 to 4 and increasing the number of tenths to 11. Now 7 tenths are removed, leaving 4 tenths.

d. Finally, 3 units are removed.

Actions

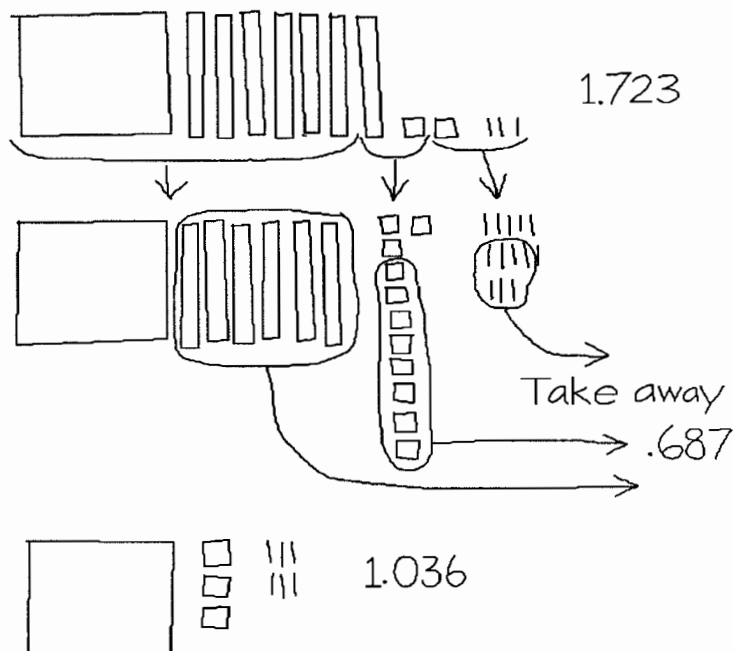
Comments

9. (Optional) Ask the students to draw sketches of number pieces to compute the following sums and differences.

$$2.507 + 1.624$$

$$1.723 - .687$$

9. See Action 9 in Activity 6, Unit IV, *Introduction to Decimals*. This extends addition and subtraction of decimals to numbers involving thousandths. There are many possible ways to sketch solutions. Here is one way for the subtraction $1.723 - .687$:



10. (Optional) Have the students add and subtract decimals using a calculator. Discuss the relative advantages and disadvantages for adding and subtracting decimals using (a) number pieces, (b) paper-and-pencil and (c) calculator.

10. You may wish to ask questions like: Which method is fastest? Which is easiest to understand? Which is easiest to imagine in your mind? Which method would be the best for explaining decimal addition and subtraction to someone who doesn't understand it?

Decimal Length and Area

O V E R V I E W

Linear measure with decimals is introduced and the distinction between linear measure and area measure is illustrated by finding dimensions and areas of rectangles.

Prerequisite Activity

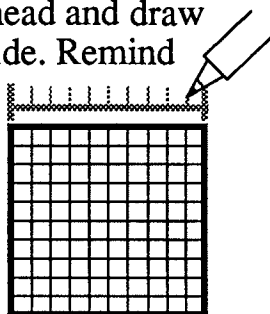
Unit IV, Activity 6, *Introduction to Decimals*

Materials

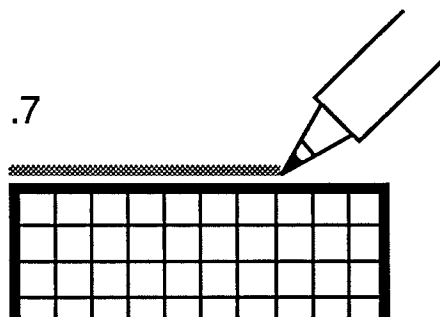
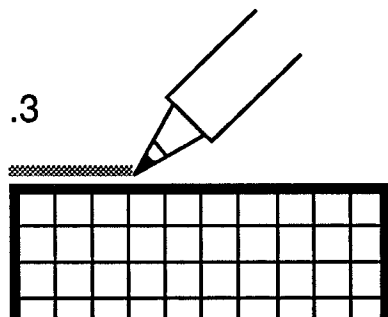
Base 10 number pieces

Actions

1. Place a decimal unit square on the overhead and draw a line segment equal to the length of one side. Remind the students that the length of the side of the unit square is 1 unit long. Then, using the grid on the unit square, subdivide the segment into 10 equal parts. Ask the students to determine the length of one of the smaller segments.



2. Use the edge of the unit square to illustrate drawing lengths of .3 and .7 units on the overhead projector.



Draw a few segments on the overhead and ask students to come forward and determine the lengths using the edge of the unit as a measuring device. Discuss ways that lengths of objects longer than the edge of a unit square can be determined.

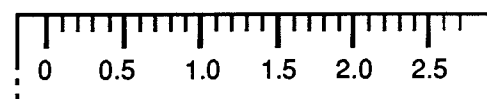
Comments

1. In this activity, as in Unit IV, Activities 6 and 7, the largest number piece represents the unit square. A transparent decimal unit would enable the students to see the grid on the unit.

One of the smaller segments is $\frac{1}{10}$ or .1 unit lengths. A note of caution: In Unit IV, Activities 6 and 7, .1 is defined in terms of the unit square and represents area. Now we are extending its meaning to length. The distinction between unit squares and unit lengths is discussed in Comment 3 of Unit III, Activity 6, *Number Piece Rectangles*.

2. To measure segments longer than 1 unit it is necessary to use the edges of more than one unit square or to use the edge of a unit repeatedly. Another option is to use a measuring tape (see Comment 3) or to construct a decimal ruler. A decimal ruler can be constructed by marking off segments along the edge of a sheet of paper. You may want your students to construct a ruler and use it to measure several objects such as their handspan, the width of a book, etc.

A 2.7 units ruler can be constructed along the longer side of an $8\frac{1}{2} \times 11$ sheet of paper:



Actions

3. (Optional) Discuss with the students methods for measuring “long” things. Have the students construct a measuring tape and use it to measure items in the classroom.

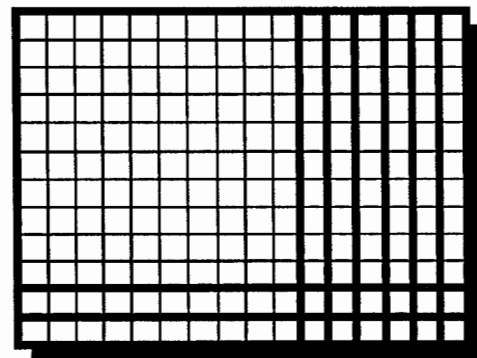
4. Distribute base 10 number pieces and a copy of Activity Sheet IV-8 to each student or group of students. Ask the students to use their number pieces to find the area of the top rectangle, doing no arithmetic other than counting. Discuss.

5. Ask the students to use their number pieces or a decimal ruler to find the dimensions of this rectangle.

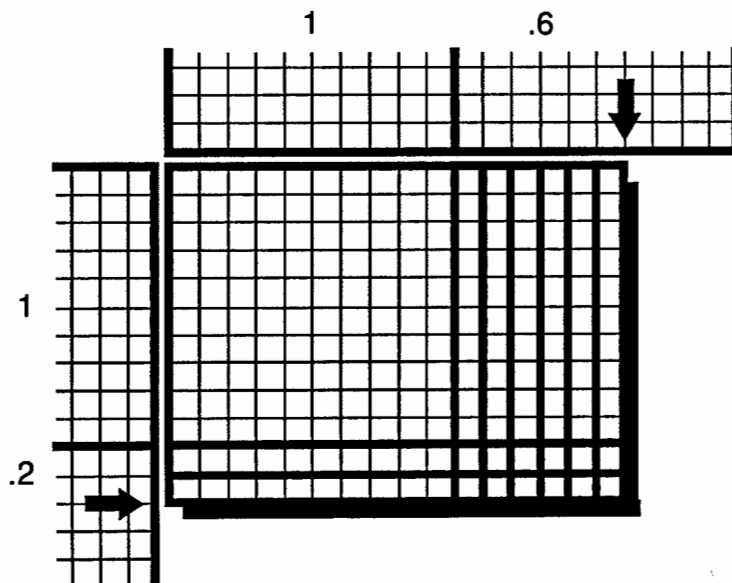
Comments

3. A pattern for a measuring tape is included at the end of this activity. One unit on the tape is the length of the side of a decimal unit square number piece—which also equals 1 decimeter. With this tape the students can measure the lengths of objects from .1 unit lengths to 12.5 unit lengths, to the nearest tenth of a unit length. Select several objects in the classroom for students to measure.

4. The rectangle can be covered with 1 unit, 8 tenths and 12 hundredths. These pieces can be traded to get a minimal collection of 1 unit, 9 tenths and 2 hundredths. So, the area is 1.92 square units. The rectangle can also be covered with other collections of decimal pieces, but the minimal collection will always be the same.



5. Decimal rulers are discussed in Comment 2. The dimensions of the rectangle are 1.2 and 1.6 unit lengths.



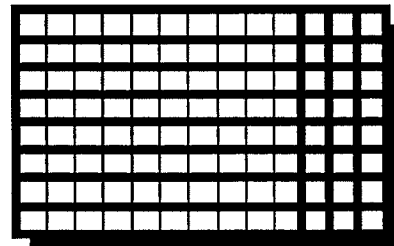
Actions

6. Ask the students to use their number pieces to find the dimensions and area of the rectangle at the bottom of the Activity Sheet, using no arithmetic other than counting. Discuss.

7. Have the students select a collection of 2 units, 6 tenths and 4 hundredths. Ask them to form a rectangle using all of these pieces and then determine its area and dimensions. After they have completed one rectangle, ask them to find other rectangles of the same area and determine their dimensions.

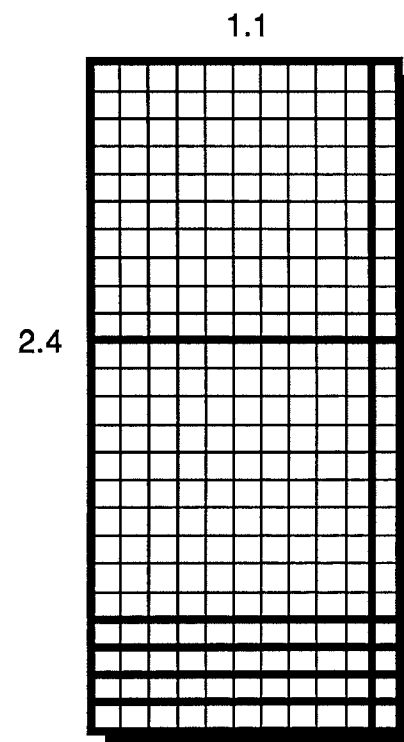
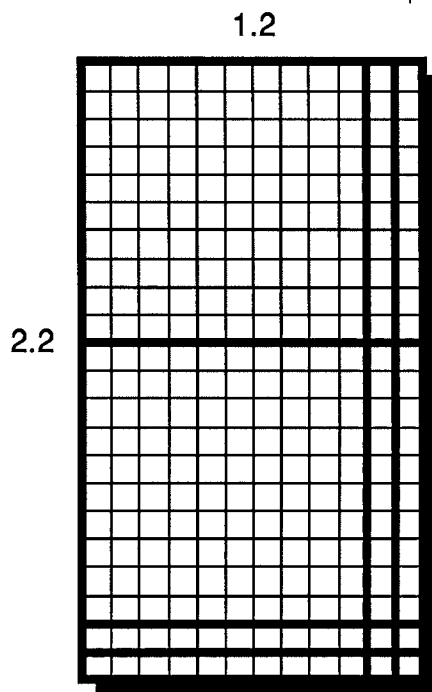
Comments

6. The dimensions of this rectangle are .8 and 1.3 unit lengths. The area is 1.04 square units.



7. With the given number pieces, or their equivalent, the area of any rectangle constructed will be 2.64 square units. The following rectangles can be constructed with these pieces without trading.

By trading 2 units for tenths and two tenths for hundredths, several additional rectangles can be constructed.



Actions

8. For each of the following, have the students form a rectangle with the given collection, or an equivalent collection, and determine its area and dimensions.

- a. 2 units, 8 tenths, 6 hundredths
- b. 3 units, 3 tenths, 6 hundredths
- c. 3 tenths, 8 hundredths
- d. 3 units, 1 tenth, 2 hundredths

9. Ask the students to construct the following number piece rectangles and provide the requested information about each.

- a. A square with area 1.69 square units. Determine its dimensions.
- b. A rectangle with dimensions 1.7 unit lengths by 2.3 unit lengths. Record its area.
- c. A rectangle with area 4.37 square units and one dimension 2.3 units lengths. Find the other dimension.

Comments

8. Below are some examples. In each case other rectangles are possible.

a. Area: 2.86 square units. Two rectangles can be constructed without trading. One has dimensions 1.3 unit lengths by 2.2 unit lengths and the other, 1.1 unit lengths by 2.6 unit lengths.

b. Area: 3.36 square units. Trades must be made to form a rectangle. Trading one unit for tenths and one tenth for hundredths, it is possible to form a rectangle that is 1.4 unit lengths by 2.4 unit lengths.

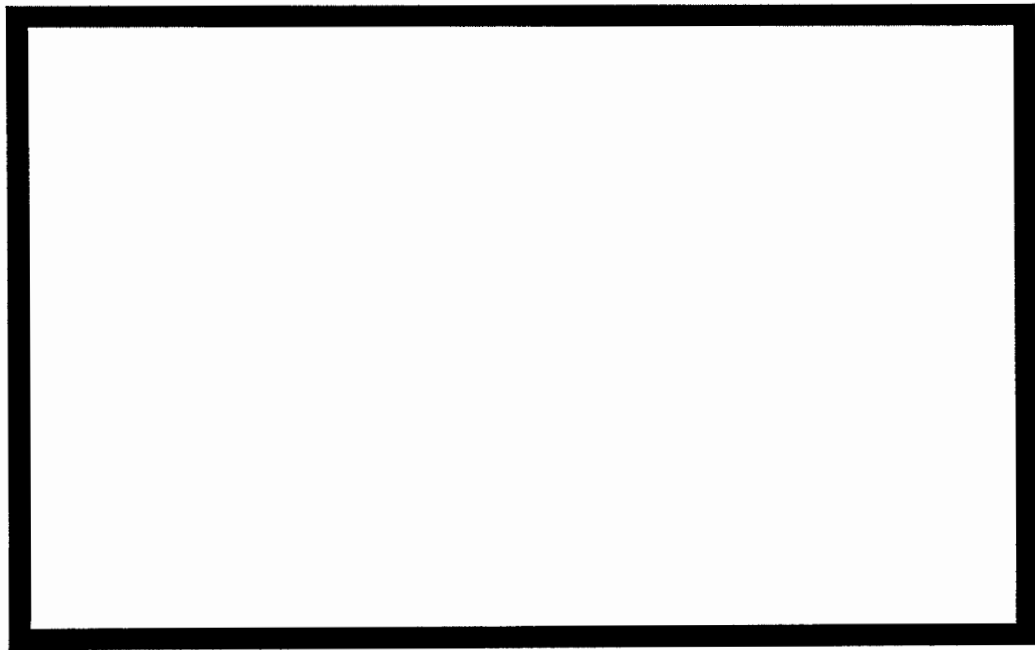
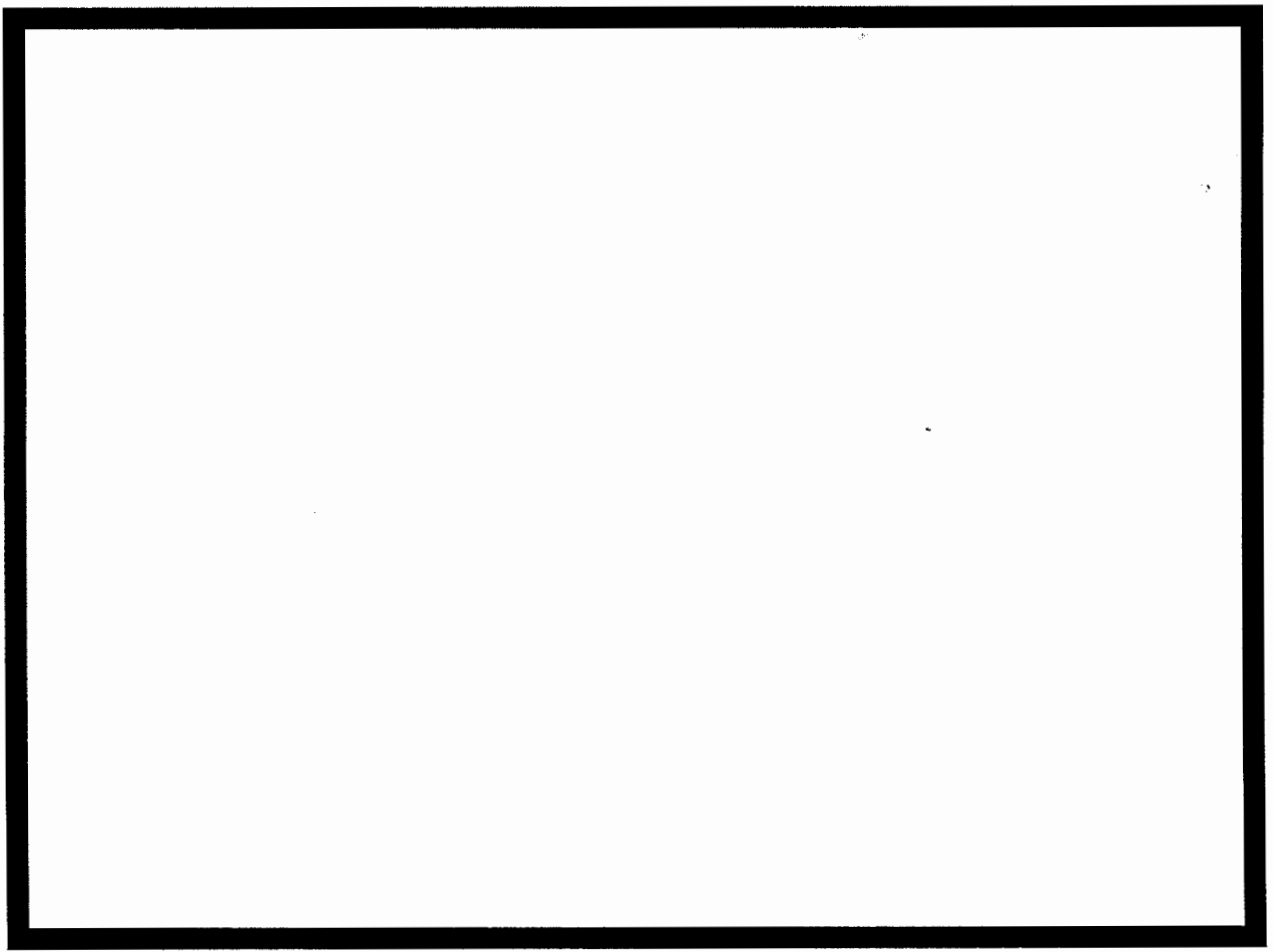
c. Area: .38 square units. The pieces must be traded to form a rectangle. One rectangle has dimensions .2 unit lengths by 1.9 unit lengths.

d. Area: 3.12 square units. Making trades, one rectangle is 1.3 unit lengths by 2.4 unit lengths.

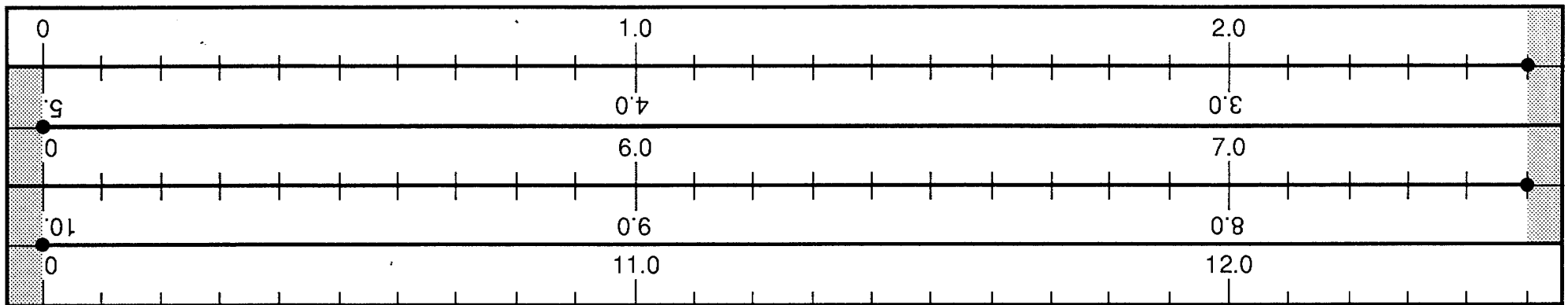
9. a. A collection of 1 unit, 6 tenths, and 9 hundredths can be arranged into a square with sides 1.3 unit lengths.

b. The area is 3.91 square units.

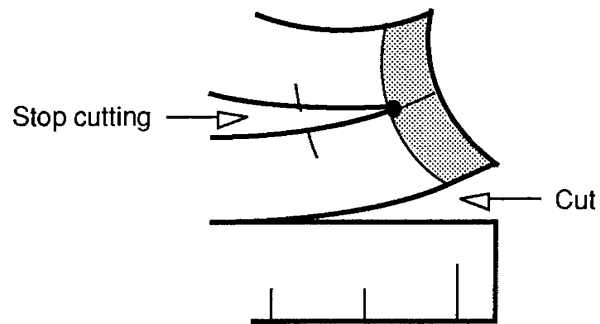
c. The other dimension is 1.9 units lengths.



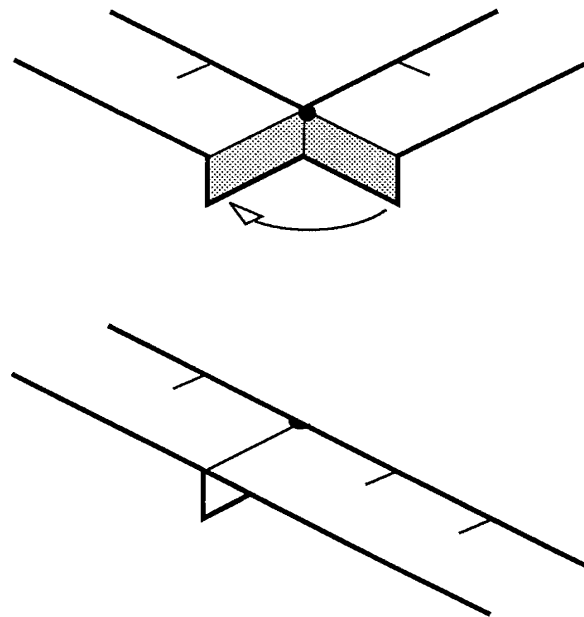
Pattern for Base 10 Measuring Tape



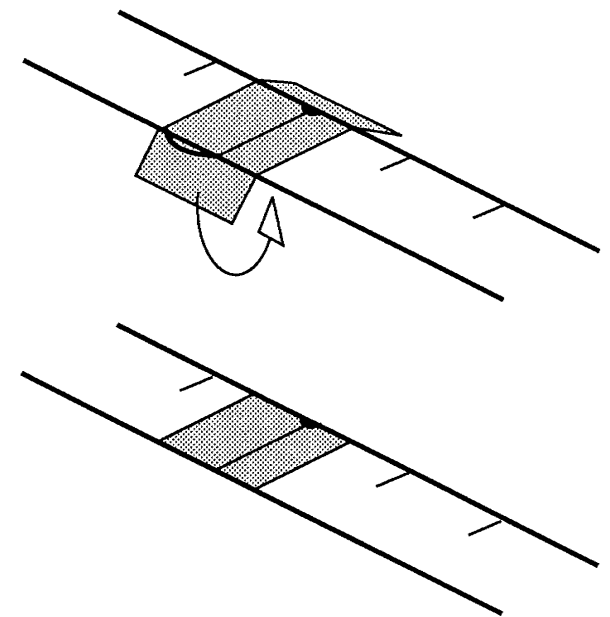
1. Cut along all heavy lines.



2. Fold in shaded areas:



3. Flatten tab and wrap connection with scotch tape:



Decimal Multiplication and Division

O V E R V I E W	Prerequisite Activity Unit IV/Activity 8 <i>Decimal Length and Area</i>
Base 10 number pieces are used to find products and quotients of decimals.	Materials Base 10 number pieces; transparencies for teacher use (see Comments 3 and 4)

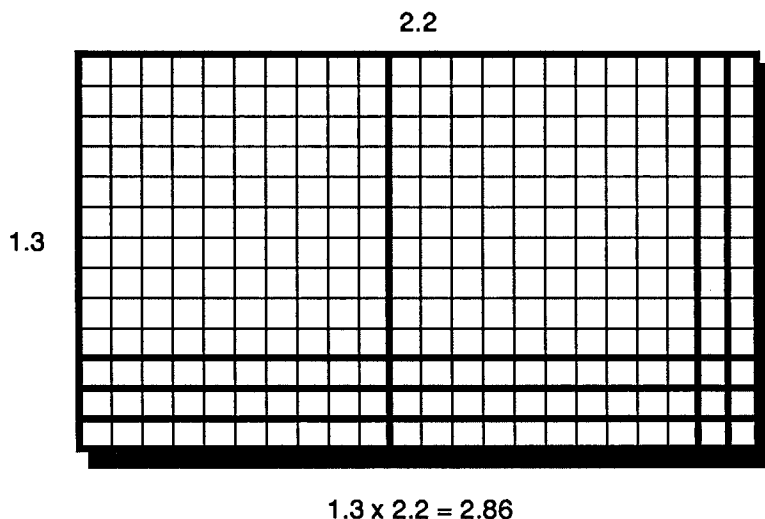
Actions

1. Distribute base ten pieces to each student or group of students. Ask the students to construct a number piece rectangle whose dimensions are 1.3 and 2.2. Discuss with them how this rectangle can be used to find the product 1.3×2.2 .

Comments

1. In this activity, as in the previous decimal activities, the largest number piece represents the unit square.

You may wish to briefly review whole number multiplication (Unit III, Activity 7) before discussing multiplication of decimals. The product of two numbers can be represented as the area of a rectangle which has the two numbers as dimensions. A number piece rectangle with dimensions 1.3 by 2.2 has an area of 2.86 square units.



Actions

2. Ask the students to use their number pieces to build rectangles which represent the following products and, then, using no arithmetical procedures other than counting, to determine the products.

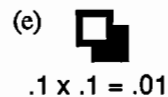
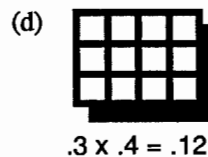
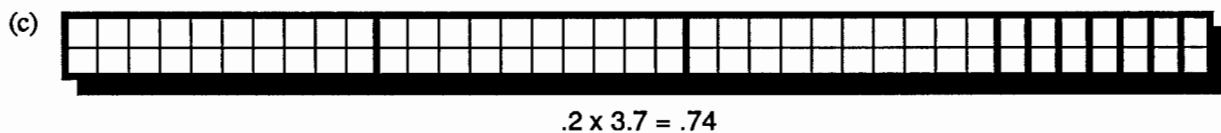
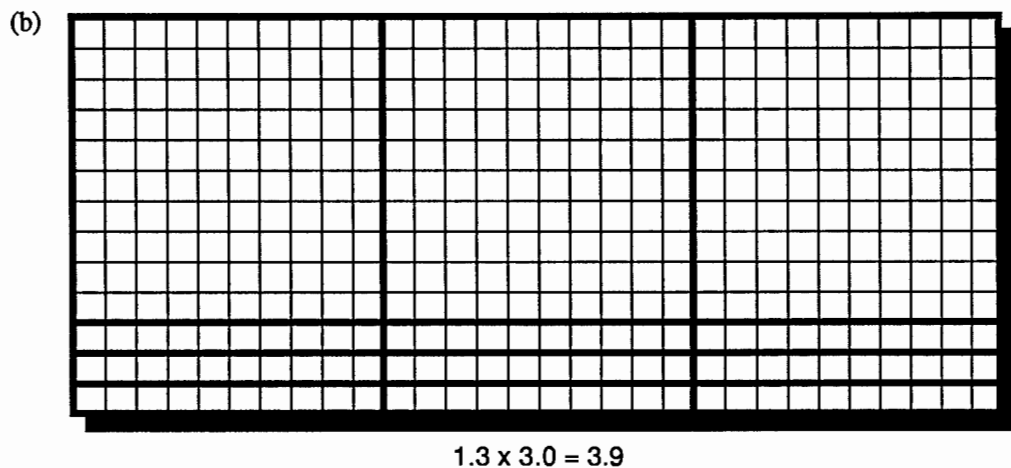
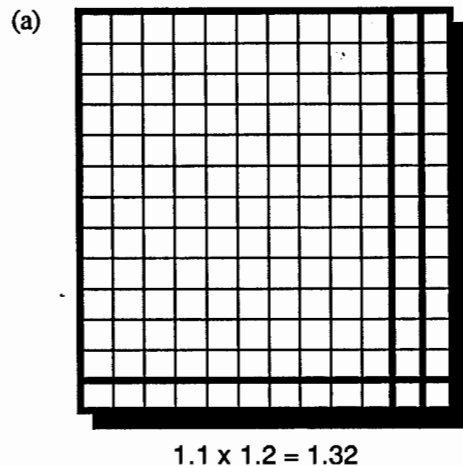
(a) 1.1×1.2 (b) 1.3×3.0

(c) $.2 \times 3.7$ (d) $.3 \times .4$

(e) $.1 \times .1$

Comments

2. Only parts c and d require number piece trading to obtain the minimal set for the area.

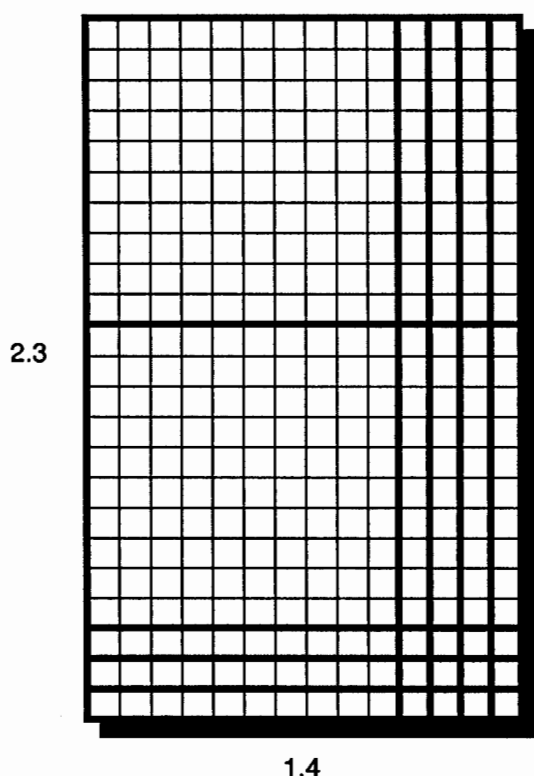


Actions

3. Project a copy of the rectangle shown below on the screen.

(a) Ask the students to determine the number of hundredth squares (small squares) in the rectangle. Discuss.

(b) Relate the multiplication of decimals to the multiplication of whole numbers, using the rectangle as a visual model.



Comments

3. The rectangle can either be sketched, as shown, on a base 10 grid transparency or constructed on the overhead using transparent base 10 number pieces.

Some students may think of the rectangle as a collection of number pieces containing 2 unit squares, 11 tenth strips and 12 hundredth squares and mentally convert this collection to 322 hundredth squares.

Other students may view the rectangle as a 23 by 14 array of hundredth squares, and find the number of hundredth squares by finding the product 23×14 . If no one does this, call this method to the students' attention.

In this last method, the number of hundredth squares in the rectangle, and hence its area, has been found by multiplying two whole numbers. To represent this area as a decimal, one can imagine trading number pieces to convert a set of 322 hundredth squares into a minimal set containing 3 unit squares, 2 tenth strips and 2 hundredth squares. Hence the area is 3.22.

The above suggests the common practice of multiplying two decimal numbers by disregarding the decimal points, multiplying them as though they were whole numbers and then placing the decimal point.

The placement of the decimal can generally be accomplished by estimation. In the above case, placing the decimal point before the 3 gives a product of .322 which is much too small; placing the decimal point after 32 gives a product of 32.2 which is much too large. Hence the only reasonable choice for placing the decimal is after the 3.

You may wish to have the students find other decimal products by treating them as whole number multiplications and then placing the decimal by estimation.

Actions

4. (Optional) Show the students a magnified decimal unit square. Discuss with them how the unit square can be further subdivided to obtain areas representing decimals smaller than a hundredth.

Comments

4. Page 9 of this activity is a master for an overhead transparency showing a decimal unit square magnified 10 times. The shaded hundredth square on the upper decimal unit corresponds to the shaded square on the lower diagram after magnification.

If each hundredth square is divided into ten equal rectangular strips, the unit square will be divided into 10×100 or 1000 of these strips. Hence each of them has an area of one thousandth (.001). Page 10 of this activity is a master for an overhead transparency.

If each thousandth strip is divided into 10 equal squares, the decimal unit square is divided into 10×1000 or ten thousand small squares. Hence each of these squares has an area of one ten-thousandth (.0001). Page 11 of this activity is a master for an overhead transparency showing a magnified decimal unit square divided into ten-thousandth squares.

During your discussion, you may want to ask the students to determine:

- The length of the side of a ten-thousandth square (.01 unit lengths).
- The number of ten-thousandth squares in a hundredth square (100).
- The dimensions of a rectangular thousandth strip (.1 \times .01).
- The number of squares into which a decimal unit square would be divided if each ten-thousandth square were divided into 100 squares (one million).

Actions

5. (Optional). Place a transparency of page 11 of this activity on the overhead. Ask for a volunteer to come to the overhead and sketch a rectangle that represents the product $.05 \times .1$. Ask another volunteer to come forward and determine the area of that rectangle.

Ask for other volunteers to compute the following products by drawing sketches.

(a) $.1 \times .11$

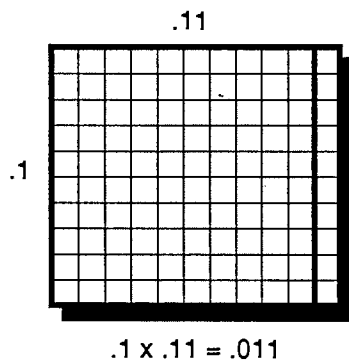
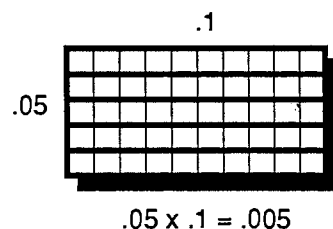
(b) $.02 \times .03$

(c) $.12 \times .12$

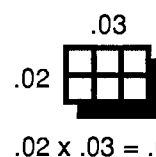
(d) $.14 \times .25$

Comments

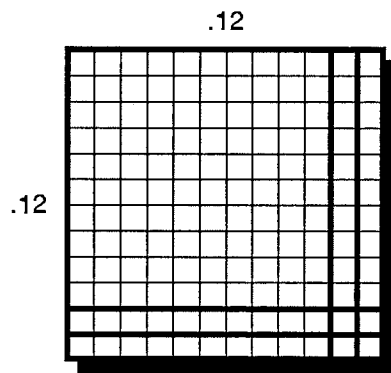
5. A $.1 \times .05$ rectangle contains 5 thousandths strips. Hence $.05 \times .1 = .005$.



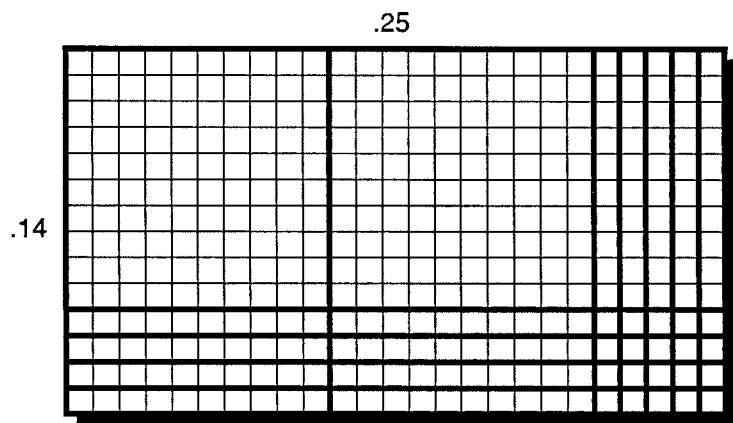
(a) A $.1 \times .11$ rectangle contains 1 hundredth square and one thousandth strip.



(b) A $.02 \times .03$ rectangle contains 6 ten-thousandths squares.



(c) A $.12 \times .12$ square contains 1 hundredth square, 4 thousandth strips and 4 ten-thousandth squares. Hence $.12 \times .12 = .0144$.



(d) A $.14 \times .25$ rectangle contains 2 hundredth squares, 13 thousandth strips and 20 ten-thousandth squares. This is equivalent to 3 hundredth squares and 5 thousandth strips. Hence $.14 \times .25 = .035$.

Actions

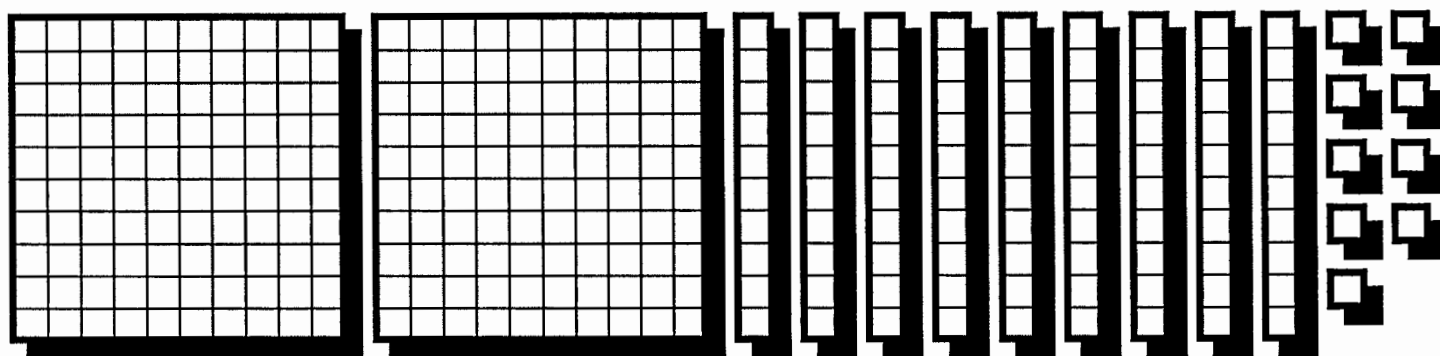
6. Ask the students to construct a number piece rectangle of area 2.99 which has one dimension 2.3 units in length. Discuss with them how this rectangle can be used to find the quotient $2.99 \div 2.3$

Comments

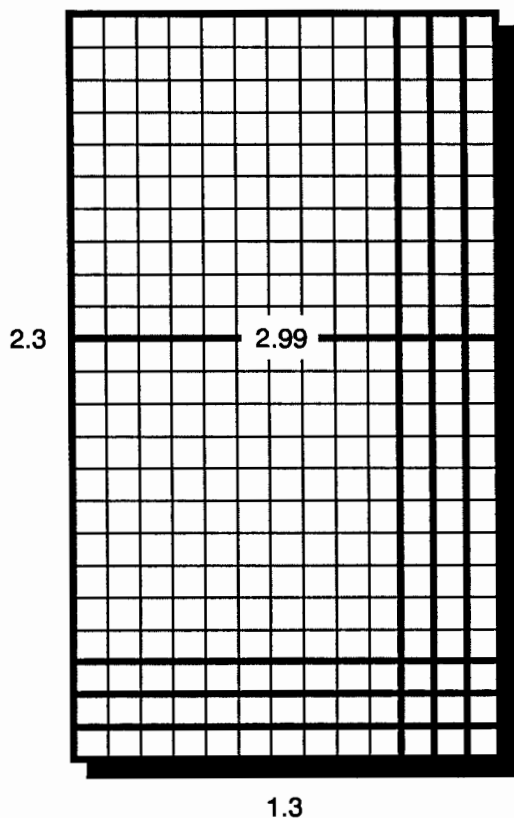
6. You may wish to review whole number division using base 10 number pieces before extending the idea to decimals.

To divide 2.99 by 2.3, construct a rectangle of area 2.99 square units which has one dimension 2.3 units in length. The other dimension will be the quotient. In this case 2 units, 9 tenths and 9 hundredths can be arranged to form a rectangle which is 2.3 units by 1.3 units. So,

$$2.99 \div 2.3 = 1.3$$



Decimal pieces representing area 2.99



Actions

7. Ask the students to use their number pieces, and no arithmetical procedures other than counting, to find the following quotients:

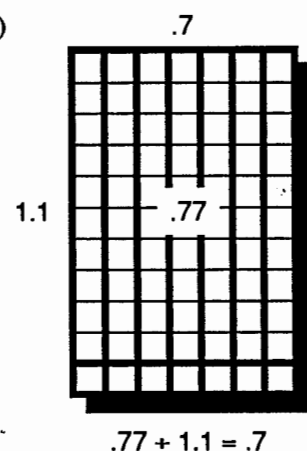
(a) $.77 \div 1.1$

(b) $3.08 \div 1.4$

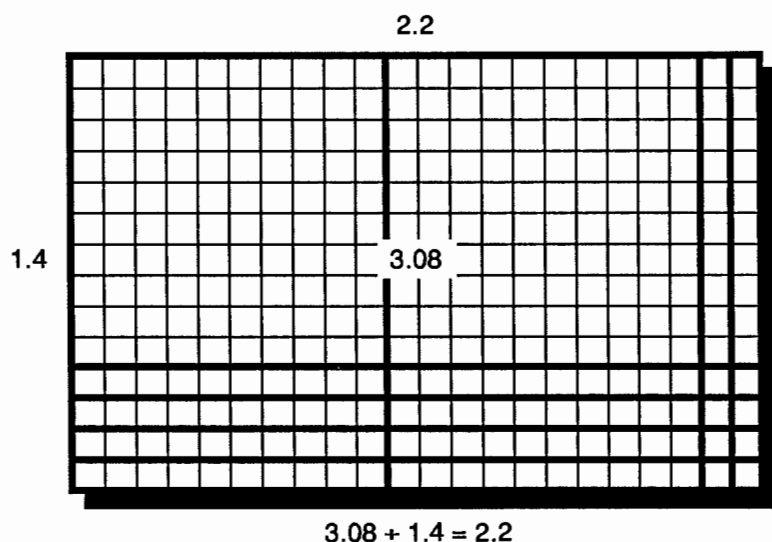
(c) $2.04 \div 1.2$

Comments

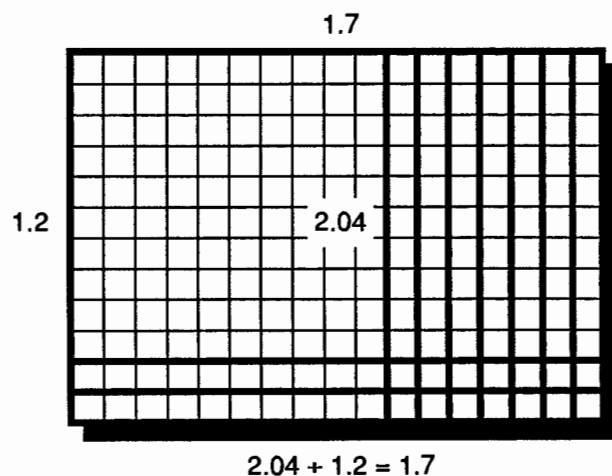
7. (a)



(b) Starting with 3 units and 8 hundredths, it is necessary to trade 1 unit for 10 tenths to build this rectangle:



(c) Starting with 2 units and 4 hundredths, 1 unit must be traded for 10 tenths and then 1 tenth must be traded for 10 hundredths.



Actions

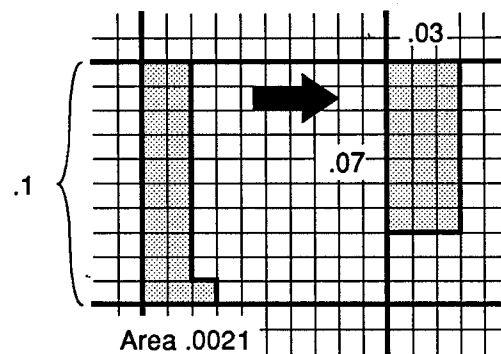
Comments

8. (Optional) Use the magnified decimal unit in Action 6 to determine these quotients:

(a) $.0021 \div .07$ (b) $.0126 \div .14$

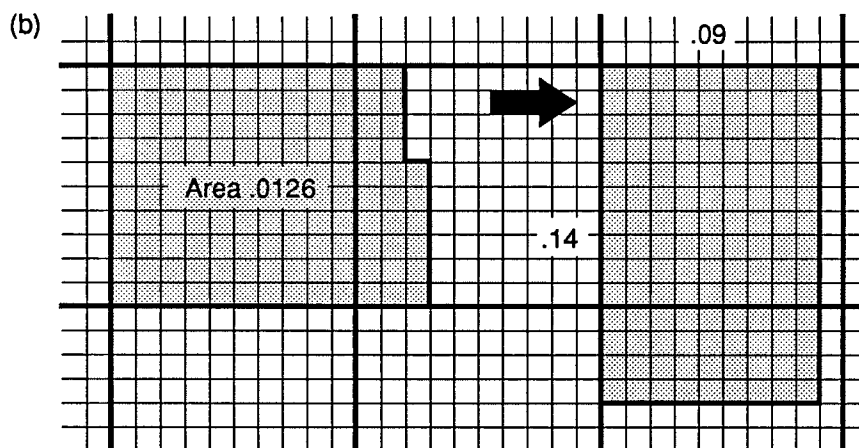
(c) $.0374 \div .22$

8. (a) If an area of 21 ten-thousandths is rearranged to form a rectangle with one dimension 7 hundredths, the other dimension is 3 hundredths.



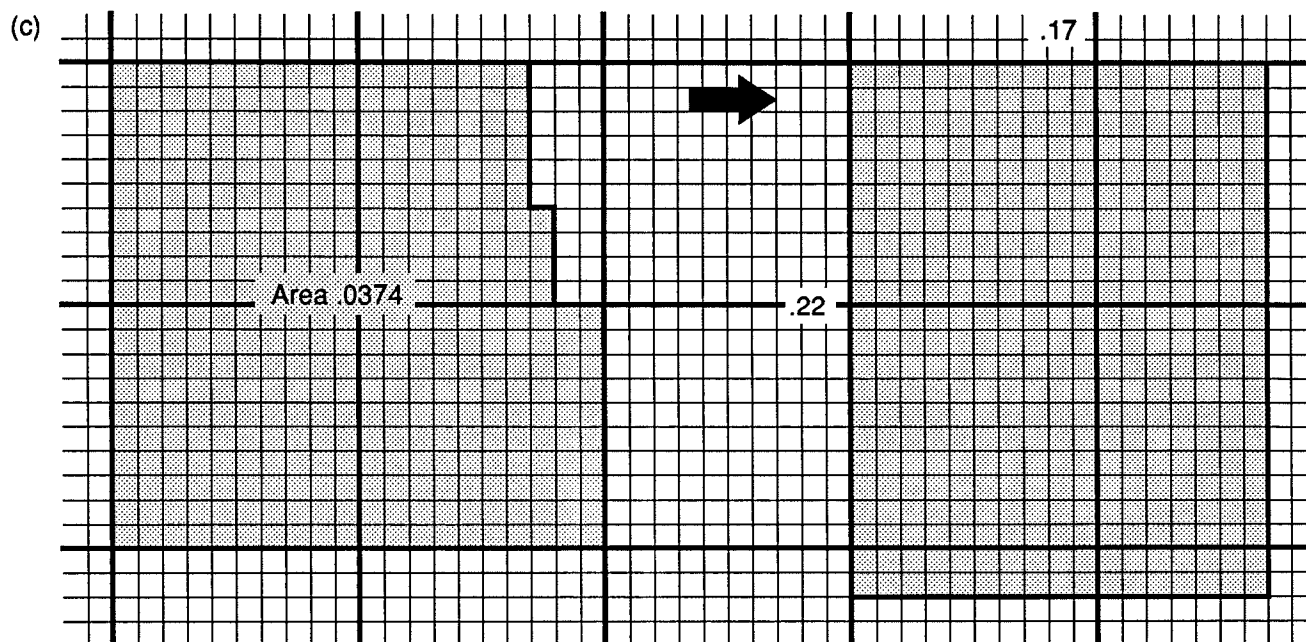
Area .0021

$$.0021 \div .07 = .03$$



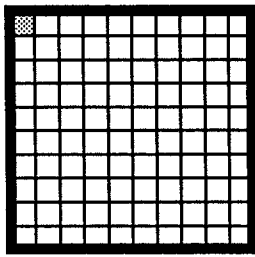
Area .0126

$$.0126 \div .14 = .09$$



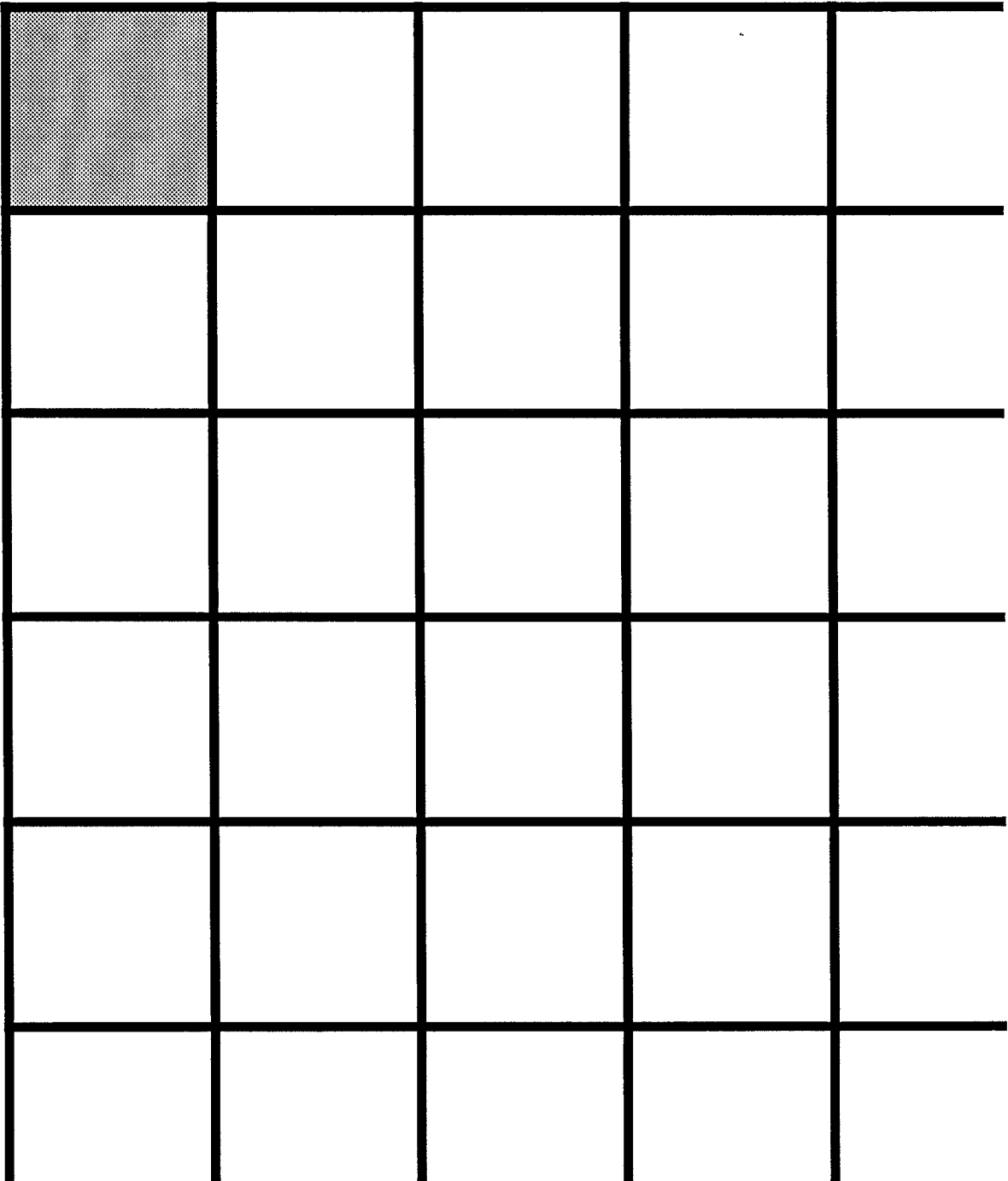
Area .0374

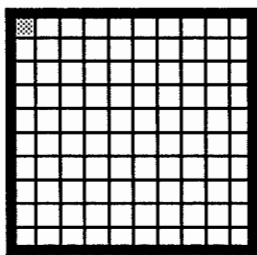
$$.0374 \div .22 = .17$$



**Decimal unit square
divided into hundredth squares**

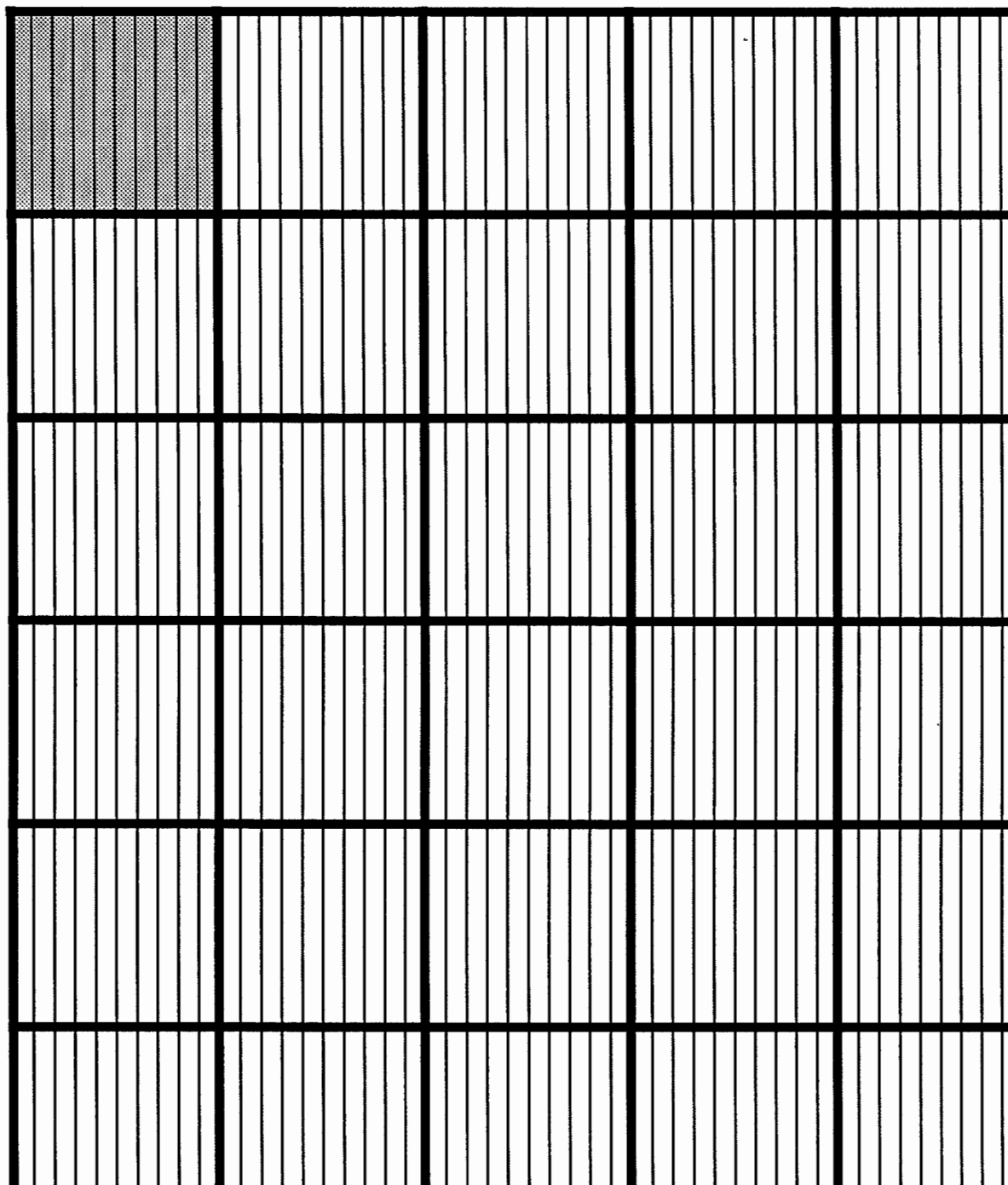
Magnified decimal unit square divided into hundredth squares

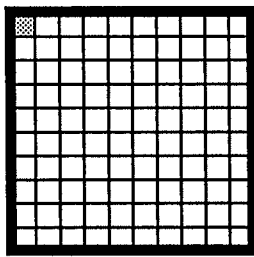




Decimal unit square
divided into hundredth squares

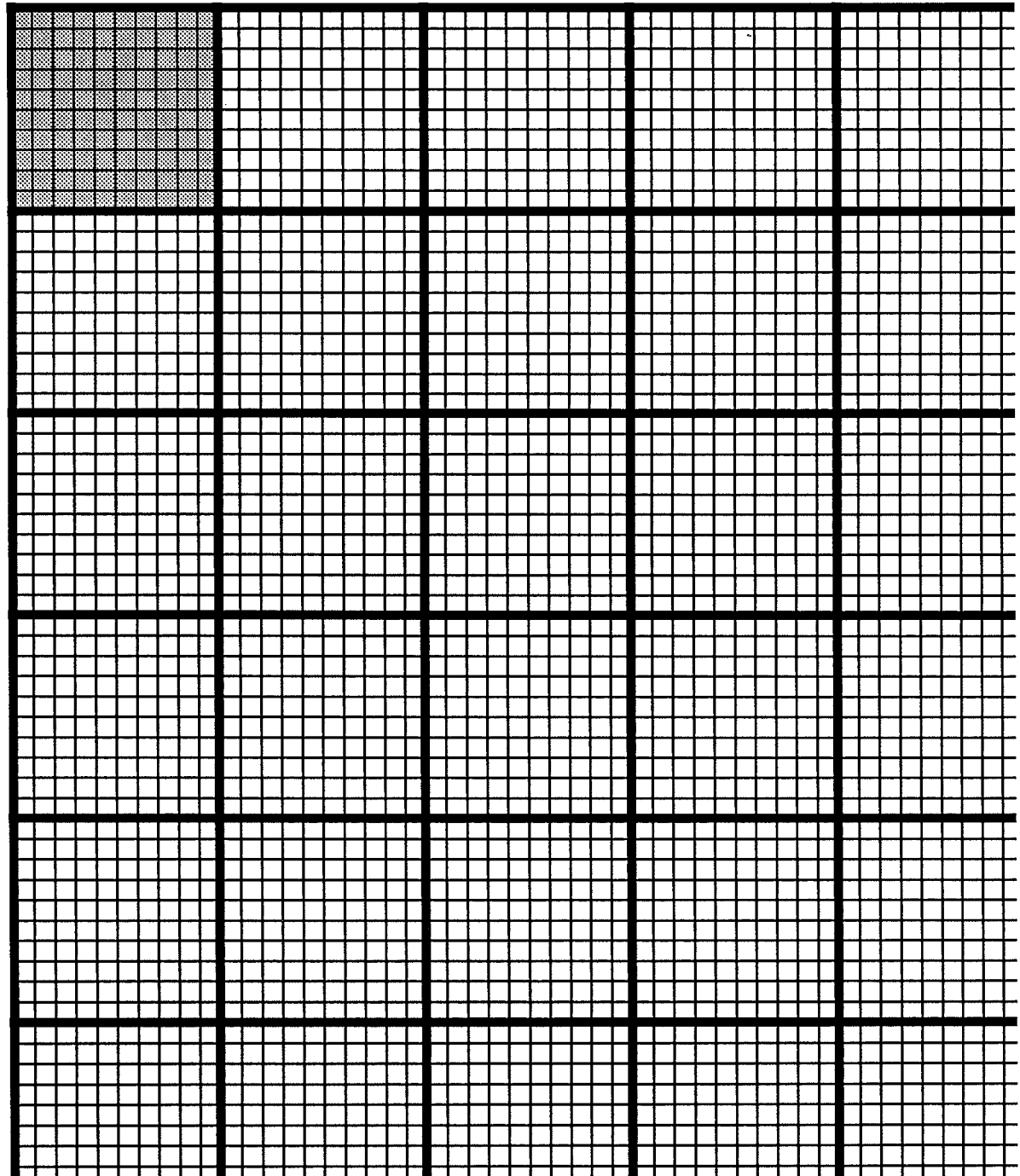
Magnified decimal unit square divided into thousandth strips





Decimal unit square
divided into hundredth squares

Magnified decimal unit square divided into ten-thousandth squares



Fraction Operations via Area Addition and Subtraction

O V E R V I E W

A fraction is represented as the area of a rectangular region. The sum and difference of two fractions is obtained by finding the sum and difference of the corresponding areas.

Prerequisite Activity

None, however previous work with fractions, such as Unit 4, Activity 2, *Fractions on a Line*, or Unit 4, Activity 3, *Fraction Bars*, may be helpful.

Materials

Copies of Activity Sheets for each student (see Comment 1)

Actions

1. Distribute a copy of Activity Sheet IV-10-A to each student. Ask the students to subdivide each region into 4 or more smaller regions which have the same size and shape.

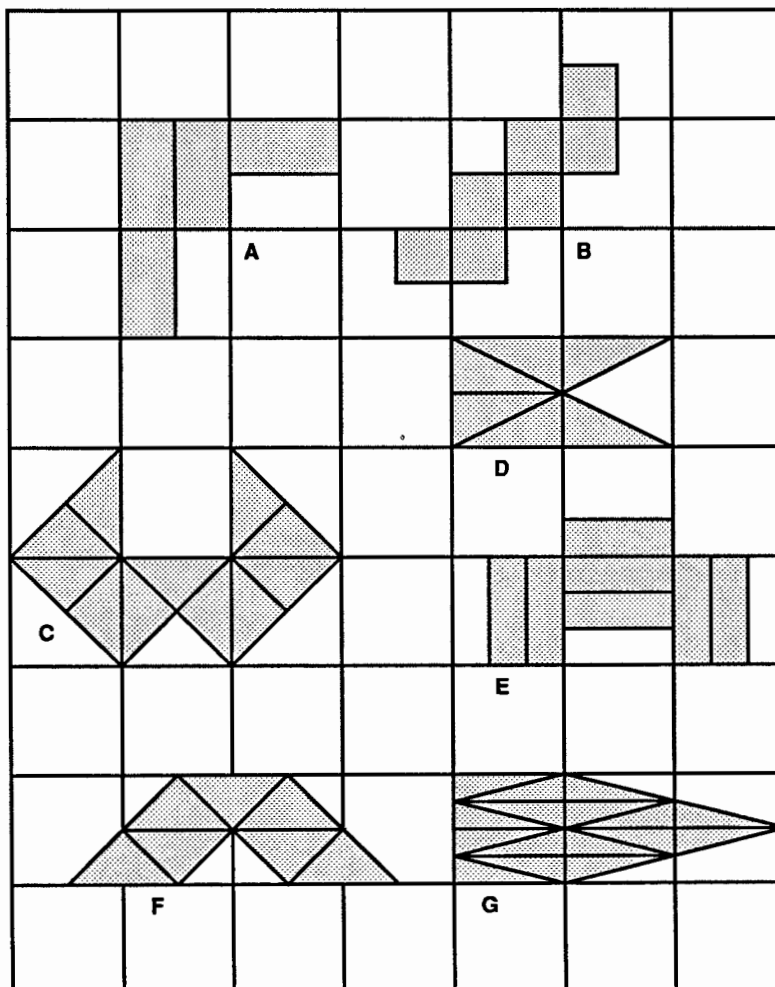
Comments

1. Masters of Activity Sheets IV-10-A and IV-10-B are found at the end of this unit. Activity Sheet IV-10-B will be used in Action 4.

You may want to demonstrate this action by subdividing one of the regions. This can

be done on the overhead using a transparency of the activity sheet. The students may find straightedges helpful in drawing subdivisions.

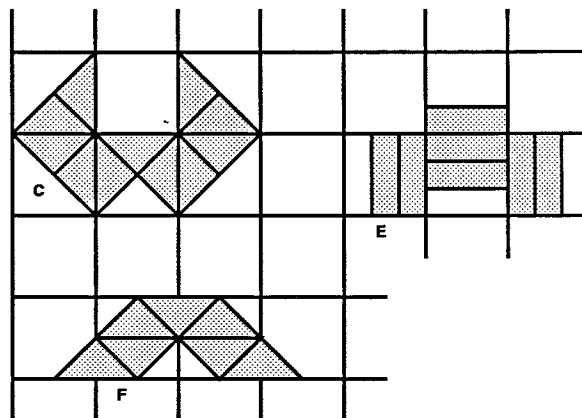
Regions which have the same size and shape are said to be *congruent*. There is more than one way to subdivide a region into four or more congruent subregions. For each region on the activity sheet, one way to subdivide it is shown here.



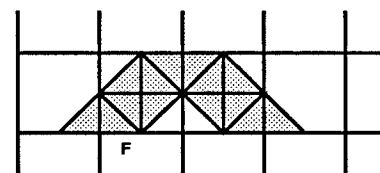
2. For each region, ask the students to find the area of the subregions into which it has been subdivided, given that the unit area is 1 square of the activity sheet grid.

2. In the subdivision of E shown below, 3 of the subregions fit into a unit square. Hence, the area of one subregion is $\frac{1}{3}$.

Note that the subregions in F are congruent to the subregions in C. Looking at the subdivision of C, it is seen that 4 of these subregions fill a unit square. Hence, each subregion has area $\frac{1}{4}$.

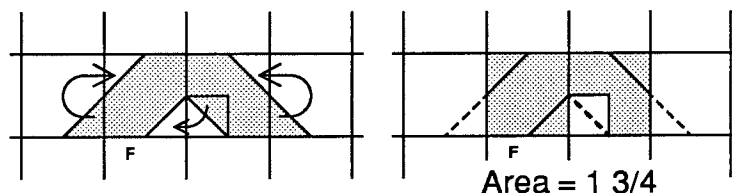
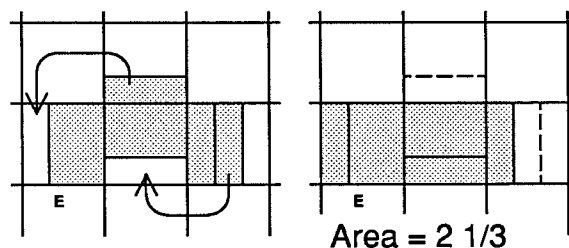


Other subdivisions are possible. If F is subdivided as shown below, the subregions have area $\frac{1}{8}$.



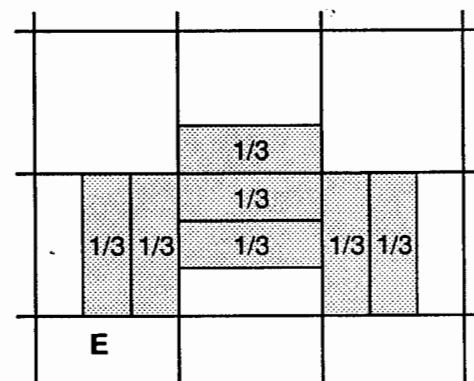
3. Ask the students to find the area of each region. Discuss their methods.

3. One way to find the area of a region is to imagine moving portions of the region to obtain another region whose area is apparent. For example, if the portions of E and F are moved as shown, one sees their areas are $2\frac{1}{3}$ and $1\frac{3}{4}$, respectively.



Continued next page.

3. (Continued.) Alternately, the area of a region can be found from the area of the subregions into which it is divided. Since E is divided into 7 subregions each of area $\frac{1}{3}$, its area is $7 \times \frac{1}{3}$ or $\frac{7}{3}$.



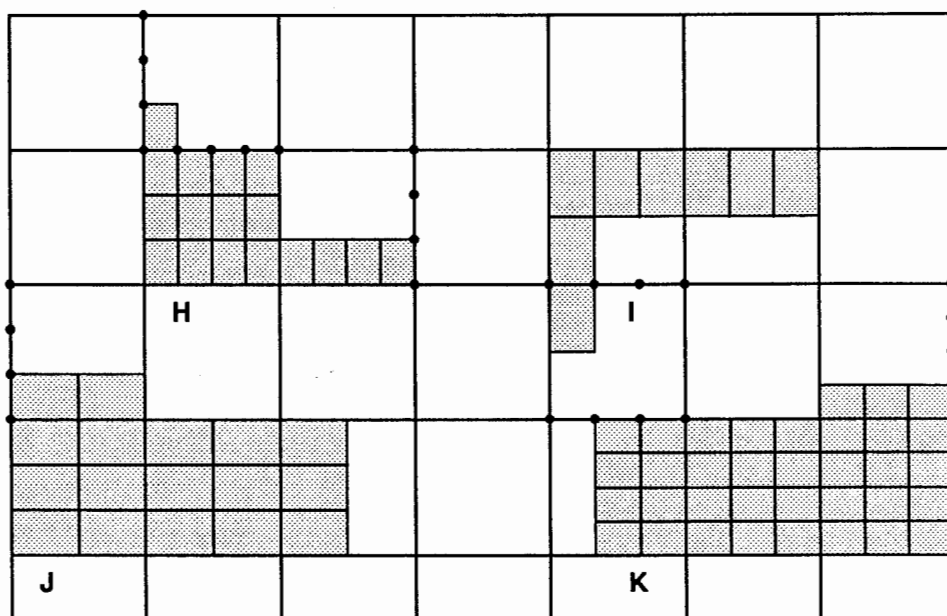
Area = $\frac{7}{3}$

Note that, using one method, the area of E was found to be $2\frac{1}{3}$ and, using another method, it was found to be $\frac{7}{3}$. Hence $\frac{7}{3} = 2\frac{1}{3}$. Similar fraction equalities can be obtained by finding the area of other regions in more than one way. In this activity, correct answers can be reported in a variety of forms. No particular form need be preferred.

4. Distribute a copy of Activity Sheet IV-10-B to each student. Ask the students to subdivide region H into 4 or more subregions of the same size and shape and then find its area. Repeat these instructions for I, J and K.

4. The dots on the activity sheet are intended to help the student recognize divisions into thirds and fourths.

Here is one way to subdivide H, I, J and K:



Actions

5. Call the students attention to pair L. Ask them to find the areas of L_1 and L_2 by inspection. Then ask them to subdivide L_1 and L_2 so that all subregions, those in L_1 and those in L_2 , have the same size and shape. After the students have completed the subdivision, ask them to find the sum of the areas of L_1 and L_2 . Discuss the corresponding arithmetical statement.

6. Repeat Action 5 for pair M.

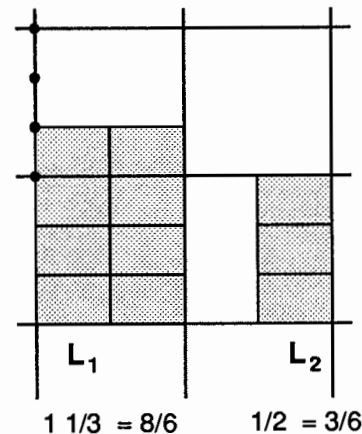
Comments

5. The area of L_1 is $1 \frac{1}{3}$. Some students may report it as $\frac{4}{3}$. The area of L_2 is $\frac{1}{2}$.

In the sketch shown below, the area of each subdivision is $\frac{1}{6}$. There are 8 subdivisions in L_1 and 3 in L_2 , hence the sum of their areas is $\frac{11}{6}$. That is,

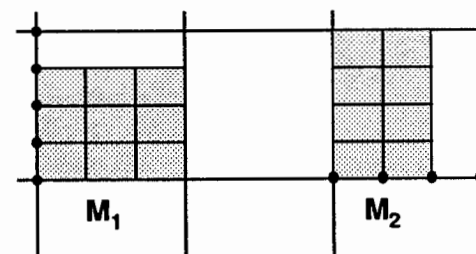
$$1 \frac{1}{3} + \frac{1}{2} = \frac{8}{6} + \frac{3}{6} = \frac{11}{6}.$$

Dividing the two regions into subregions of the same size and shape is a geometric equivalent to finding common denominators.



$$1 \frac{1}{3} + \frac{1}{2} = \frac{11}{6}$$

6. M_1 has area $\frac{3}{4}$ and M_2 has area $\frac{2}{3}$. They can be subdivided into subregions with area $\frac{1}{12}$:



$$\frac{3}{4} + \frac{2}{3} = \frac{9}{12} + \frac{8}{12} = \frac{17}{12}$$

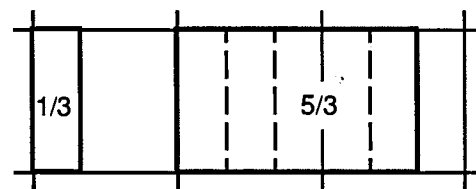
Actions

7. Distribute grid paper to each student. Ask the students to sketch a rectangle whose area is $1/3$. Then ask them to sketch a rectangle of area $5/3$. Discuss.

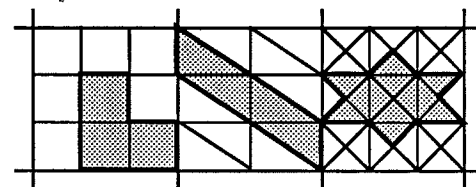
8. Ask the students to draw rectangles of areas $5/4$ and $2/3$. Then have them use the subdivision method to find the sum of the areas of these two rectangles. Discuss the corresponding arithmetical statement.

Comments

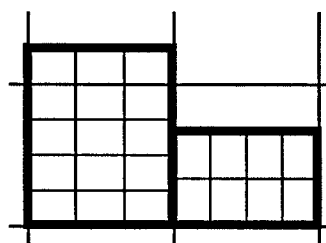
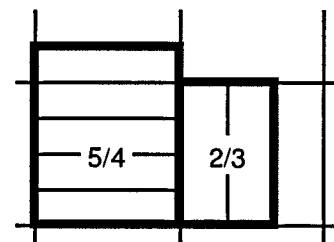
7. A master for grid paper is included. One way to obtain a rectangle whose area is $5/3$ is to adjoin 5 rectangles of area $1/3$:



The students may be interested in sketching other regions whose area is $1/3$. Here are some examples:

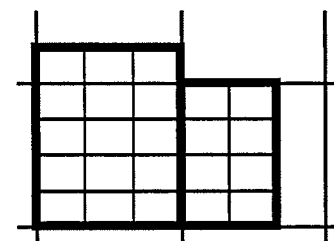


8. The rectangles can be drawn separately or adjoining each other:



It is easier to subdivide the rectangles into congruent subregions if the rectangles are drawn as shown above, with the sides of length 1 perpendicular to one another. If the rectangles are drawn as shown on the left, the subdivision into congruent regions is not as apparent.

In finding the sum of the areas, the students have added $5/4 + 2/3$ by converting these fractions to fractions with a common denominator:



$$5/4 + 2/3 = 15/12 + 8/12 = 23/12$$

Actions

9. Have the students use the method discussed in Action 7 to find these sums:

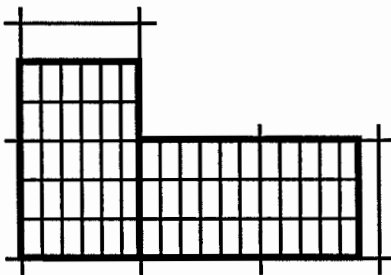
(a) $4/3 + 3/2$

(b) $7/5 + 1/2$

(c) $2/3 + 3/8$

(d) $1 \frac{2}{3} + 1 \frac{5}{6}$

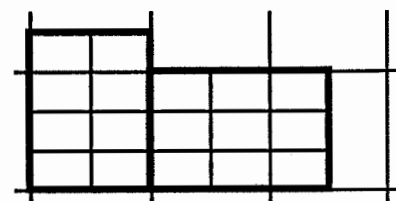
(d)



$$1 \frac{2}{3} + 1 \frac{5}{6} = \frac{30}{18} + \frac{33}{18} = \frac{63}{18}$$

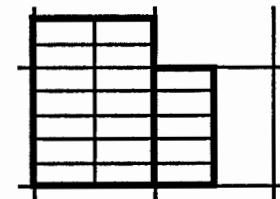
Comments

9. (a)



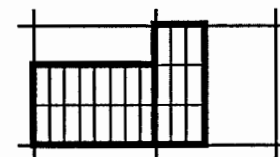
$$4/3 + 3/2 = 8/6 + 9/6 = 17/6$$

(b)

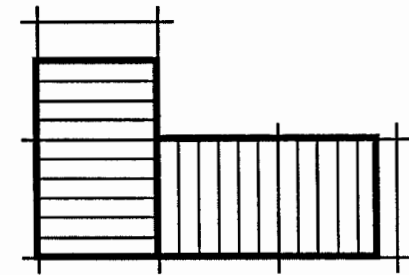


$$7/5 + 1/2 = 14/10 + 5/10 = 19/10$$

(c)



$$2/3 + 3/8 = 16/24 + 9/24 = 25/24$$



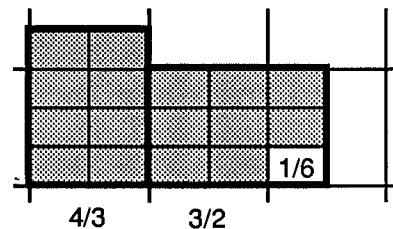
$$\text{or } 1 \frac{2}{3} + 1 \frac{5}{6} = \frac{10}{6} + \frac{11}{6} = \frac{21}{6}$$

Actions

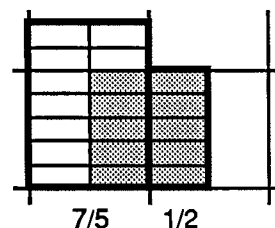
10. Discuss with the students how they can use their sketch for part (a) of Action 8, or another sketch, to determine $3/2 - 4/3$. Then ask them to find $7/5 - 1/2$ and $2/3 - 3/8$.

Comments

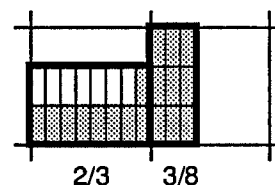
10. $3/2 - 4/3$ can be viewed as the difference in area between a rectangle of area $3/2$ and one of area $4/3$. In the sketch below, the rectangle of area $4/3$ is shaded. An equivalent area is shaded in the rectangle representing $3/2$. The unshaded portion is the desired difference.



$$3/2 - 4/3 = 9/6 - 8/6 = 1/6$$

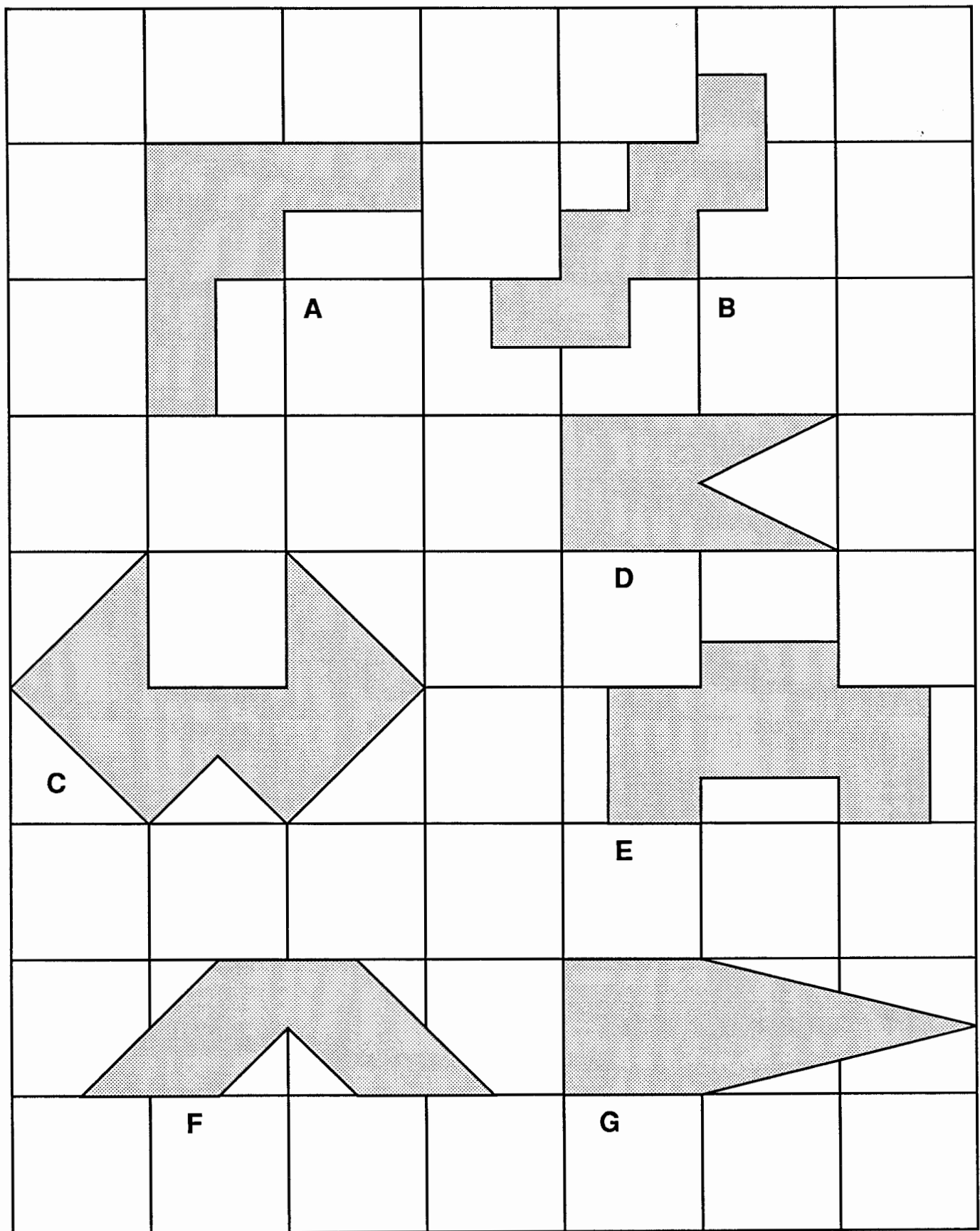


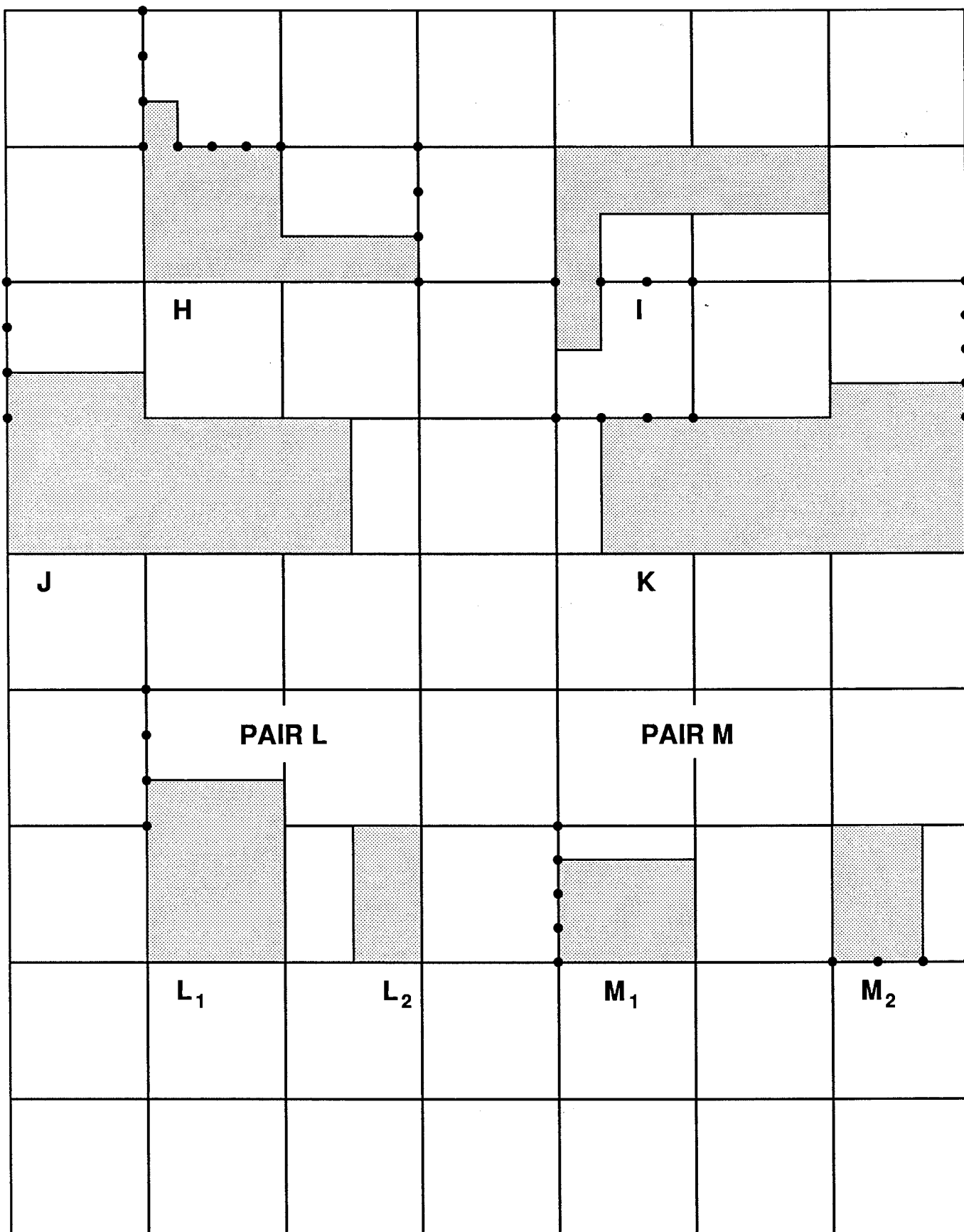
$$7/5 - 1/2 = 14/10 - 5/10 = 9/10$$

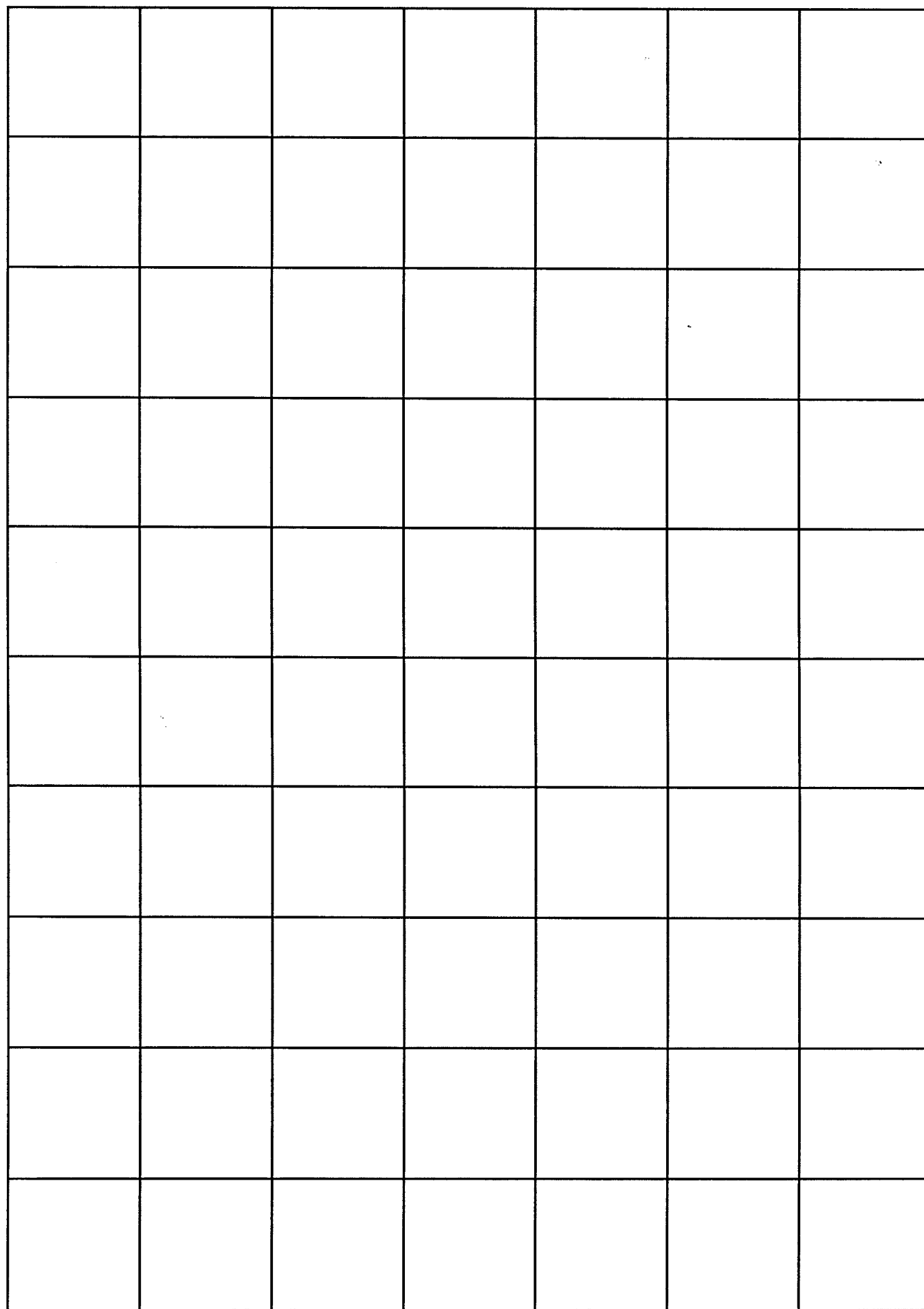


$$2/3 - 3/8 = 40/24 - 9/24 = 31/24$$

The students may devise other sketches for finding the differences.







Fraction Operations via Area: Multiplication

O V E R V I E W

Two fractions are multiplied by viewing the fractions as the dimensions of a rectangle and their product as the area of that rectangle.

Prerequisite Activity

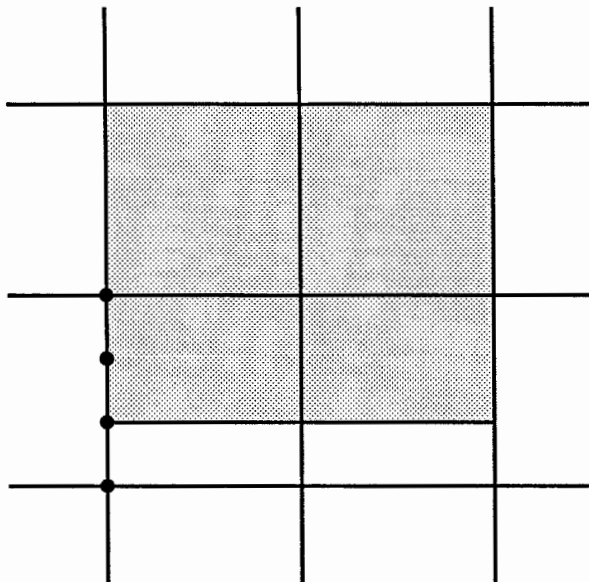
Unit IV/Activity 10, *Fractions Via Area: Addition and Subtraction*

Materials

A copy of Activity Sheet IV-11 and grid paper for each student

Actions

1. Distribute a copy of Activity Sheet IV-11 to each student. Discuss with the students how the area and dimensions of rectangle A can be determined.



Rectangle A

Comments

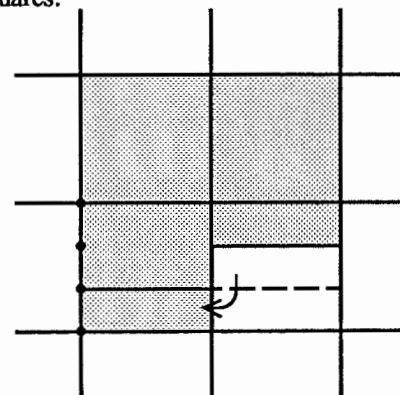
1. A master for Activity Sheet IV-11 is found at the end of this activity. The dots on the activity sheet are intended to help the students recognize divisions into thirds and fourths.

You may want to review the distinction between area measure and linear measure with the students. See Comments 2 and 3 of Unit III, Activity 6, *Number Piece Rectangles*, for a discussion of area and length.

In the comments that follow, the unit area is one square of the grid and the unit length is the length of an edge of this square.

The area of rectangle A can be found by either dissection or subdivision. (See comment 3, Unit IV, Activity 10, *Fraction Operations Via Area: Addition and Subtraction*.)

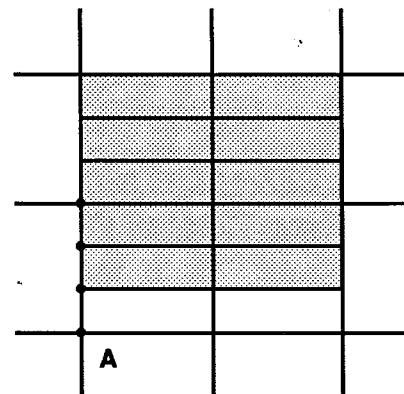
If rectangle A is dissected and reassembled as shown, one sees that its area is $3 \frac{1}{3}$ unit squares:



Area = $3 \frac{1}{3}$

Continued next page.

1. (*Continued.*) Alternately, A may be subdivided into 10 congruent subregions each of area $\frac{1}{3}$, as shown below. Hence the area of A is $\frac{10}{3}$.



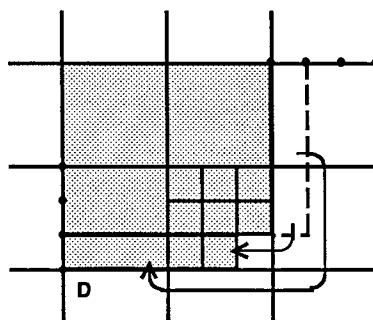
Area = $\frac{10}{3}$

For purposes of this activity, either form of the area, $3\frac{1}{3}$ or $\frac{10}{3}$, or any other equivalent form, is acceptable.

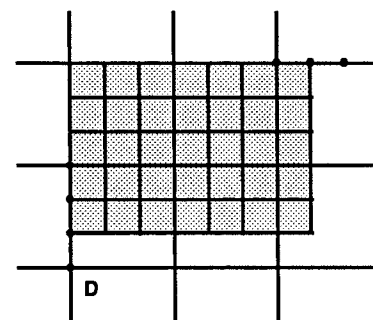
The dimensions of rectangle A are $1\frac{2}{3}$ and 2.

2. Some students may want to find the area of a rectangle by using a computational procedure to find the product of its dimensions. In this case, ask the students to see if they can find the areas without doing any arithmetic other than counting. Each of the areas can be readily found by either dissection or subdivision.

Depending on the methods used, the students may report the areas in varying forms. For example, a student who finds the area of rectangle D by dissection may report the area as $3\frac{8}{9}$, while a student who finds this area by subdivision may report it as $\frac{35}{9}$.



Area = $3\frac{8}{9}$

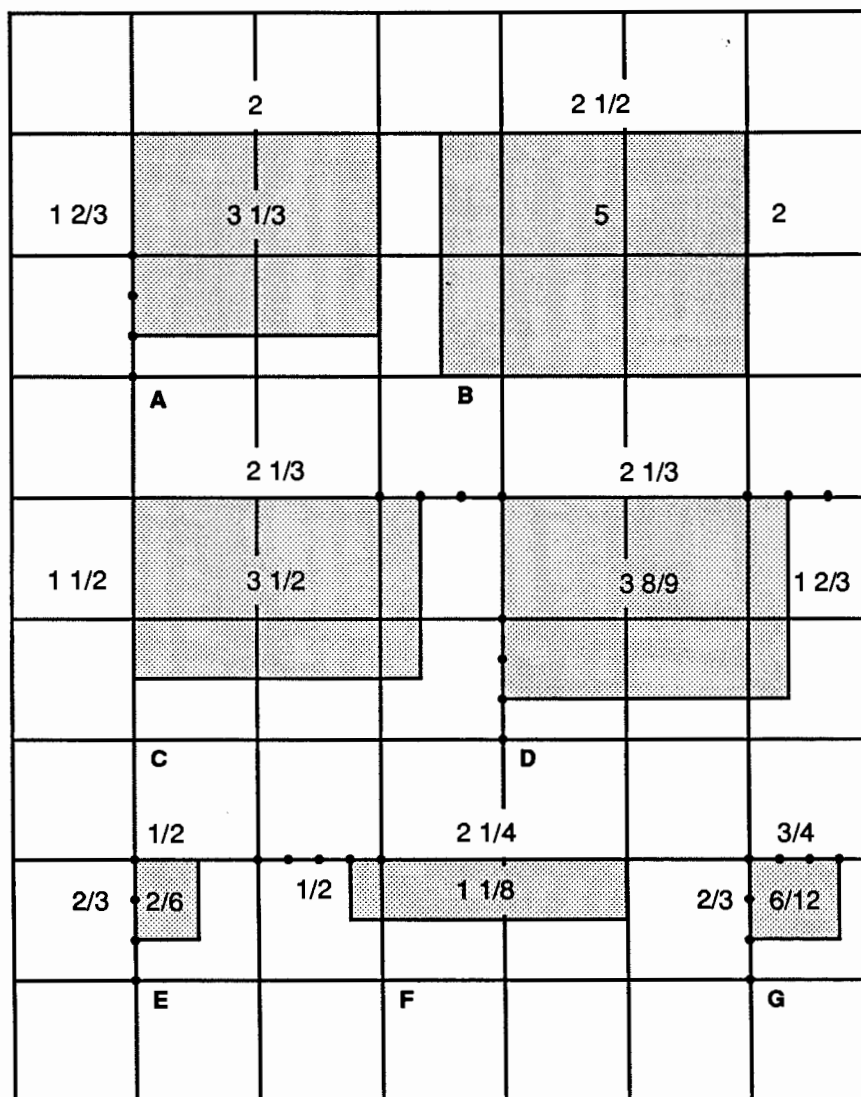


Area = $\frac{35}{9}$

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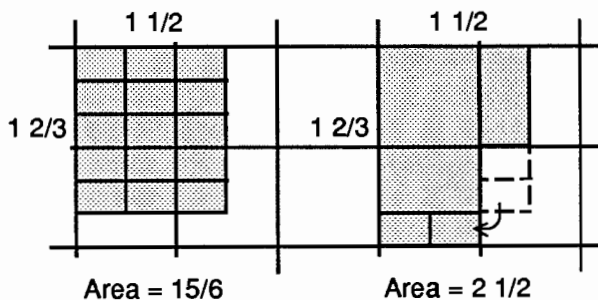
2. Ask the students to find the areas and dimensions of the remaining rectangles on the activity sheet.

2. (Continued.) The areas and dimensions of the rectangles on the activity sheet are shown below. Other forms of the answers are acceptable.



3. Distribute grid paper to each student. Ask the students to sketch a rectangle with dimensions $1\frac{2}{3}$ and $1\frac{1}{2}$. Then have the students find the area of this rectangle.

3. A master for grid paper is included as the last page of Unit IV, Activity 10, *Fraction Operations Via Area: Addition and Subtraction*.



The area of the rectangle can be found by subdividing it into 10 sub-regions of area $\frac{1}{3}$, or by dissection. Some students may use other methods.

Actions

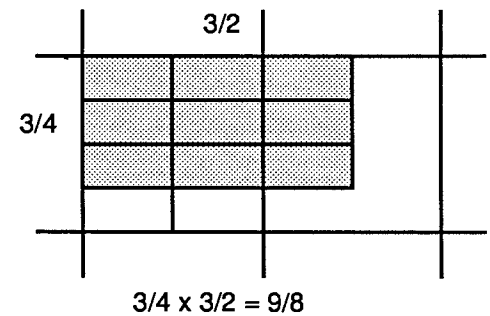
4. Discuss with the students the model for multiplication illustrated in Action 3.

5. Ask the students to find the product $\frac{3}{4} \times \frac{3}{2}$ by sketching a rectangle with dimensions $\frac{3}{4}$ and $\frac{3}{2}$ and finding its area, using no arithmetic other than counting.

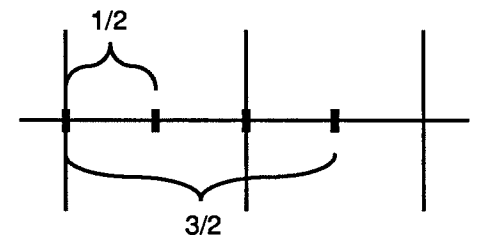
Comments

4. The product of two numbers can be thought of as the area of a rectangle which has these two numbers as its dimensions. In Action 3, it is found that a rectangle with dimensions $1\frac{2}{3}$ and $1\frac{1}{2}$ has an area of $2\frac{1}{2}$. Hence $1\frac{2}{3} \times 1\frac{1}{2} = 2\frac{1}{2}$.

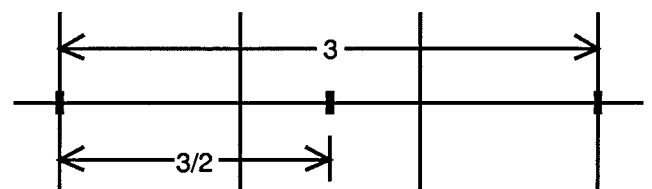
5. The area of the rectangle can be found by subdivision:



If the students have difficulty sketching a rectangle with the given dimensions, you may want to discuss with them how line segments of lengths $\frac{3}{4}$ and $\frac{3}{2}$ can be obtained. For example, a line segment of length $\frac{3}{2}$ can be obtained by thinking of $\frac{3}{2}$ as 'three halves', i.e. $\frac{1}{2} + \frac{1}{2} + \frac{1}{2}$, and marking off 3 segments of length $\frac{1}{2}$ end-to-end:



Alternately, a line segment of length $\frac{3}{2}$ can be obtained by thinking of $\frac{3}{2}$ as 'three divided by two' and dividing a line segment of length 3 into two equal parts:



Actions

6. Ask the students to use the method of Action 5 to find the following products:

(a) $2/3 \times 4/3$

(b) $7/4 \times 5/3$

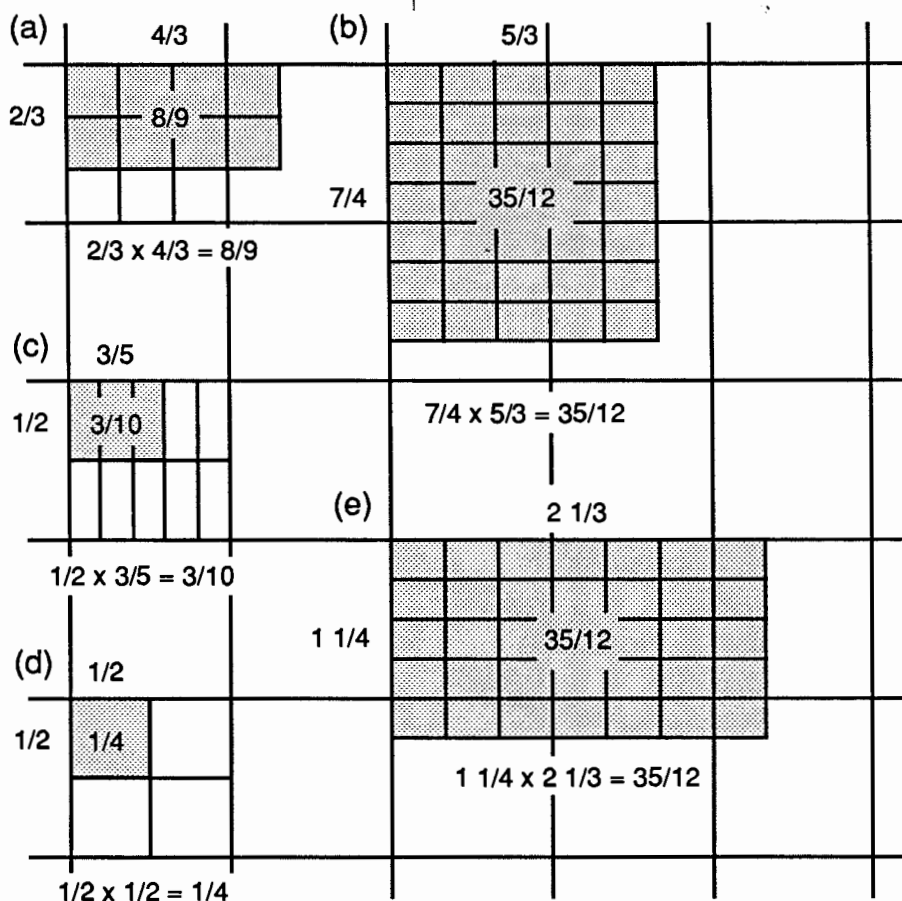
(c) $1/2 \times 3/5$

(d) $1/2 \times 1/2$

(e) $1 \frac{1}{4} \times 2 \frac{1}{3}$

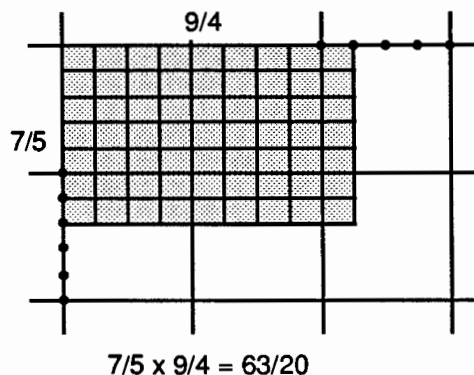
Comments

6. In the sketches shown below, the areas were found by the subdivision method. If students find areas by some other method, they may report their answers in different forms.



7. (Optional) Discuss with the students a general procedure for multiplying fractions.

7. A procedure for multiplying fractions can be based on the area subdivision method used in Actions 5 and 6.



One way to begin a discussion is to show the students a sketch like the one at left showing that $7/5 \times 9/4 = 63/20$.

Ask the students to make observations relating the numerators and denominators of the fractions to features of the sketch. Here are some possible observations:

- The numerators, 7 and 9, indicate the number of segments into which the sides of the rectangle are divided, that is, one side of the rectangle is divided into 7 equal segments and an adjacent side is divided into 9 equal segments.
- The number of subregions in the rectangle is the product of the numerators.
- The denominators indicate the number of segments into which sides of a unit square are divided, that is, one side of a unit square is divided into 5 equal segments and an adjacent side is divided into 4 equal segments.
- The number of subregions in a unit square is the product of the denominators.
- Since there are 20 subregions in a unit square, the area of each subregion is $1/20$.
- The area of the rectangle is $63/20$ since it has 63 subregions each of area $1/20$.

Combining these observations, one sees:

(a) the product of the numerators = the total number of subregions = the numerator of the area,

(b) the product of the denominators = the number of subregions in a unit square = the denominator of the area.

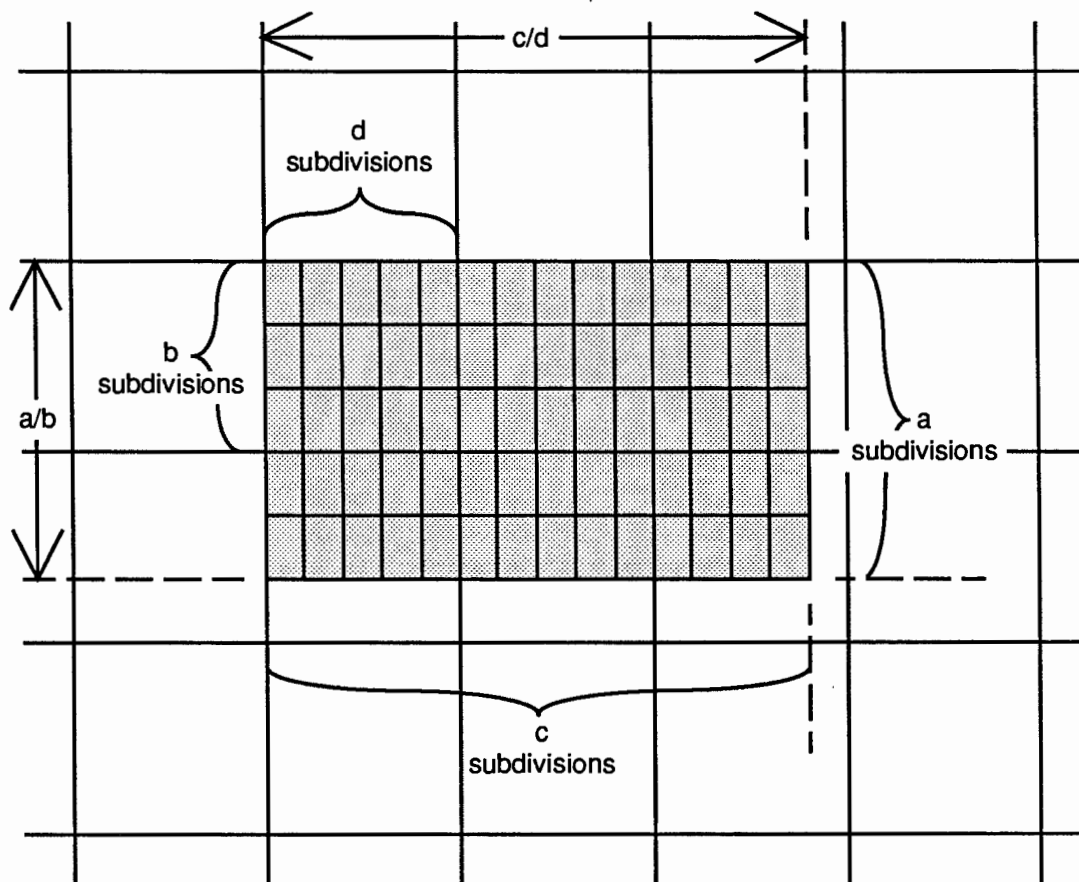
That is, $7/5 \times 9/4 = (7 \times 9)/(5 \times 4) = 63/20$.

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7. (Continued.) Some students may be able to develop a general formula for the product of two fractions. Here is one way to do this:

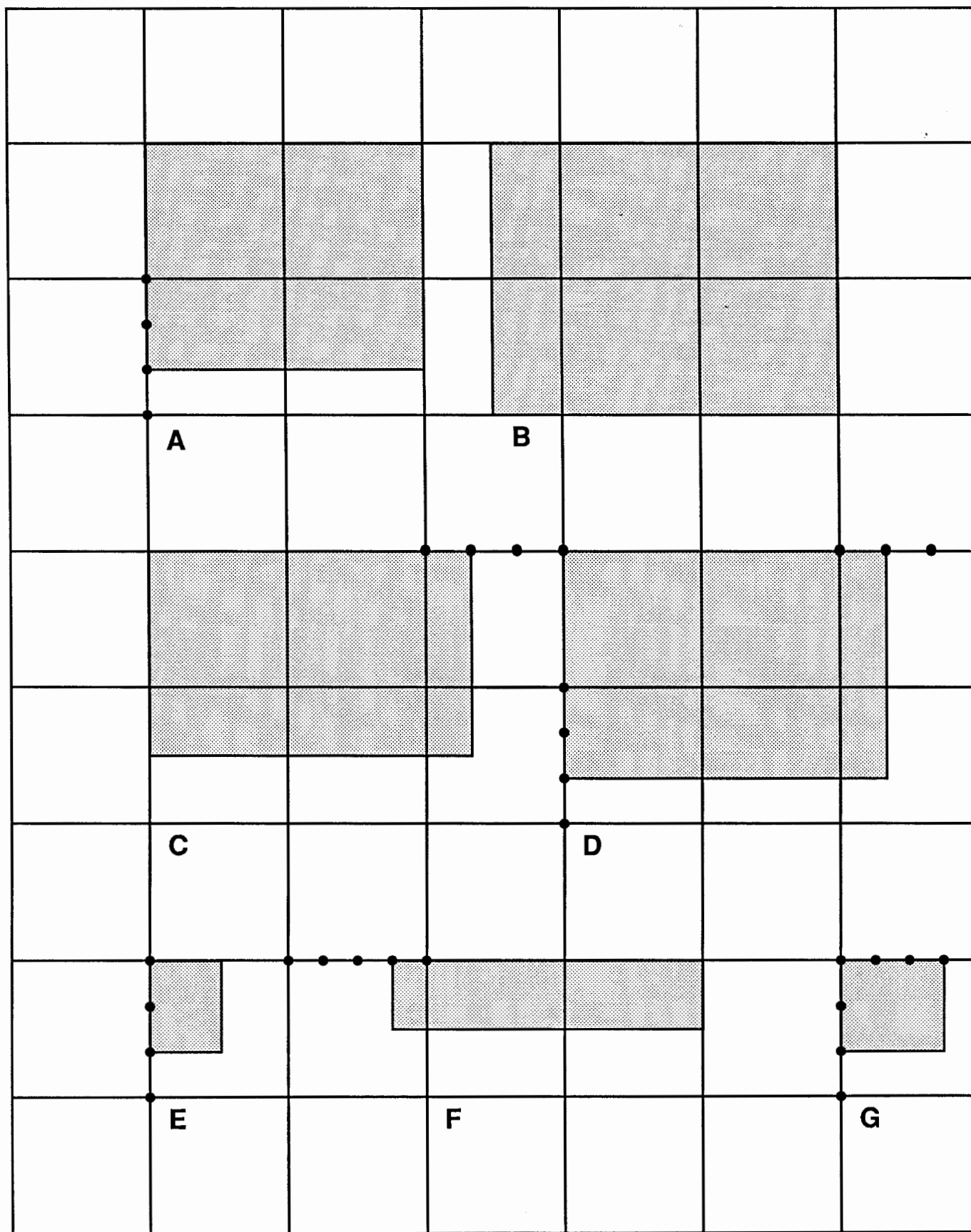
Given any two fractions, a/b and c/d , a rectangle can be constructed with these fractions as dimensions. If this rectangle is subdivided as shown, there will be $a \times c$ subregions in the entire rectangle and $b \times d$ subregions in one unit square. Thus each subregion has area $1/(b \times d)$ and the area of the rectangle is $(a \times c)/(b \times d)$. Since the area of a rectangle is the product of its dimensions,

$$(a/b) \times (c/d) = (a \times c)/(b \times d).$$



$b \times d$ subregions in one unit square; area of one subregion = $1/(b \times d)$

$a \times c$ subregions in rectangle; area of $a \times c$ subregions = $(a \times c)/(b \times d)$



Fraction Operations via Area: Division

O V E R V I E W

One fraction is divided by a second fraction by viewing the first fraction as the area of a rectangle and the second fraction as one of the rectangle's dimensions. The other dimension is the desired quotient.

Prerequisite Activity

Unit IV, Activity 1, *Fraction Operations via Area: Multiplication*

Materials

Grid paper

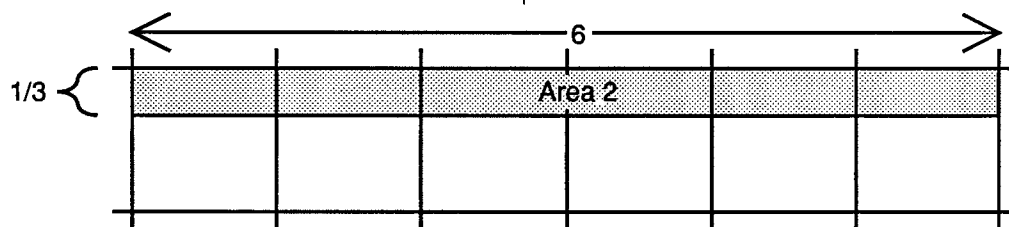
Actions

1. Distribute grid paper to each student. Ask the students to construct a rectangle that satisfies the following conditions: its area is 2 and one of its dimensions is $\frac{1}{3}$. Ask them to determine the other dimension of the rectangle. Repeat these directions for a rectangle with area 2 and one dimension $\frac{2}{3}$. Discuss the methods the students use.

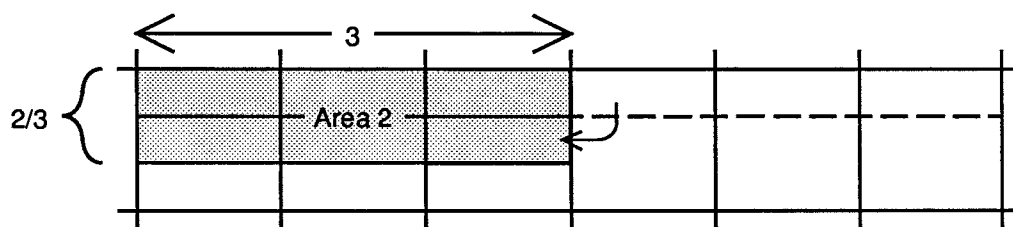
Comments

1. A master for grid paper can be found at the end of Unit IV, Activity 10, *Fraction Operations via Area: Addition and Subtraction*.

Six rectangles of area $\frac{1}{3}$ placed end to end form a rectangle with area 2 and one dimension $\frac{1}{3}$. Its other dimension is 6.



Dividing the $\frac{1}{3} \times 6$ rectangle in two and rearranging produces a $\frac{2}{3} \times 3$ rectangle of area 2:



The students may use other methods to construct these rectangles.

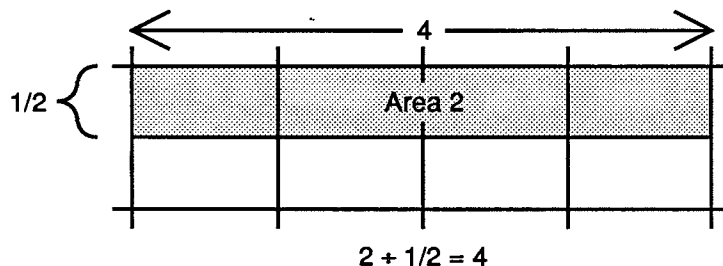
Actions

2. Discuss the area model of division with the students.
Ask them to use this model of division to find the quotients $2 \div \frac{1}{2}$ and $\frac{4}{3} \div \frac{1}{2}$.

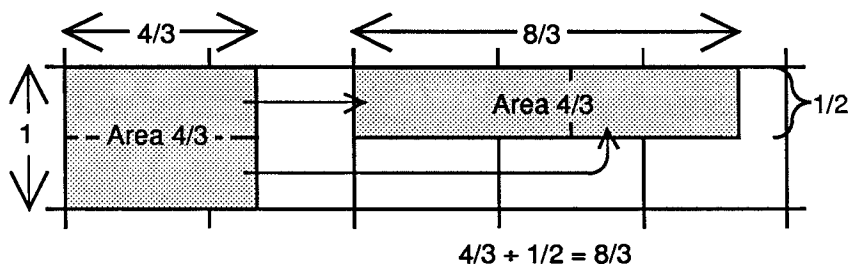
Comments

2. The area model for division, like that for multiplication, is based on the assumption that the area of a rectangle is the product of its dimensions. Hence, dividing the area by one dimension gives the other dimension. (See Comment 5, Unit II, Activity 1, *Basic Operations*.)

To use this model to find $2 \div \frac{1}{2}$, a rectangle is constructed that has area 2 and one dimension $\frac{1}{2}$. The other dimension is the desired quotient.



There are a number of ways to construct a rectangle of area $\frac{4}{3}$ and a dimension of $\frac{1}{2}$. One way is to first construct a $1 \times \frac{4}{3}$ rectangle. This rectangle has area $\frac{4}{3}$. If it is cut in half lengthwise and the two halves rearranged as shown, the resulting rectangle has area $\frac{4}{3}$ and one dimension $\frac{1}{2}$. Its other dimension is $2 \times \frac{4}{3}$ or $\frac{8}{3}$.



3. Ask the students to find the following quotients:

(a) $3 \div 3/4$

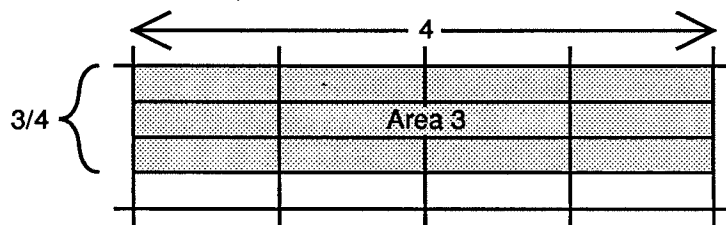
(b) $1/2 \div 2$

(c) $5/3 \div 2/3$

(d) $3/2 \div 1/3$

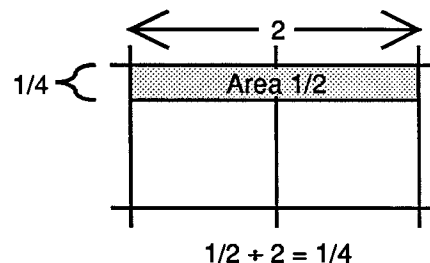
3. The students may use a variety of methods to construct appropriate rectangles and may report their answers in various forms. In this activity, there is no need for answers to be reported in a particular form.

(a) The shaded rectangle shown below has one dimension $3/4$. It has area 3 since it is composed of 12 quarter-units of area. Its other dimension is 4.



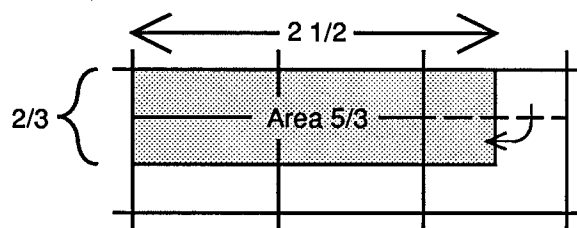
$$3 \div 3/4 = 4$$

(b)



$$1/2 \div 2 = 1/4$$

(c)



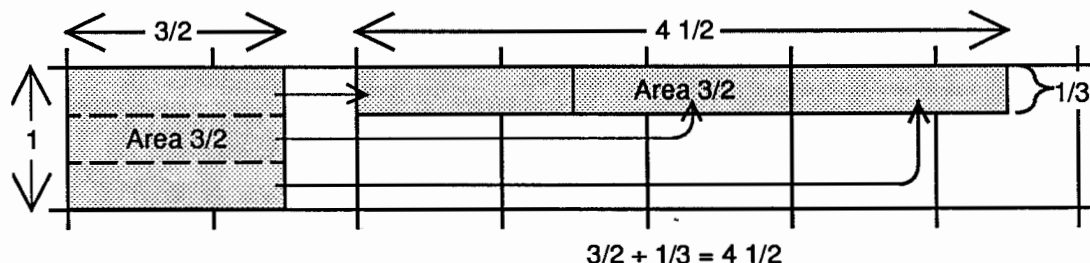
$$5/3 \div 2/3 = 2 \frac{1}{2}$$

The shaded rectangle is composed of 5 regions, each of area $1/3$.

Continued next page.

3. (Continued.)

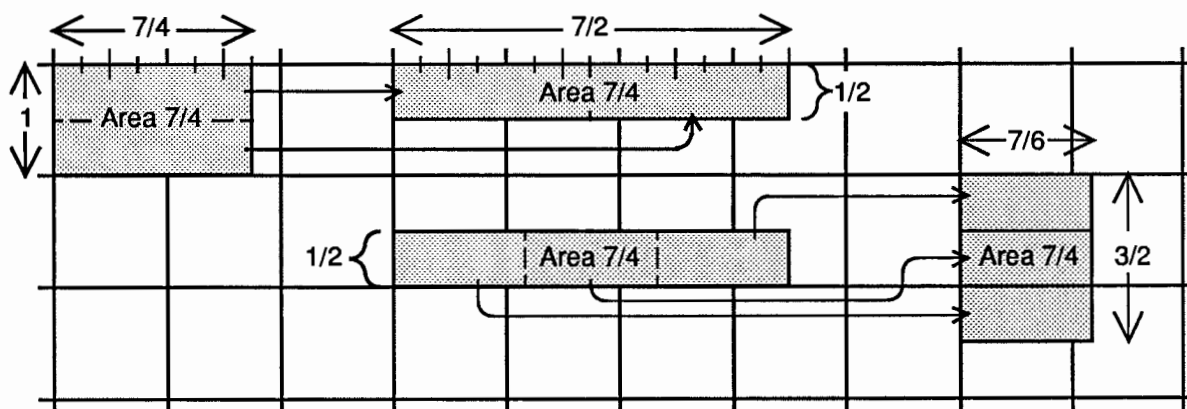
(d) A $1 \times \frac{3}{2}$ rectangle has area $\frac{3}{2}$. If it is divided into thirds and the parts rearranged as shown, the result is a rectangle which has area $\frac{3}{2}$ and one dimension $\frac{1}{3}$. The other dimension is $3 \times 1\frac{1}{2}$ or $4\frac{1}{2}$.



4. (Optional) Ask the students to construct a rectangle whose area is $\frac{7}{4}$ and has $\frac{3}{2}$ as one dimension. Then ask them to determine the other dimension of the rectangle.

4. The students will use various methods to construct the rectangle.

One method is illustrated in the sequence shown below. A $1 \times \frac{7}{4}$ rectangle is cut in half and the two parts rearranged to obtain a rectangle of area $\frac{7}{4}$ and one dimension $\frac{7}{2}$. A copy of this rectangle is then divided into thirds and the three parts rearranged to obtain the desired rectangle.

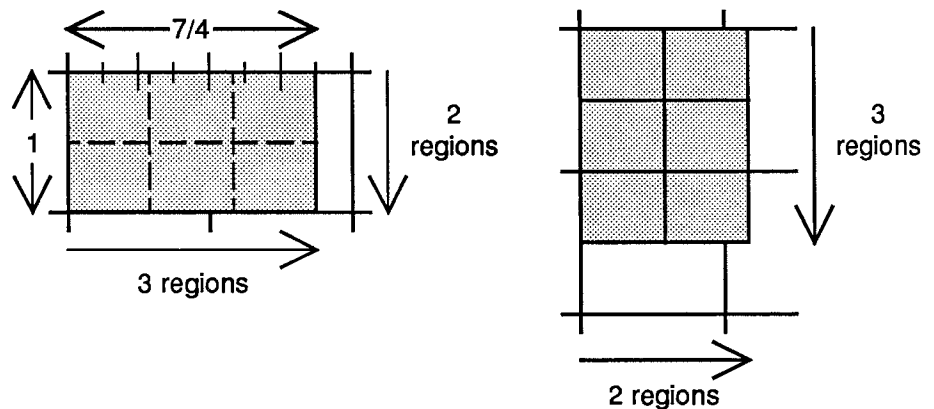


The horizontal dimension of the final rectangle is a third of the horizontal dimension of the middle rectangle, that is, it is a third of 7 halves. Dividing each of these 7 halves into 3 equal parts converts them into 21 sixths. Hence a third of 7 halves is the same as a third of 21 sixths, or 7 sixths. Thus the other dimension of the desired rectangle is $\frac{7}{6}$.

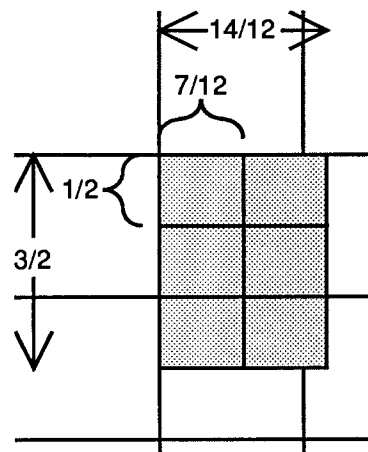
Continued next page.

4. (Continued.) The above method can be summarized as follows:
1. Construct a $1 \times 7/4$ rectangle.
 2. Divide this rectangle horizontally into halves and rearrange.
 3. Divide resulting rectangle vertically into thirds and rearrange.

Omitting the rearrangement in step 2 leads to another method for constructing the rectangle. In this method, a $1 \times 7/4$ rectangle is divided into halves horizontally and thirds vertically. This subdivides the rectangle into a 2×3 array of small rectangular regions. These regions are rearranged so that the first row of regions becomes the first column and the second row becomes the second column. The resulting 3×2 array of small regions is the desired rectangle.



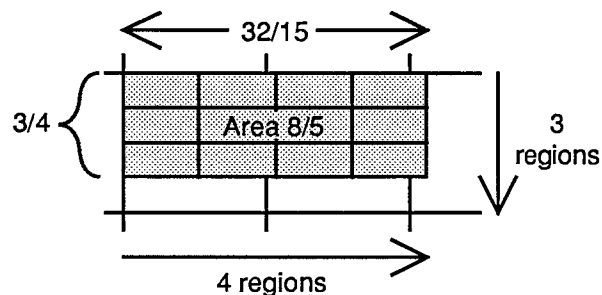
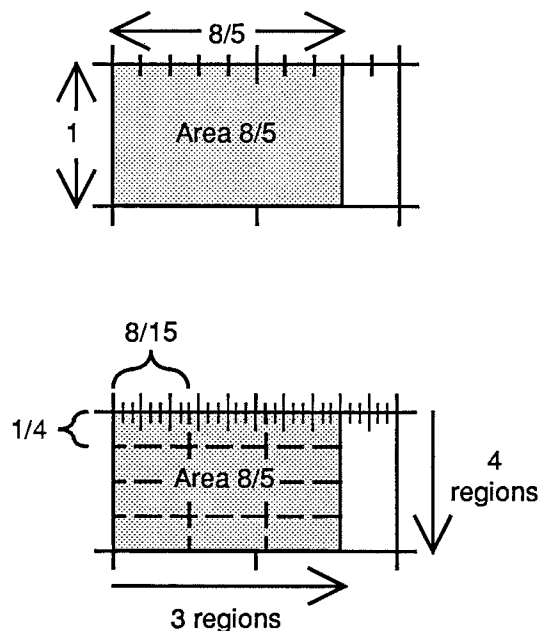
The dimensions of the final rectangle can be found from the dimensions of one of the small rectangular regions. Since the vertical dimension of one of the small rectangles is $1/2$, the vertical dimension of the final rectangle is $3 \times 1/2$ or $3/2$.



The horizontal dimension of each small rectangle is a third of 7 fourths or 7 twelfths. (Dividing each of the 7 fourths into 3 equal parts, converts the 7 fourths into 21 twelfths. A third of 21 twelfths is 7 twelfths.) Hence the horizontal dimension of the final rectangle is $2 \times 7/12$ or $14/12$.

Actions

5. (Optional) Discuss with the students a general procedure for constructing a rectangle which has a given fraction as area and a given fraction as one dimension. From this develop a general method for dividing fractions.



The vertical dimension of a small rectangle is $1/4$ and the horizontal dimension is $8/15$. Hence the dimensions of the final rectangle are $3 \times 1/4$, or $3/4$, and $4 \times 8/15$, or $32/15$. Thus, $32/15$ is the other dimension of a rectangle that has area $8/5$ and one dimension $3/4$. That is

$$8/5 \div 3/4 = 32/15.$$

Continued next page.

Comments

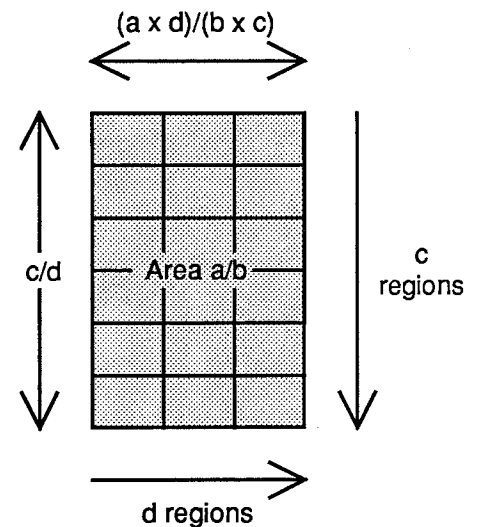
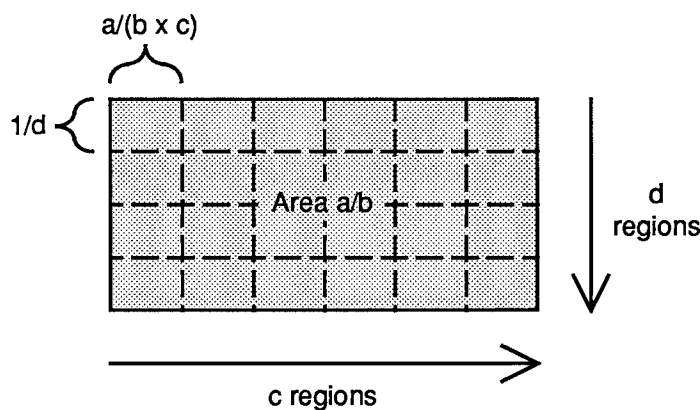
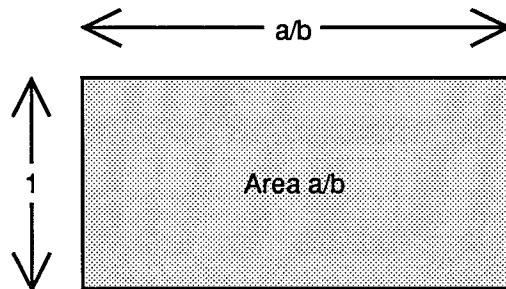
5. There are various methods for constructing a rectangle given its area and one dimension. One procedure is suggested by the second method discussed in Comment 4. Its use is illustrated here to construct a rectangle of area $8/5$ with one dimension $3/4$.

First, construct a $1 \times 8/5$ rectangle. Divide this rectangle into 4 equal parts horizontally and 3 equal parts vertically, giving a 4×3 array of small rectangles. Rearrange this 4×3 array of small rectangles into a 3×4 array:

5. (Continued.) In general, to find $a/b + c/d$, construct a rectangle of area a/b and one dimension c/d . The other dimension will be the desired quotient.

To construct the rectangle, begin with a $1 \times a/b$ rectangle. Divide this rectangle into d equal parts horizontally and c equal parts vertically. Then rearrange this $d \times c$ array of small rectangles into a $c \times d$ array. The result is a rectangle with area a/b and one dimension c/d . The other dimension is $(a \times d)/(b \times c)$. That is,

$$a/b + c/d = (a \times d)/(b \times c).$$

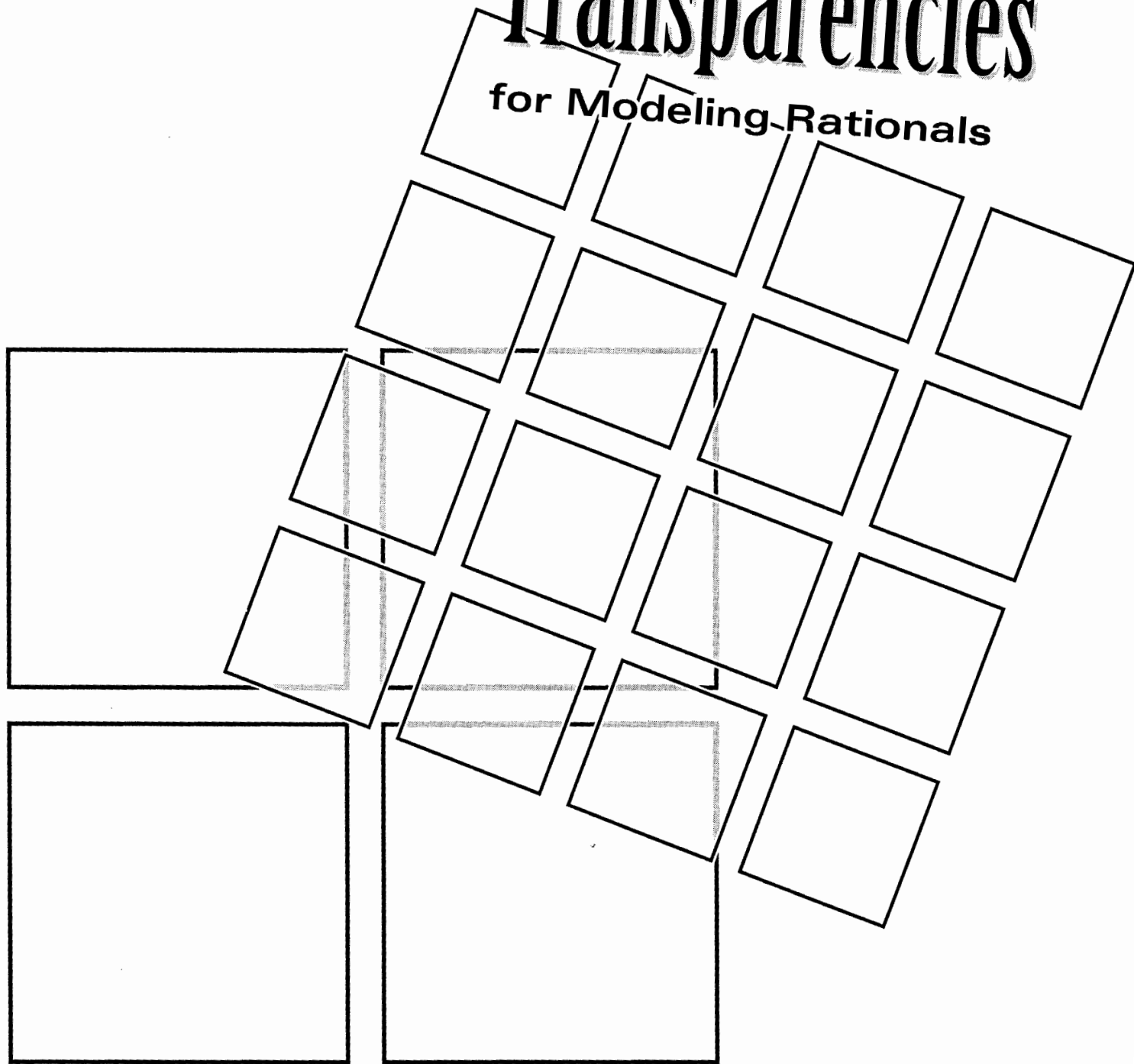


$$a/b + c/d = (a \times d)/(b \times c)$$

Unit IV / Math and the Mind's Eye

Transparencies

for Modeling Rationals



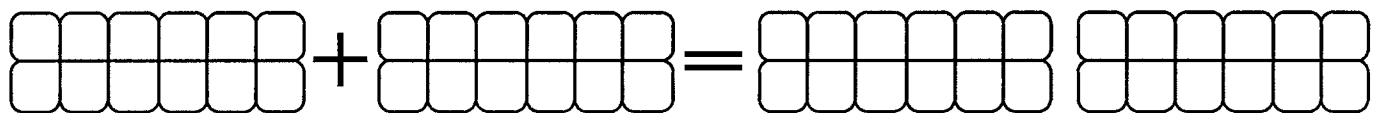
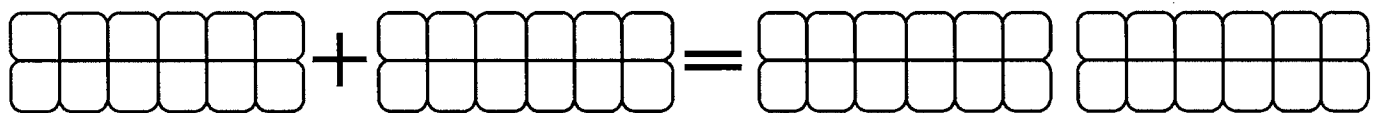
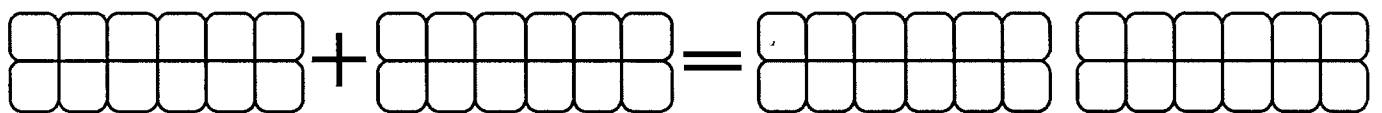
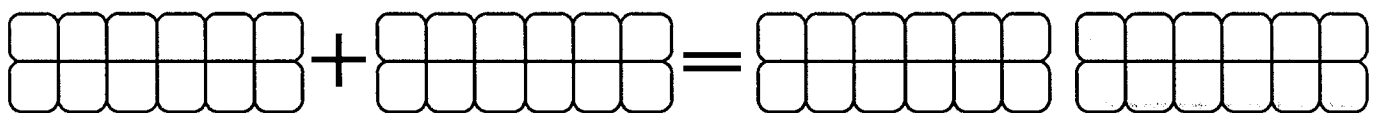
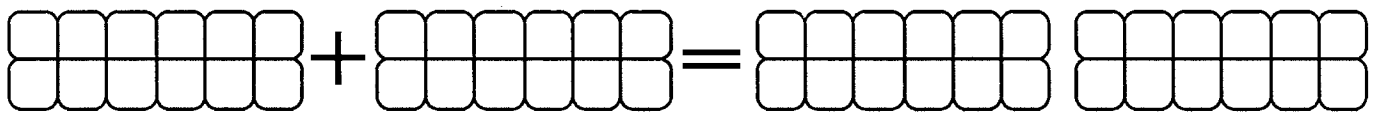
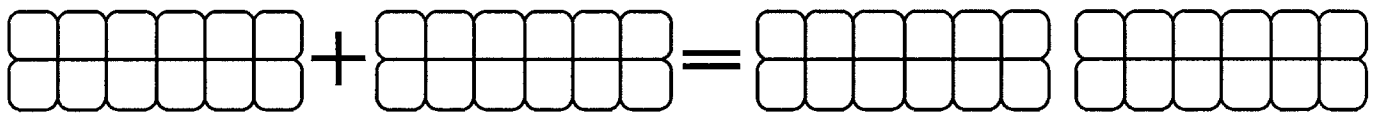
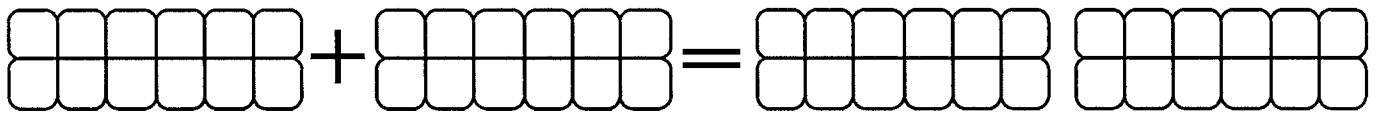
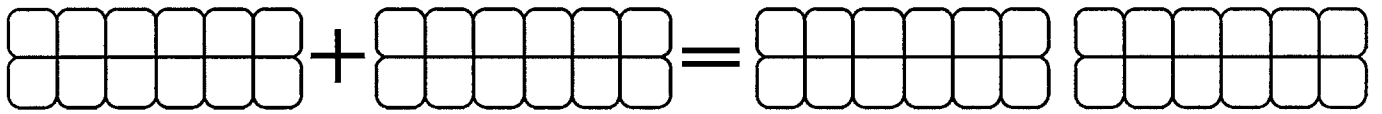
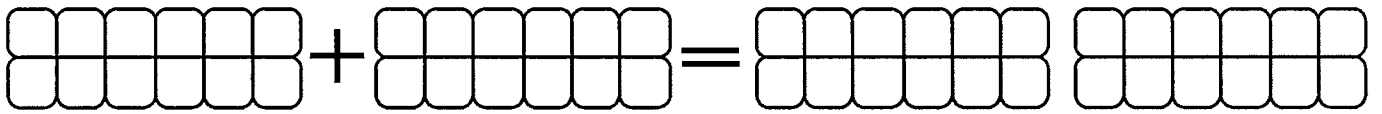
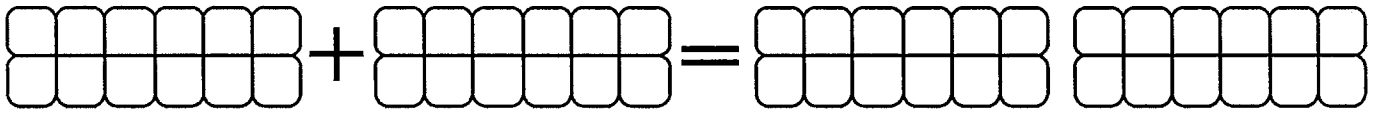
The Math Learning Center
PO Box 3226
Salem, Oregon 97302

Catalog #MET4

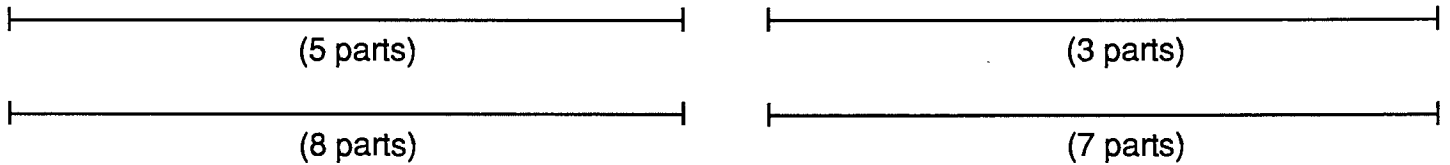
Egg Carton Recording Paper

A large grid consisting of 10 rows and 3 columns. Each of the 30 cells in this grid is an empty 2x6 grid, designed for students to write multiplication problems and show their work.

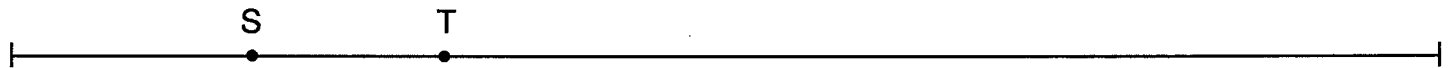
Egg Carton Addition Paper



1. Use the parallel line sheet to divide each segment into the indicated number of parts.



2. Locate points to the right of T and to the left of S so that distance between adjacent points is the same as ST.



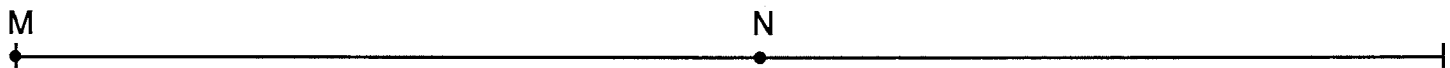
3. If the distance from X to Y is 1 unit, what is the distance from X to Z?



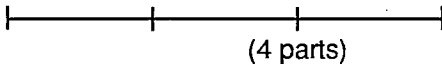
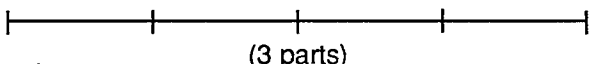
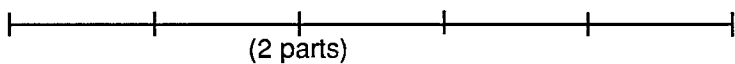
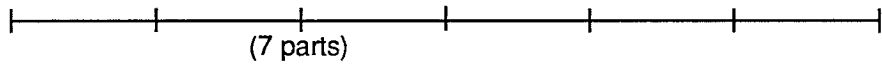
4. If the distance from A to B is 7 units, locate a point P which is 5 units from A.



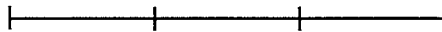
5. If MN is 3 units, find point Q so that MQ is 5 units.





1. Use the parallel line sheet to divide each segment into the indicated number of parts. Then

		Length of One Part
3 units		a. <input type="text"/>
4 units		b. <input type="text"/>
5 units		c. <input type="text"/>
6 units		d. <input type="text"/>

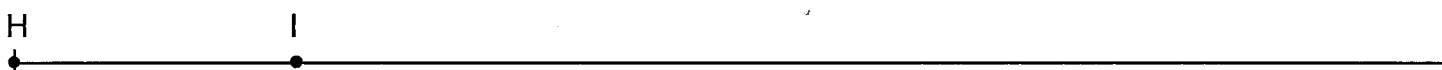
2. Use the parallel lines to locate the indicated fraction on the given number line.

3 units  $\frac{3}{5}$

5 units  $\frac{5}{2}$

7 units  $\frac{7}{10}$

3. HI is $\frac{1}{4}$ of a unit. Find point J so that HJ is 1 unit.



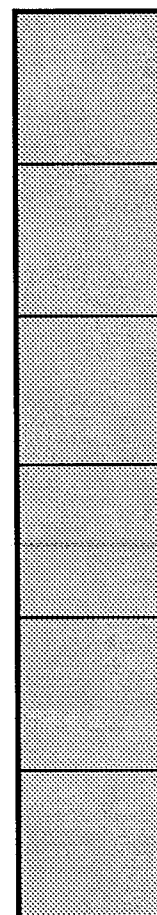
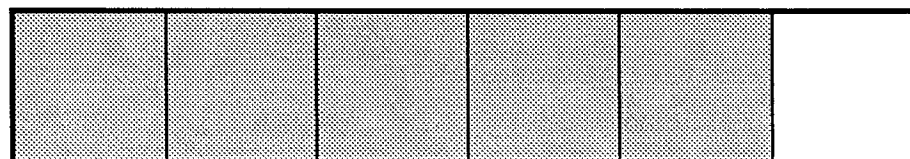
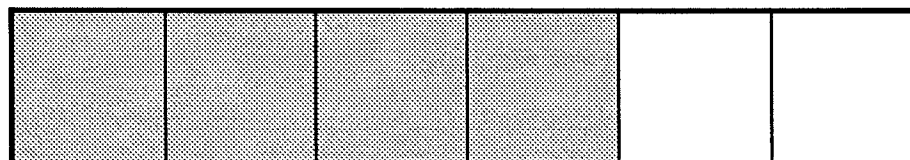
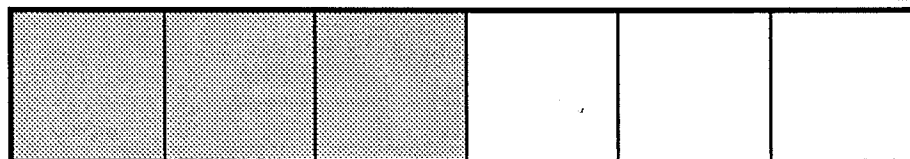
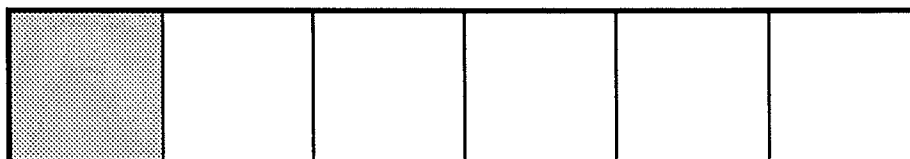
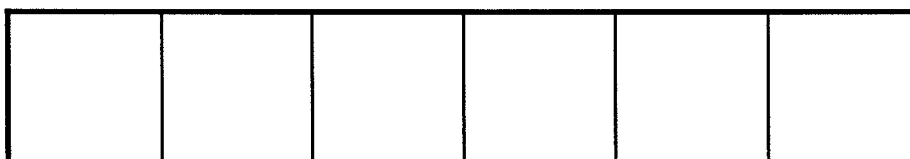
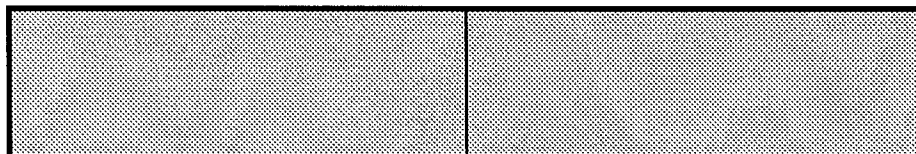
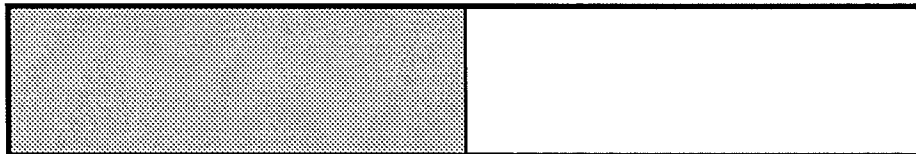
4. UV is $\frac{3}{5}$ of a unit. Find point M so that UM is 3 units.



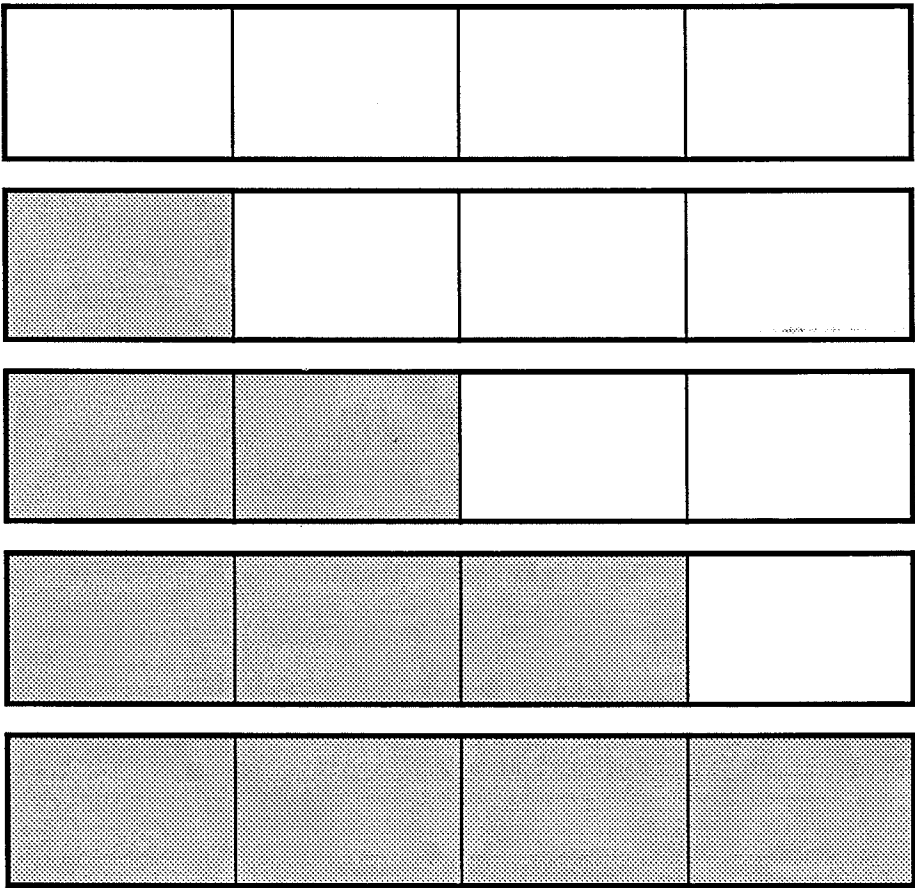
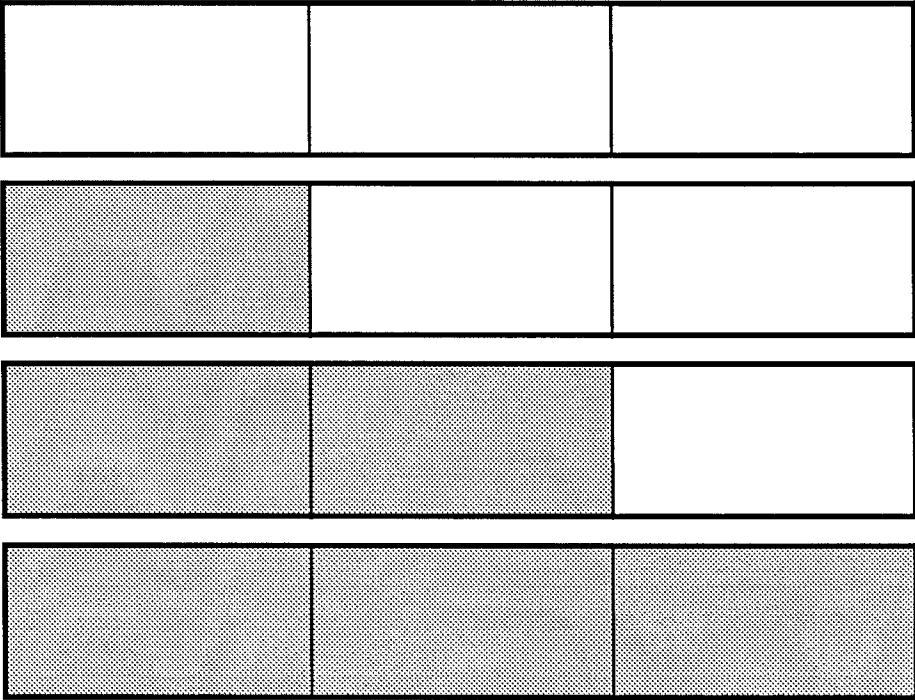
5. UV is $\frac{3}{5}$ units. Find point W so that UW is 1 unit.



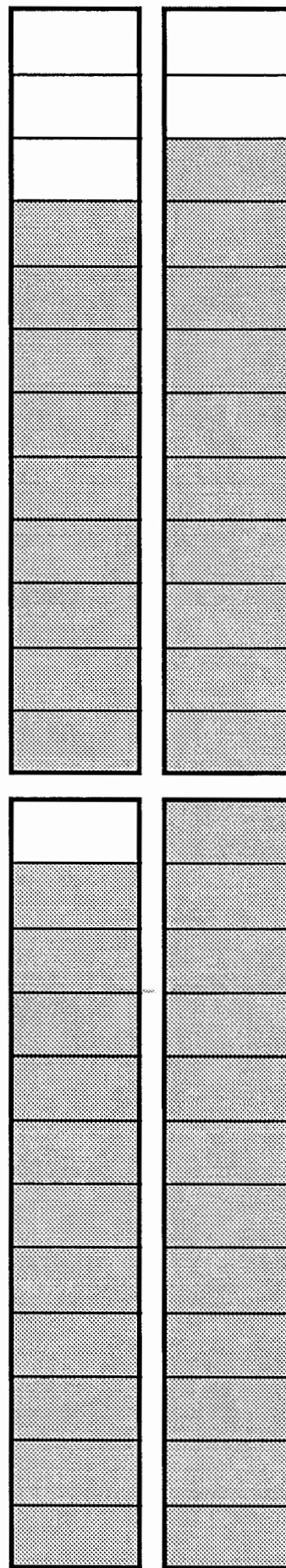
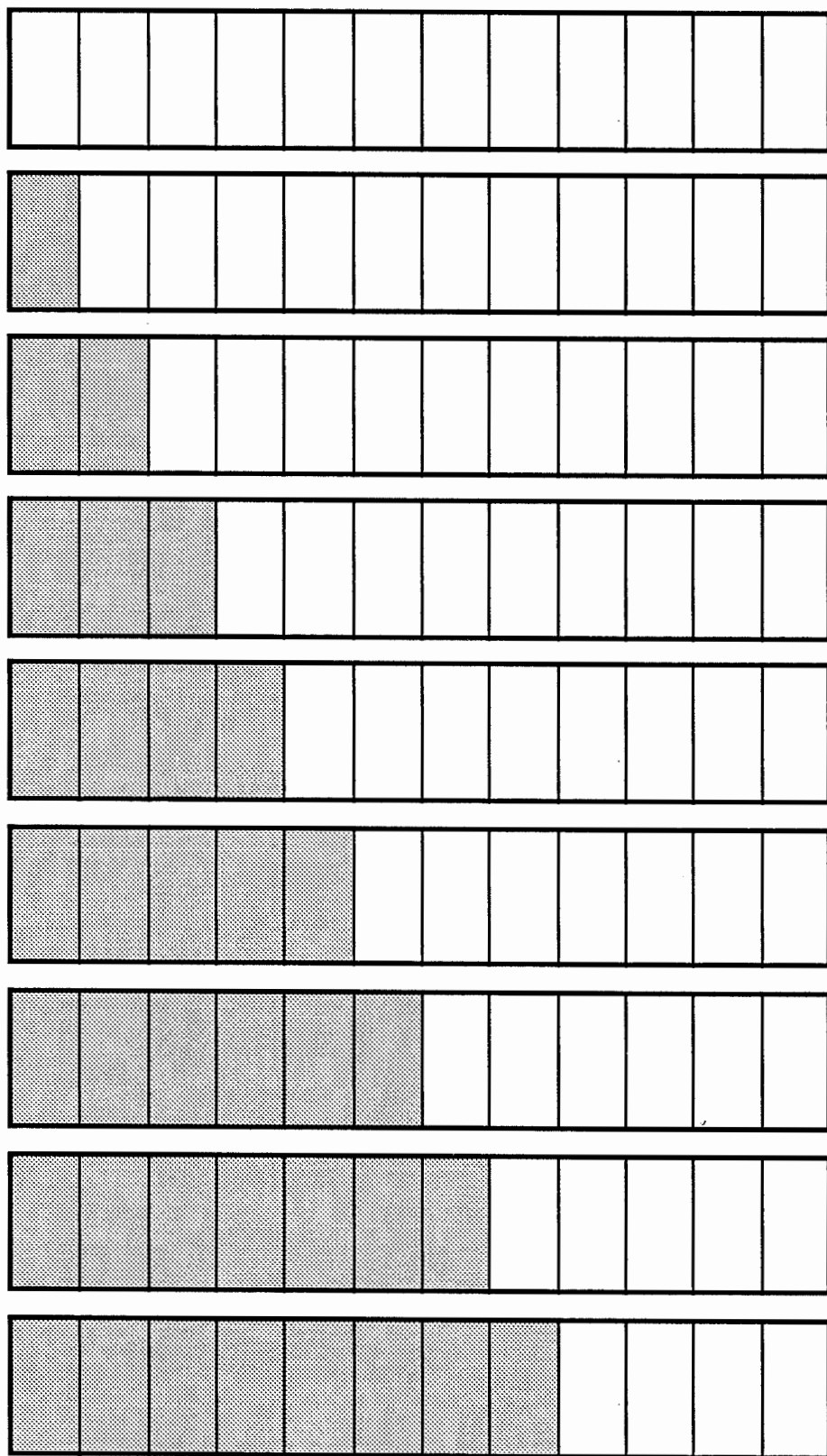
Fraction Bars for Halves and Sixths

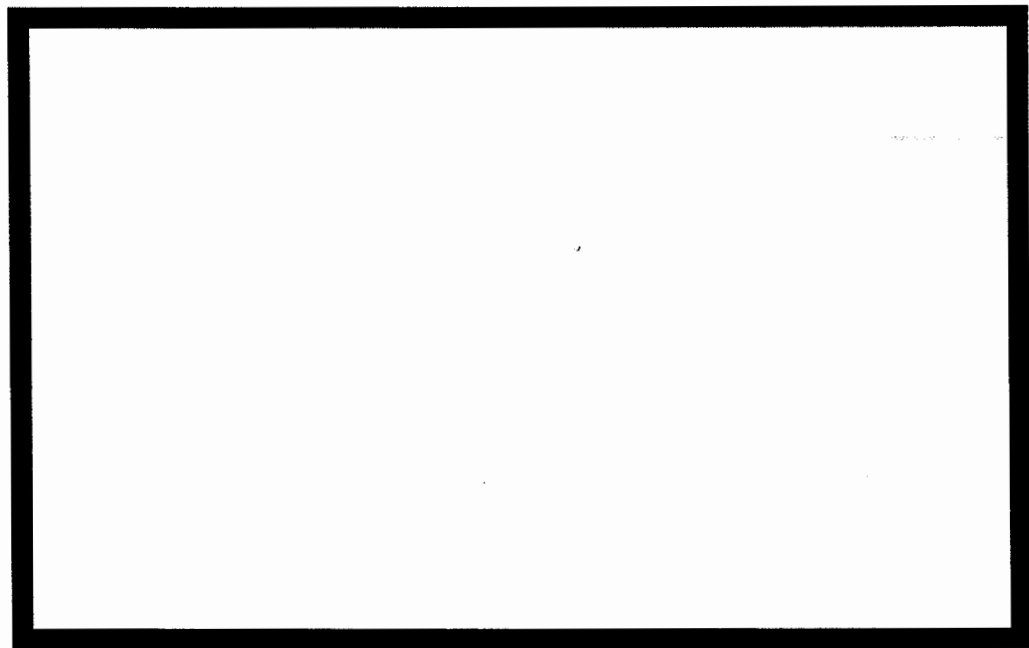
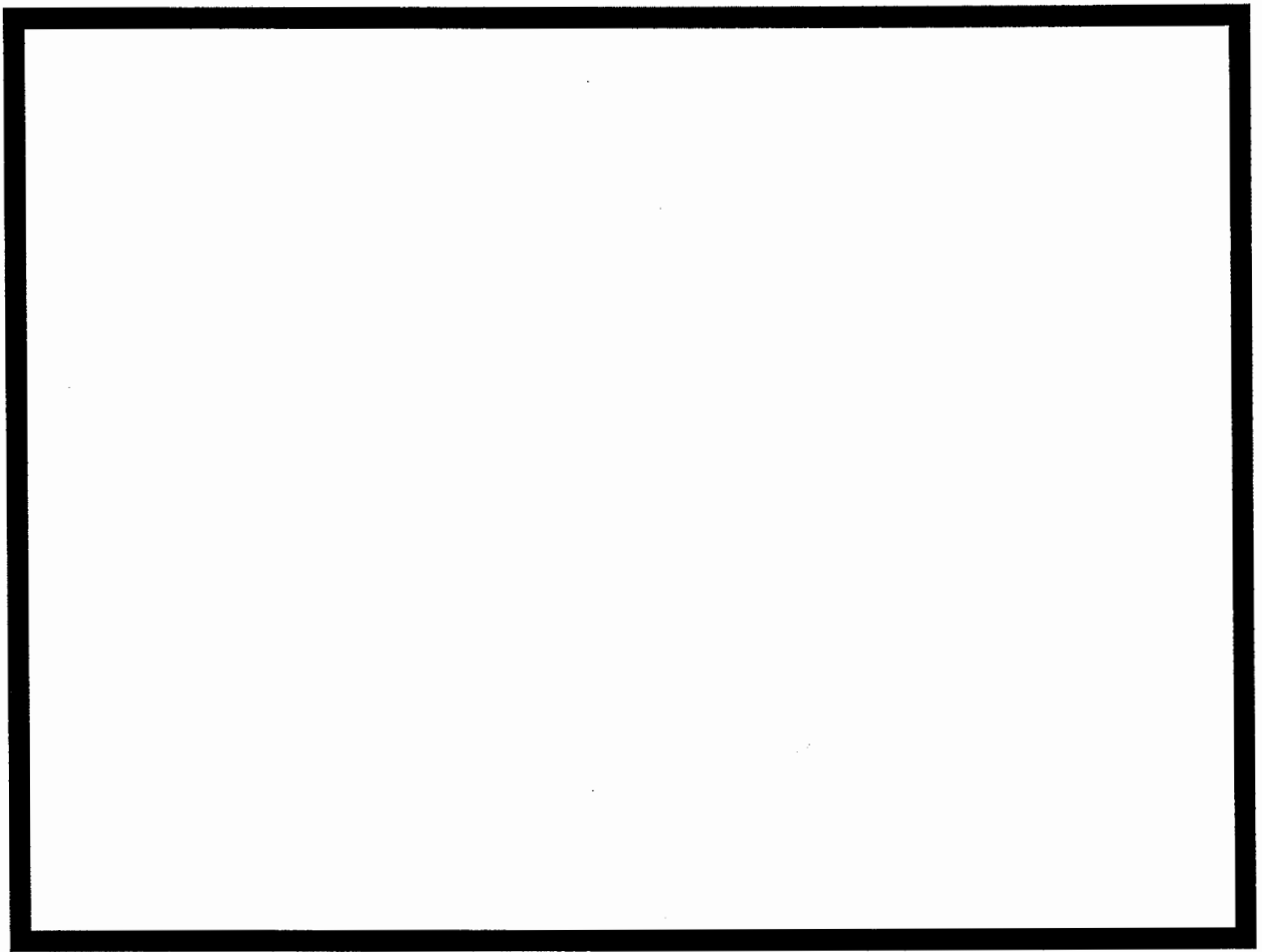


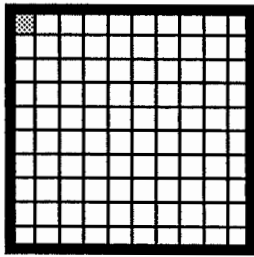
Fraction Bars for Thirds and Fourths



Fraction Bars for Twelfths

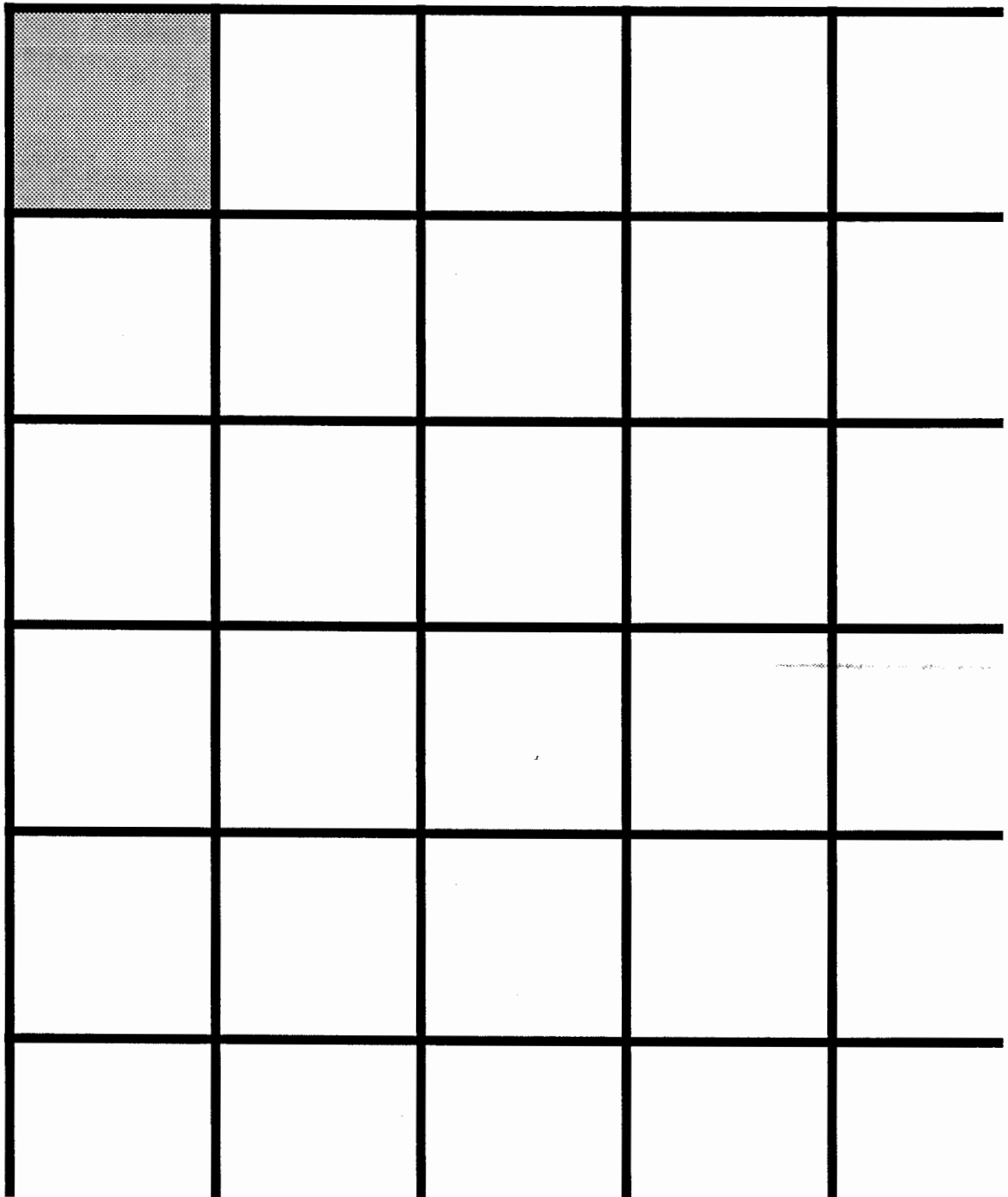


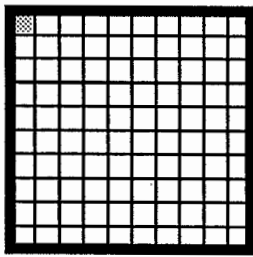




**Decimal unit square
divided into hundredth squares**

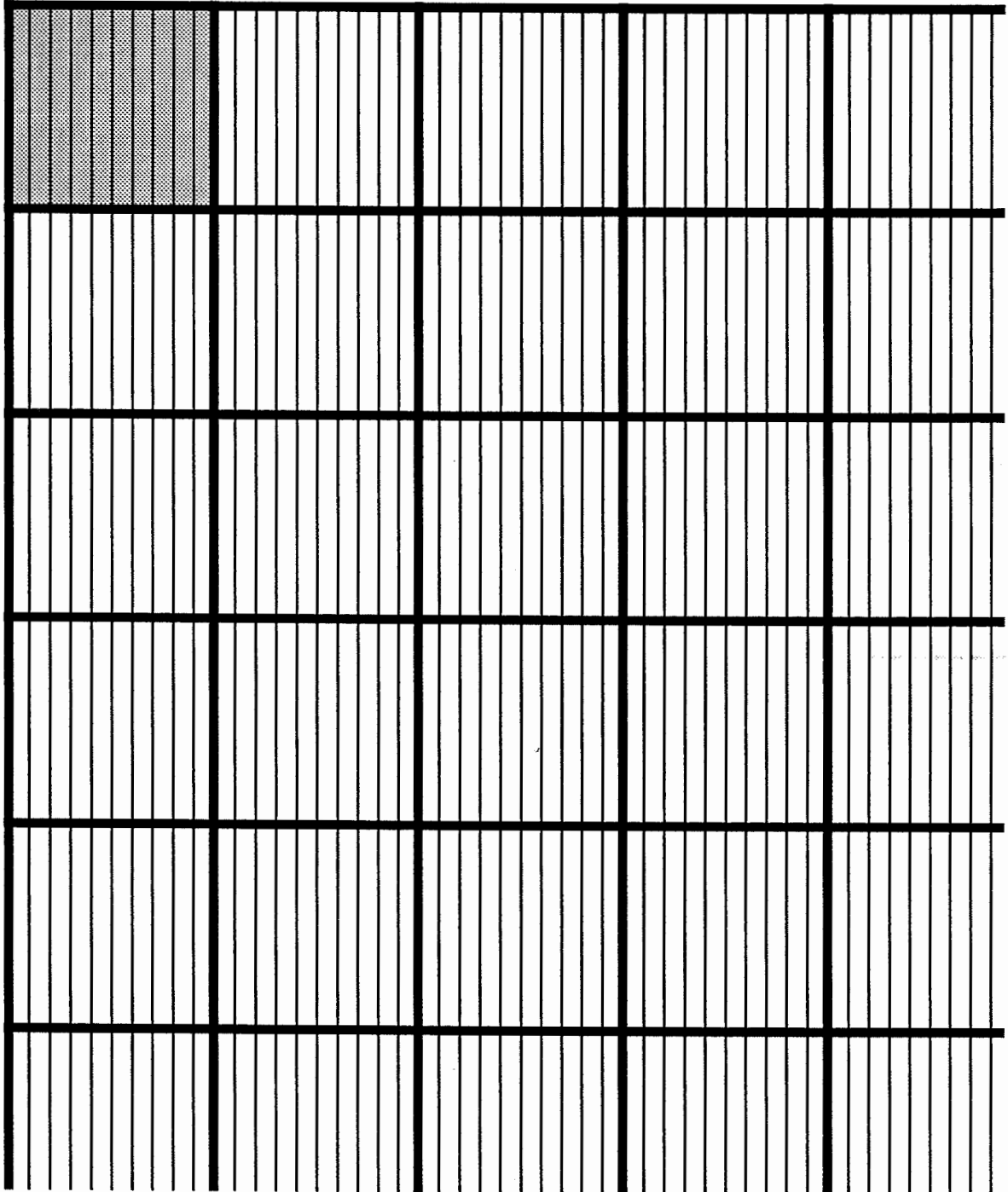
Magnified decimal unit square divided into hundredth squares

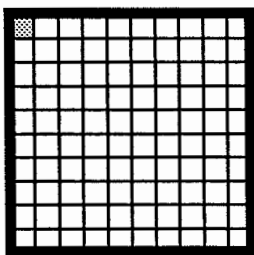




Decimal unit square
divided into hundredth squares

Magnified decimal unit square divided into thousandth strips





Decimal unit square
divided into hundredth squares

Magnified decimal unit square divided into ten-thousandth squares

