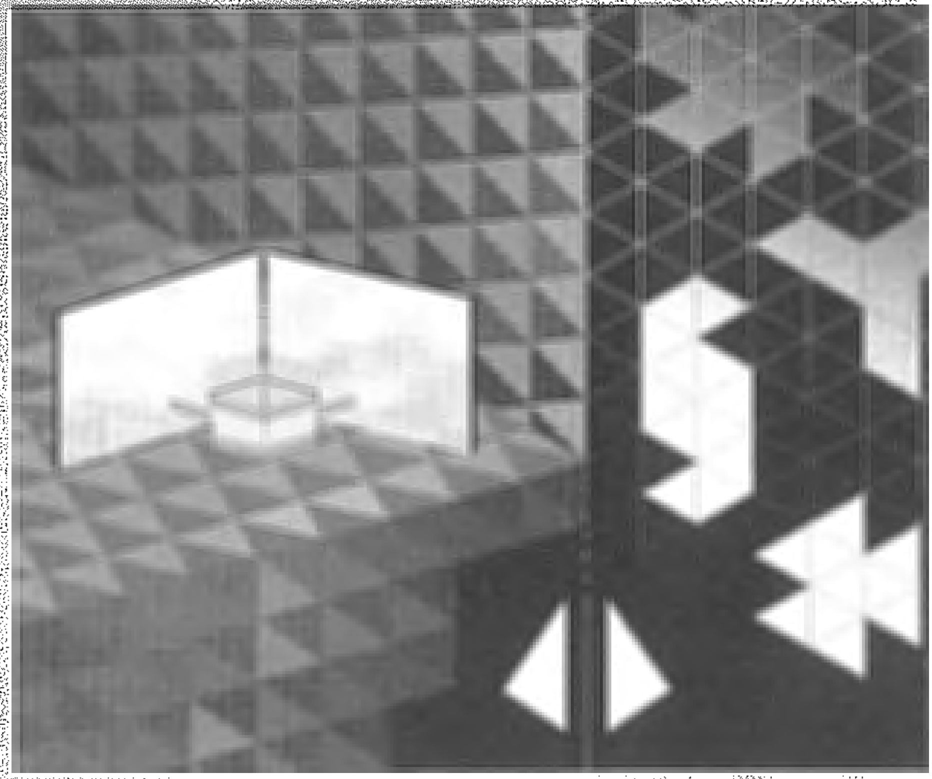
Unit X / Math and the Mind's Eye Activities



Seeing Symmetry

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Seeing Symmetry

1 **Paperfolding**
Students predict and describe the results of several paper-folding and cutting problems. The accompanying discussion develops geometric language and an awareness of concepts such as congruence, angle and symmetry.

2 **Mirrors and Shapes**
Students investigate possible shapes that can be seen as the mirror is moved about on various geometric figures.

3 **Shapes and Symmetries**
The students begin by drawing "frames" around shapes and exploring different ways of fitting them in. This is followed by identification of lines of reflection and centers of rotation. The students are encouraged to make generalizations, to classify, to make conjectures, and to pose and solve problems.

4 **Strip Patterns**
The students examine strip patterns and classify them according to their symmetries.

5 **Combining Shapes**
To review and extend ideas about symmetry and develop prohlem-solving strategies, students explore ways of producing symmetrical figures by joining given shapes together.

6 **Symmetries of Polygons**
Students classify hexagons and other polygons according to their symmetries.

7 **Polyominoes and Polyamonds**
The students consider shapes made by joining together squares or equilateral triangles, and some tessellations based on these shapes. They classify the shapes by symmetry and extend symmetry concepts to tessellations.

Math and the Mind's Eye materials are intended for use in grades 4-9. They are written so teachers can adapt them to fit student backgrounds and grade levels. A single activity can be extended over several days or used in part.

A catalog of Math and the Mind's Eye materials and teaching supplies is available from The Math Learning Center, PO Box 3226, Salem, OR 97302, 503-370-8130. Fax: 503-370-7961.

Math and the Mind's Eye

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Paperfolding

OVERVIEW

Students predict and describe the results of several paper-folding and cutting problems. The accompanying discussion develops geometric language and an awareness of concepts such as congruence, angle and symmetry.

Prerequisite Activities

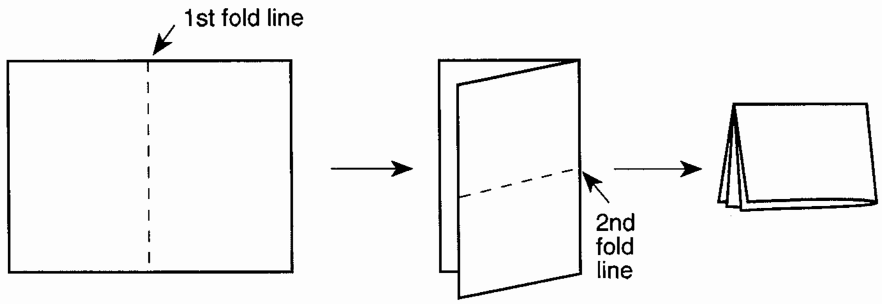
Some knowledge of or experience with angle measure will help, especially 90° and 45°.

Materials

Paper, scissors, rulers, activity sheets, masters for transparencies.

Actions

1. Fold a piece of paper in half twice as shown here and ask the students to do the same.



(a) Ask the students to imagine what the paper will look like when it is unfolded. Have them predict what will be seen, making a list of their thoughts at the overhead.

(b) While the students work in small groups, have them unfold their papers and check their predictions. Have the groups prepare a written description of their observations. Discuss.

Comments

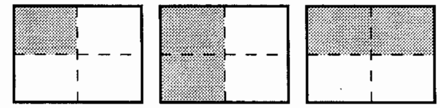
1. The initial actions of this activity are intended to provide experience with mental geometry.

(a) Students will likely predict that the paper will be divided into smaller rectangles or that they will see several right angles and lines. Ask them to describe their predictions more fully, perhaps with questions such as:

- How many rectangles will be seen? How many right angles? How many line segments?
- How are the rectangles alike? different? How are their dimensions related? How about their areas?
- How can the orientation of the lines be described?

(b) You might ask each team to post their observations and to share their thinking about some of them. Here are some possible responses that can be discussed:

- Looking at one side of the paper, I can see 9 rectangles. Here are 3 of them:



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Actions

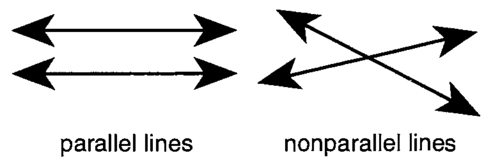
Comments

1. (b) *Continued.*

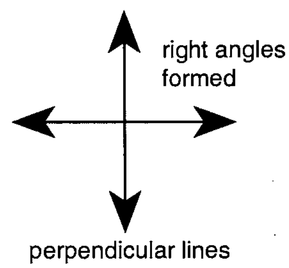
- There are 4 rectangles that are ¼ the size of the whole paper.
- The folds show 2 lines of symmetry.
- The folds are perpendicular to each other.
- There are examples of parallel sides.

Make a transparency from Master 1. It shows a picture of the paper after it has been unfolded. This is an opportunity to introduce (or review) related vocabulary. Some of the terms that might be discussed are *parallel, perpendicular, congruent* and *line of symmetry*:

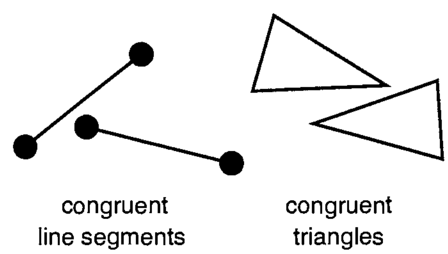
- Parallel lines: lines in a plane that will never intersect.



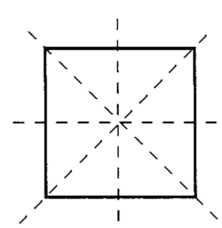
- Perpendicular lines: lines that form right angles.



- Congruent figures: figures that have the same size and shape. One can be placed exactly on top of the other.



- Lines of symmetry: lines which divide a shape into two parts that are mirror images of one another. For example, a square has 4 lines of symmetry as shown here:



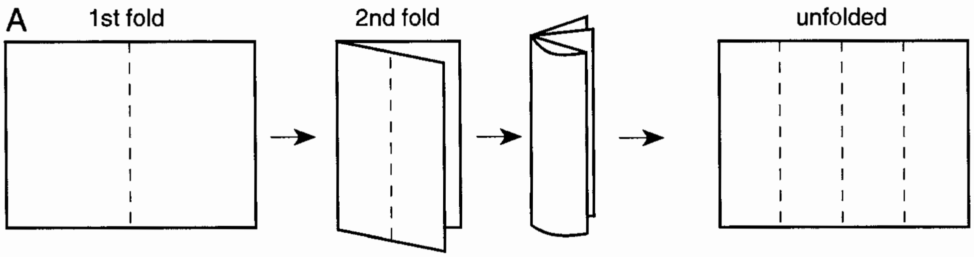
Actions

2. Repeat Action 1, only this time have the groups follow the given first fold with a second fold of their choice.

Comments

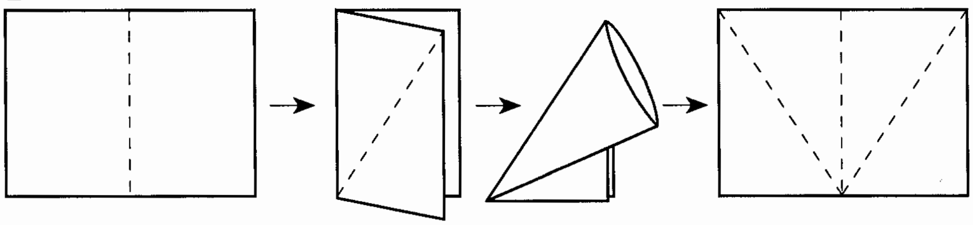
2. Here are some possibilities:

A



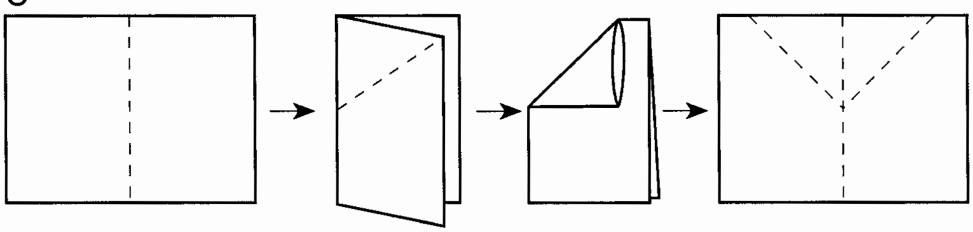
"The folds are parallel to 2 sides of the paper."

B



"There are 4 congruent right triangles."

C



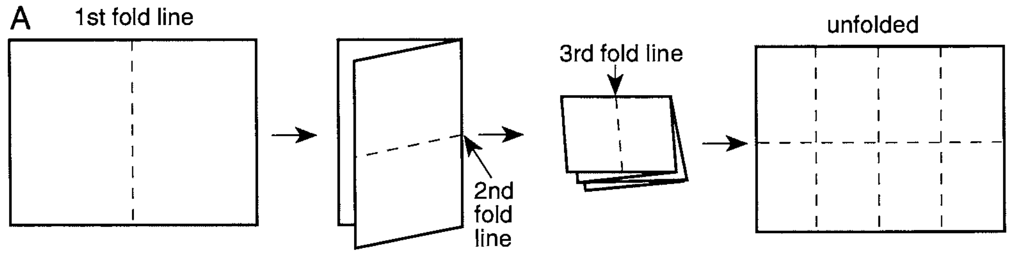
"There is an isosceles triangle that has been split into 2 right triangles. The right triangles are congruent."

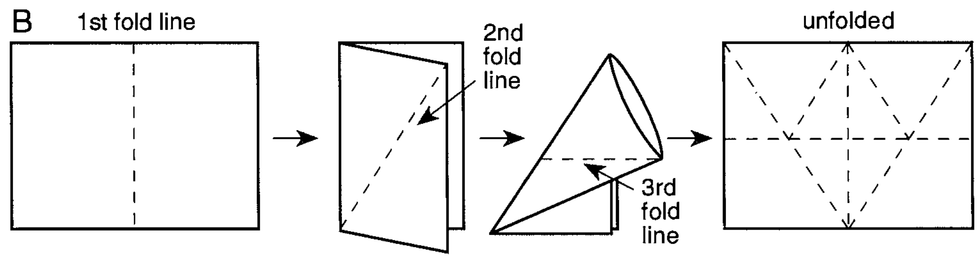
Allow time for the students to explain their observations. For example, in Illustration B, how do students decide that the 4 right triangles are congruent?

Actions	Comments

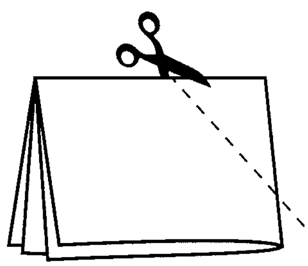
3. (Optional) Repeat Action 3, only this time have the teams make a third fold of their choice.

3. Here are two possibilities:





4. (a) Once more, fold a piece of paper as in Action 1. Begin a single, straight cut across the folds:



Without completing the cut, ask the students to describe the shape that will be formed by the triangular piece when it is completely unfolded.

4. (a) Provide plenty of time for discussion here. Encourage the students to describe the predicted shape in words. Some will suggest, for example, that the shape will be a "diamond". What is a diamond? Others will say there will be 4 sides. How did they decide this? Can these be any 4 sides?

It's important to note that the shape will have 4 sides of equal length (students will generally suggest this). Tell the class that such a shape is normally called a *rhombus*.

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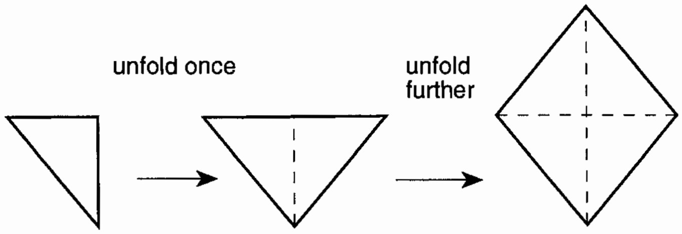
Actions

(b) Ask each student to fold a piece of paper in the above manner and make a single, straight cut across the folds as illustrated above. Discuss the shapes that are formed when cutoff triangular pieces are completely unfolded.



(c) Discuss this question: How should the cut across the folds be made so that the piece cut off will unfold into a square?

Comments

4. *Continued.*
(b) Note: It is common for some students to cut off the wrong corner in this activity. To help avoid this, have the students make the folds, then unfold the paper and mark the center of the paper with a small dot. Then ask them to refold the paper and cut off the corner where the dot is located.

Here are some likely observations:

- The shape has 4 sides that are the same length. These 4 sides were all formed by the cut.
- The folds divide the rhombus into 4 right triangles. These triangles are congruent.
- Each fold is along a line of symmetry for the rhombus.
- The folds are at right angles to each other.
- There are several examples of congruent angles.

You may wish to discuss related vocabulary such as: legs of a right triangle, hypotenuse, diagonal, congruent, parallel, perpendicular, symmetry, etc.

A transparency can be made from Master 2 to show an example of an unfolded rhombus.

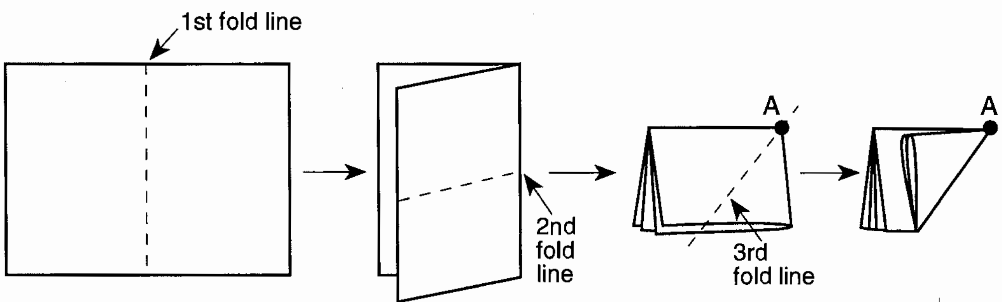
(c) In Action 4(b), some students will likely cut shapes that appear to be square. This will motivate the question of Action 4(c). Encourage verbal responses to this question. Most students will suggest cutting off equal lengths from the corner where the folds meet. Others may suggest cutting off a right triangle that has a 45° angle.

It is helpful to discuss how squares and rhombuses are related. How are they alike? Different? Is a square also a rhombus? Is a rhombus always a square?

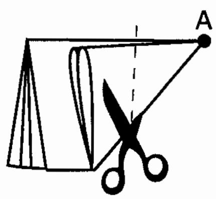
By definition, a square is always a rhombus but a rhombus is not always a square. Some students may struggle with this idea, perhaps because their feelings for the two shapes are different.

Actions

5. (a) Have each student make the folds of Action 4 once more with another sheet of paper. Do the same with a paper of your own. Discuss the right angle formed at the corner (point A in the diagram below) where the folds intersect. Ask the class to fold this angle into a half right angle.

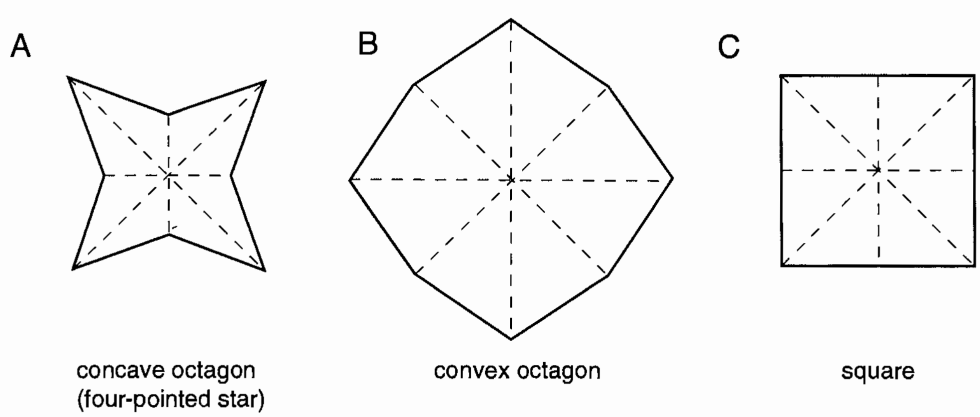


(b) Make the fold described in part (a) with your paper and begin a single, straight cut across the folds.



Without completing the cut, ask the students to predict the shapes that will be formed when the cutoff triangular piece is completely unfolded.

(c) Ask the students to make similar cuts across the folds of their papers. While the students work in small groups, have them discuss the unfolded shapes and prepare written descriptions of them. Post the results and discuss.



Comments

5. (a) The instructions of this action may need clarification. It may be helpful for students to explain their understanding of a right angle and to discuss the number of degrees such an angle contains.

(b) Several predictions may be made, the most frequent being hexagon and octagon. Encourage the students to explain the thinking behind their predictions. In this case, they often find it helpful to reflect on the number of thicknesses of paper formed with each fold.

(c) It is possible to unfold octagons and squares such as those pictured here. A transparency can be made from Master 3 to display these possibilities to the students.

Continued next page.

Actions

Comments

5. (c) *Continued.* Here are some questions for discussion: How are these shapes alike? different? Is a 4-pointed star also an octagon? How about the square—does it have 8 sides too? Are any of the shapes regular octagons?

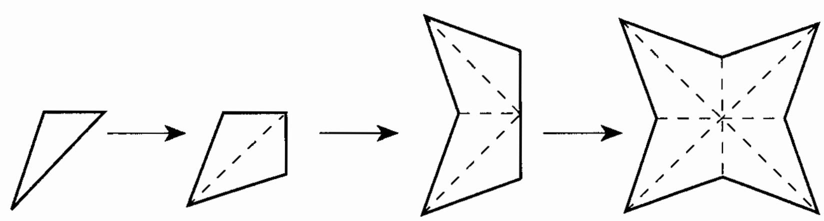
The 4-pointed stars are octagons because they have 8 sides. Students seem to find this acceptable, especially in view of the previous discussion about squares and rhombuses. See Comment 4(c).

Four-pointed stars may also be described as concave octagons (Figure B, on the previous page, is an example of a convex octagon). You may wish to use this language during the discussion.

Note: The students are likely to unfold the different shapes shown above. Should they all unfold just one type of shape, however, ask them to explore the situation further. Can they somehow cut differently and unfold something else? See also Action 5(d).

(d) Ask the groups to work the following problem: How should the cut across the folds be made so the triangular piece cut off will unfold into a 4-pointed star? How should the cut be made so as to unfold a convex octagon? a square? a regular octagon? Discuss.

(d) This problem provides a nice context for discussing acute, right and obtuse angles. A 4-pointed star will be unfolded whenever an obtuse triangle is cut off. This is illustrated here:



If a right triangle is cut off, then a square will be unfolded. An acute triangle will unfold into a convex octagon.

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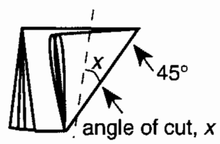
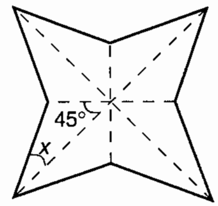
Actions

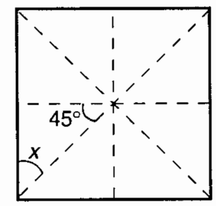
Comments

5. (d) *Continued.* The following illustration depicts the effect of varying the angle of the cut ($\angle x$). You may wish to ask the groups to discuss the values of x that will lead to each of the three shapes.

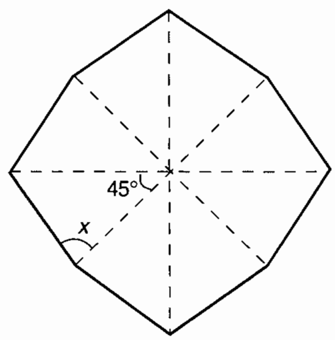




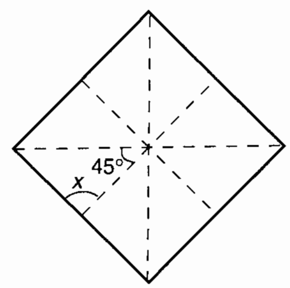
$0 < x < 45°$
(4-pointed star)



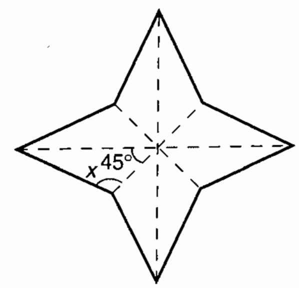
$x = 45°$
(square)



$45° < x < 90°$
(convex octagon)



$x = 90°$
(square)



$90° < x < 135°$
(4-pointed star)

This effect can also be demonstrated by superimposing transparencies (made from Masters 4a–e). A transparency made from Master 5 will show the result of this superimposition.

(e) (Optional) Using available software, have the students create computer displays of the shapes unfolded in Action 5(c).

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TO SHAPE :CUT :SIDE

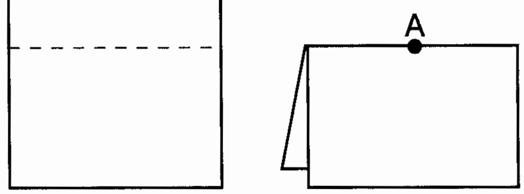
PU RT 45 FD 35 PD
REPEAT 4[FD :SIDE LT 2*:CUT-90 FD :SIDE LT 180-2*:CUT]
PU HOME PD
END
```

(e) As an example of this, here are some problems that can be investigated with Logo [the angle of cut refers to angle x in the second illustration of Comment 5(d)]:

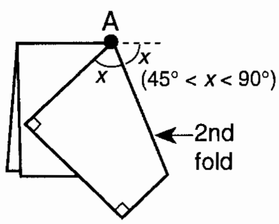
- Make an angle of cut of 30°. What shape will be unfolded? Create a Logo display of this shape. Repeat for other angles of cut.
- Create a Logo procedure that will draw the unfolded shapes in this activity. Make your procedure a variable one that will allow you to input any side length and any angle of cut (an example is given to the left). Explore your procedure using different lengths and angles.

Actions

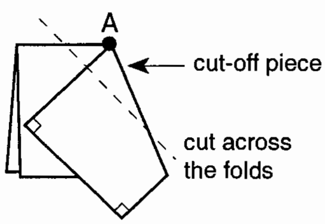
6. (a) Ask each student to fold a piece of paper in half and mark a point A on the fold as shown here.



Now ask them to make a second fold, through A, at an angle to the first fold that is between 45° and 90° (angle x in the illustration below).



Make the above folds with a paper of your own and begin a single, straight cut across the folds.



Without completing the cut, ask the students to imagine and describe the shape that will be formed when the cutoff piece is completely unfolded.

Comments

6. (a) Possible student responses include:
- The shape will be a quadrilateral.
- There will be 2 short sides and 2 long sides.
- The 2 short sides will be congruent and so will the long sides.
- Undoing the second fold will produce a triangle. Undoing the first fold will give two of these triangles.

Encourage the students to explain the thinking behind their predictions.

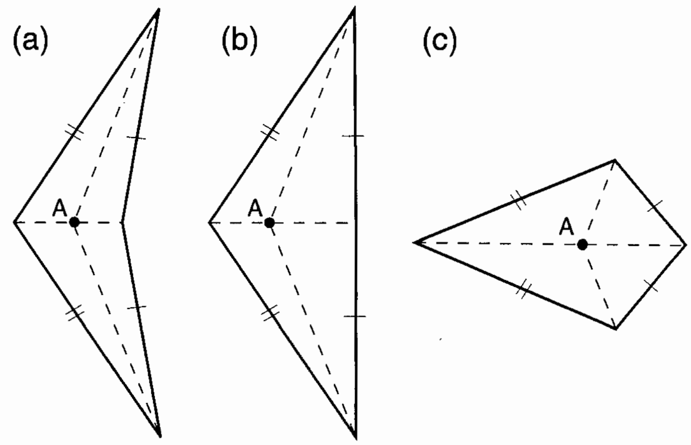
Continued next page.

Actions

(b) Ask the students to make similar cuts across the folds of their papers. While the students work in small groups, have them discuss the unfolded shapes and prepare written descriptions of them. Post the results and discuss.

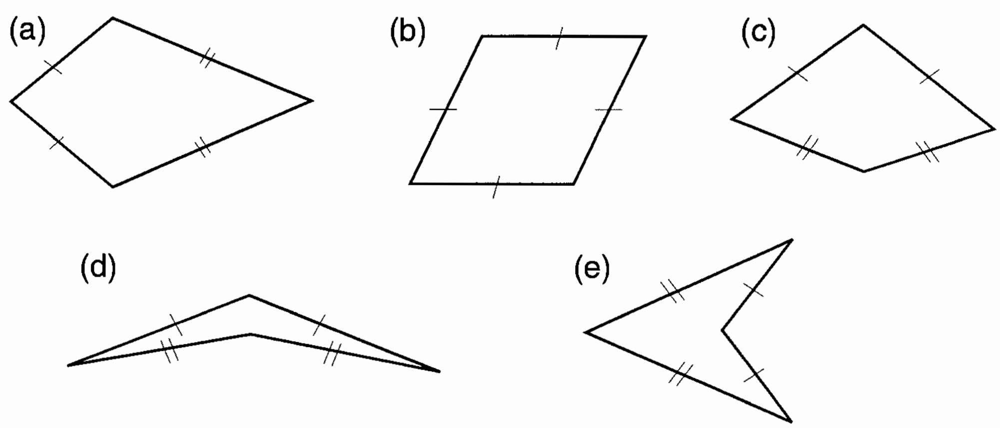
Comments

6. *Continued.*
(b) It is possible to unfold quadrilaterals and triangles such as those pictured here. (Make a transparency from Master 6 to display these on the overhead.)



Here are some suggested discussion questions: How are these shapes alike? different? Do they all have 4 sides? Even the triangles?

As illustrated above, the unfolded quadrilaterals will have two pairs of congruent adjacent sides. It is customary to call these quadrilaterals *kites*. Each shape shown below is an example of a kite.



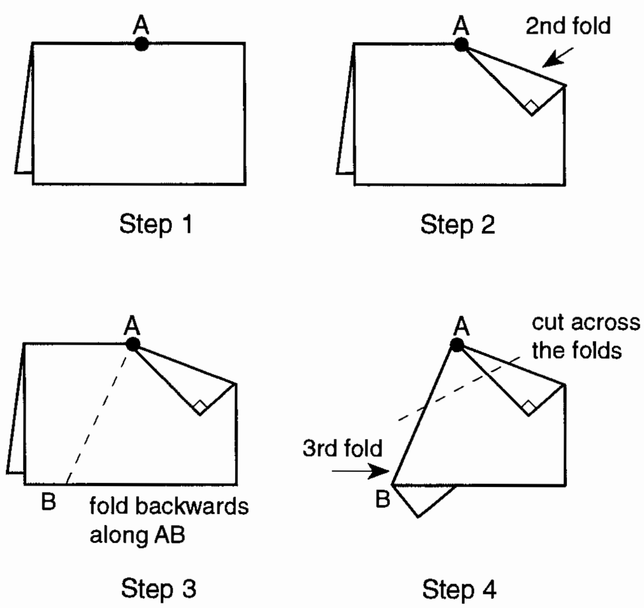
Shapes (a)–(c) in the illustration are examples of convex kites; shapes (d) and (e) are concave. Convex kites have the property that both diagonals lie entirely in the interior of the kite.

Continued next page.

Actions

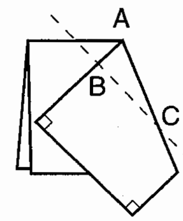
(c) Distribute copies of Activity Sheet X-1-A and have the groups work the problems. Discuss.

7. Repeat Actions 6(a) and 6(b) for the following sequence of folds:



Comments

6. *Continued.*
(c) The problems on this activity sheet offer another context for thinking about acute, right and obtuse angles. The following illustration depicts the setting of Questions 1–3. A concave kite will be unfolded whenever triangle *ABC* is obtuse.



If triangle *ABC* is acute, a convex kite will be unfolded. If it is a right triangle, then a triangle will be unfolded.

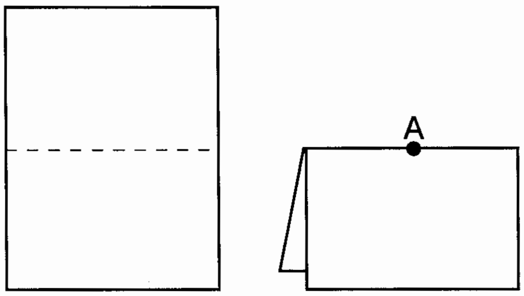
Problem 4 of the activity sheet motivates a look at the properties of an equilateral triangle.

Problem 5 reinforces the discussion of Problems 1–3. Encourage the groups to reflect on why only concave kites can be unfolded here.

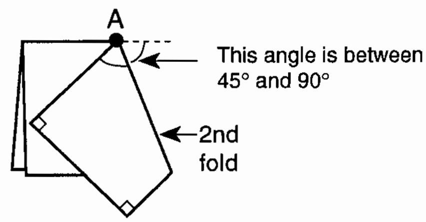
7. Various hexagons and pentagons can be unfolded here. Encourage the students to investigate the effects of changing the angles at which the second and third folds are made. You might also ask the students to describe the folds and cuts that will generate a particular shape such as a regular hexagon.

Name ___________________________

Fold a piece of paper in half and mark a point *A* on the fold as shown here:



Now, make a second fold, through *A*, at an angle between 45° and 90° to the first fold:



1. By making a single, straight cut across the folds, it is possible to unfold the cutoff piece into an arrowhead. Describe how to make such a cut.

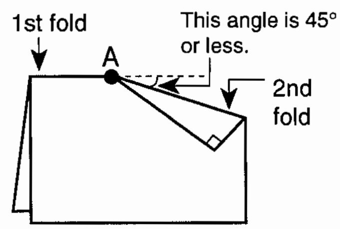
2. How should the cut be made so as to unfold a kite that is not a concave kite?

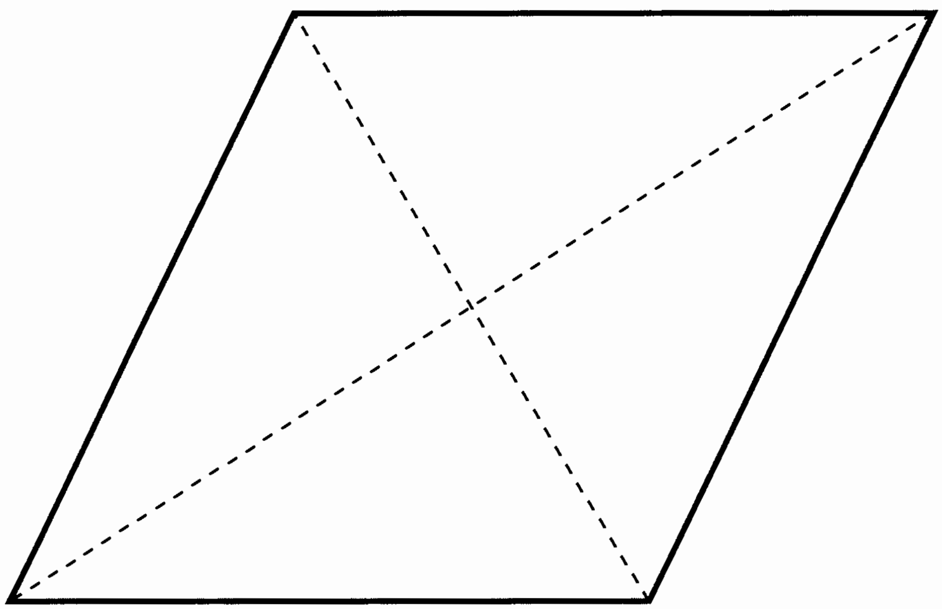
3. How should the cut be made so as to unfold a triangle?

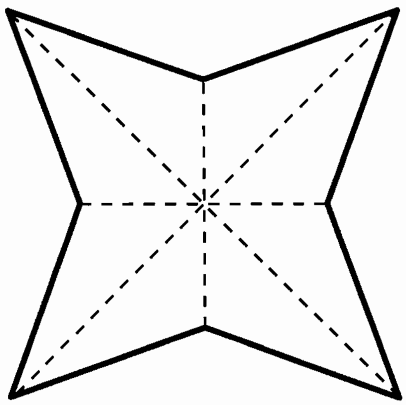
Name ______________________________

4. Describe how to make the second fold and the cut so as to unfold an equilateral triangle.

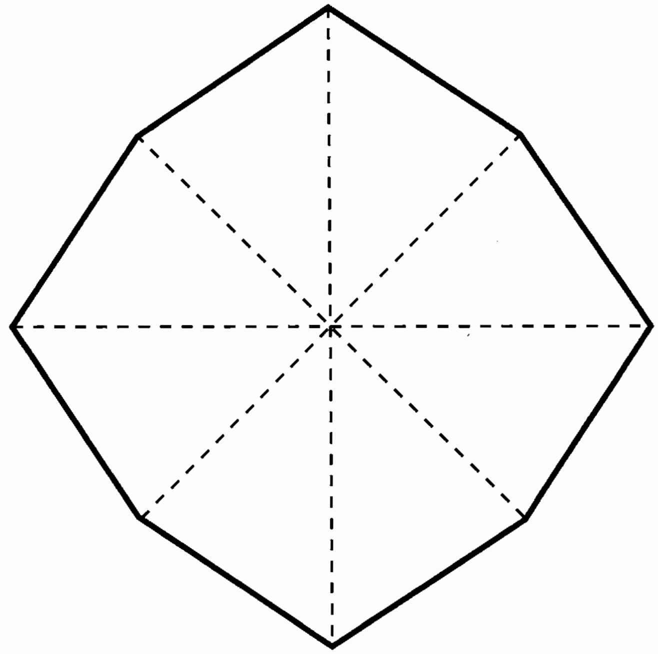
5. Investigate this situation: Make the second fold through *A* at an angle of 45° or less to the first fold (as illustrated below). What shapes(s) can be formed by making a single, straight cut across the folds and unfolding the cutoff piece?



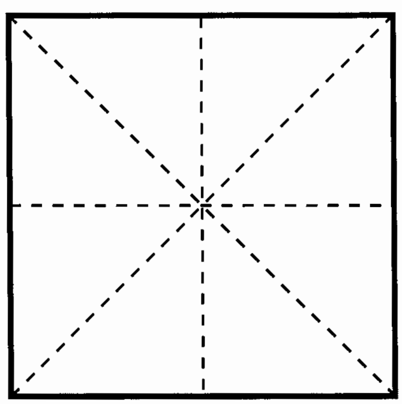




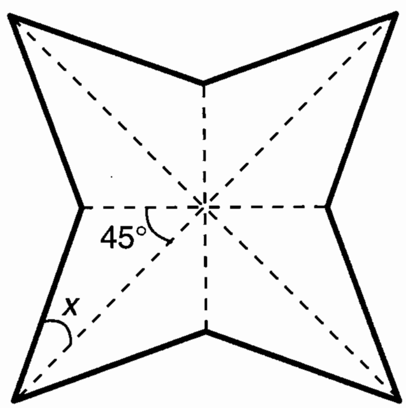
(a)

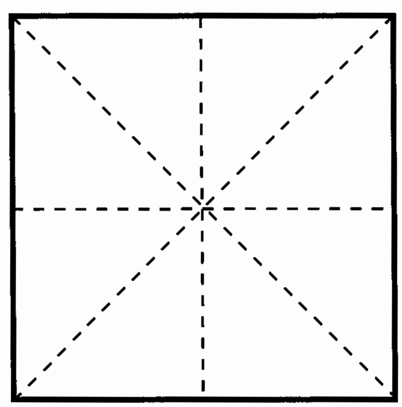


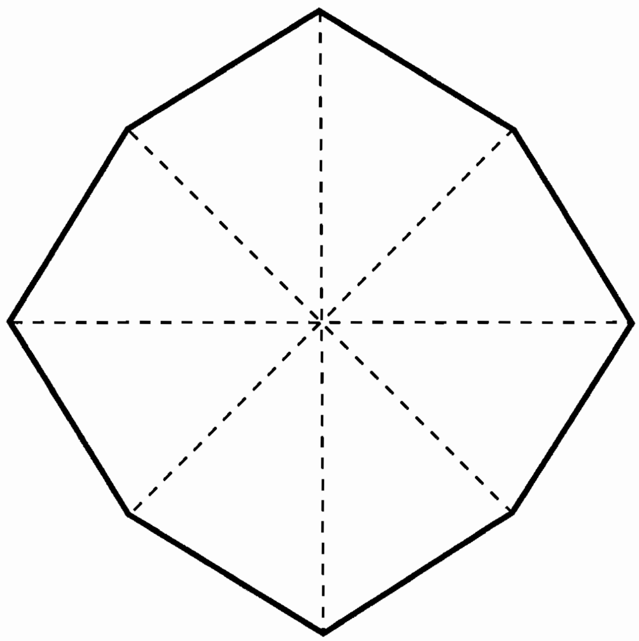
(b)

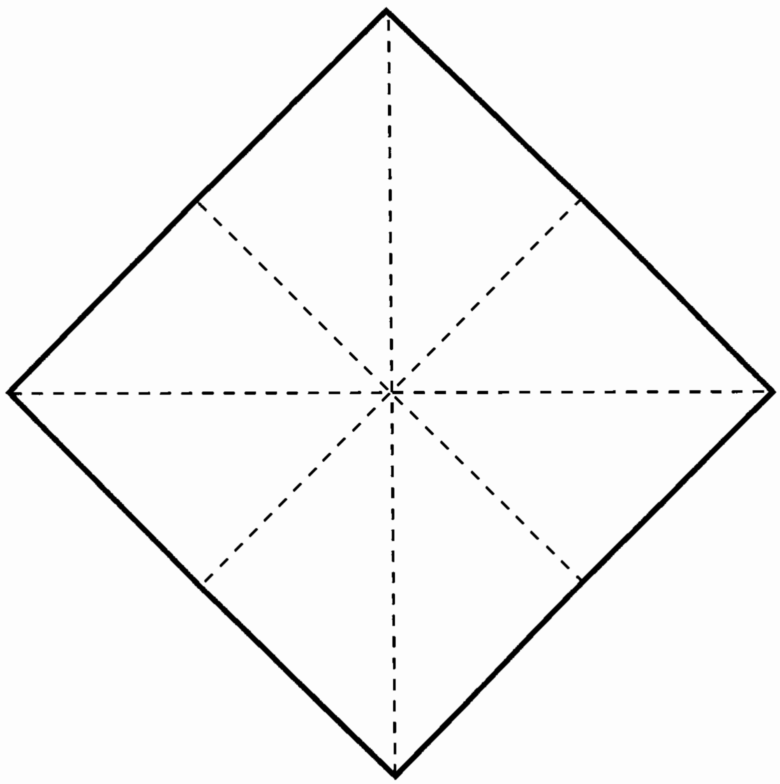


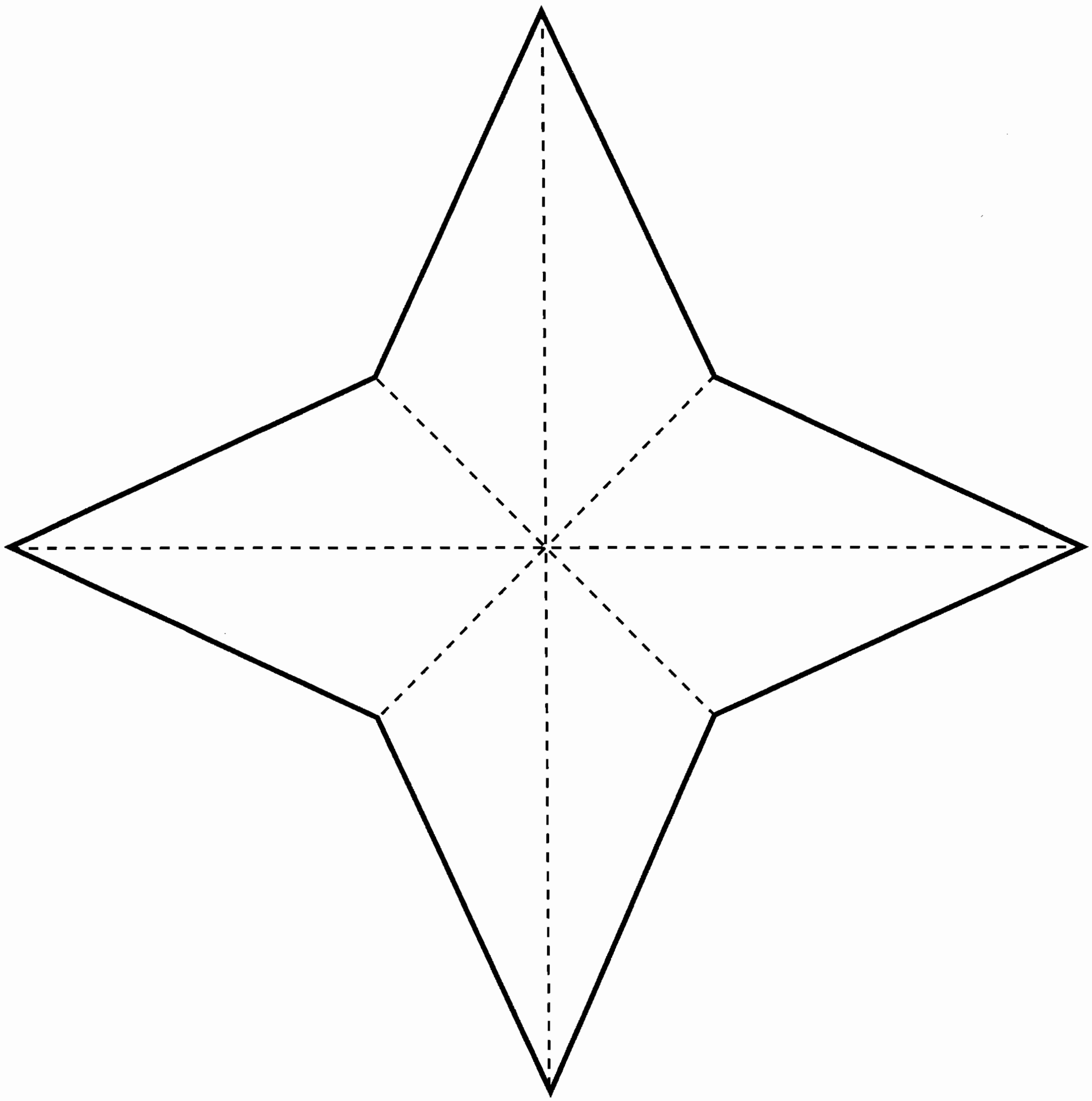
(c)

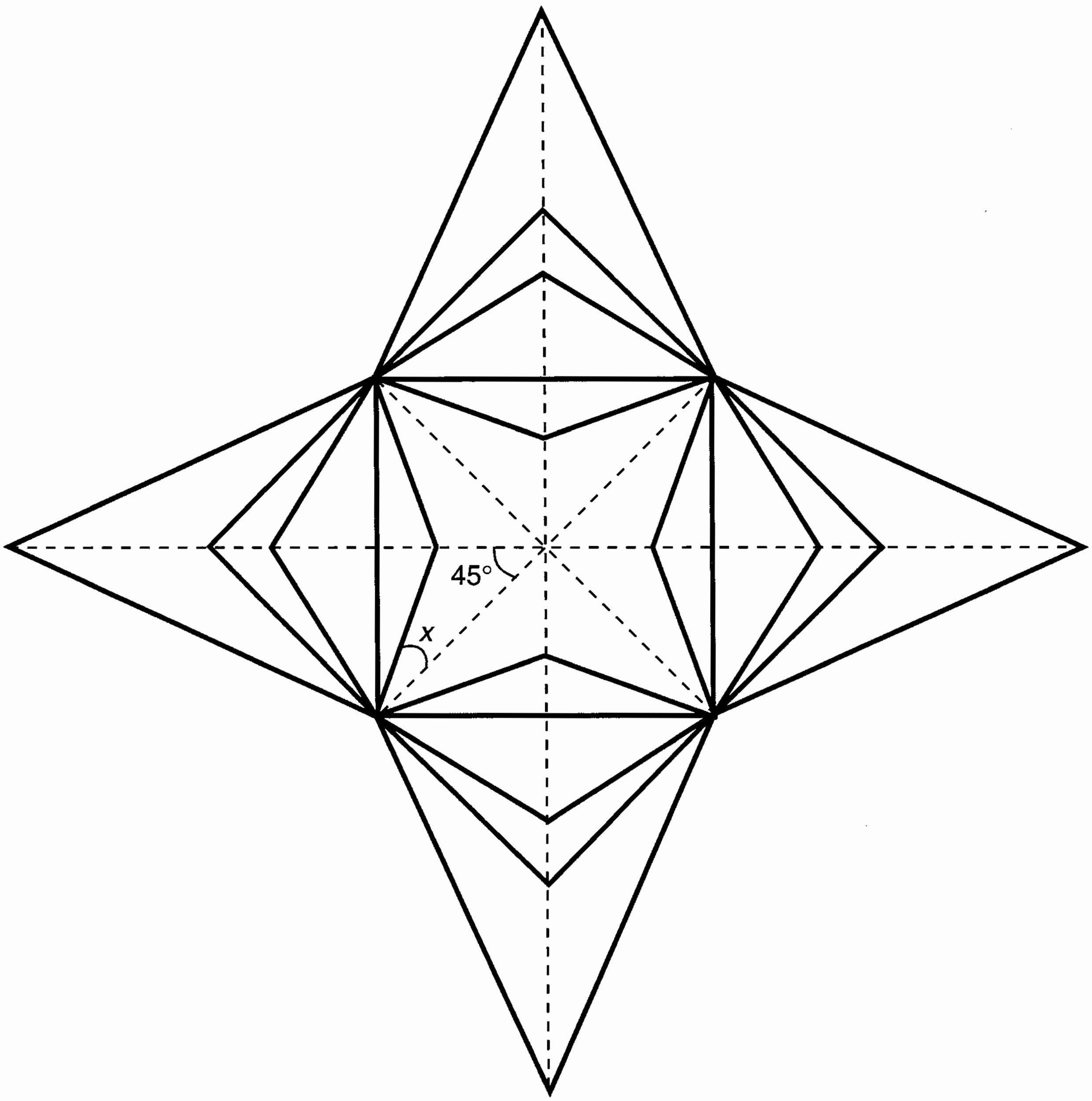

45°
x

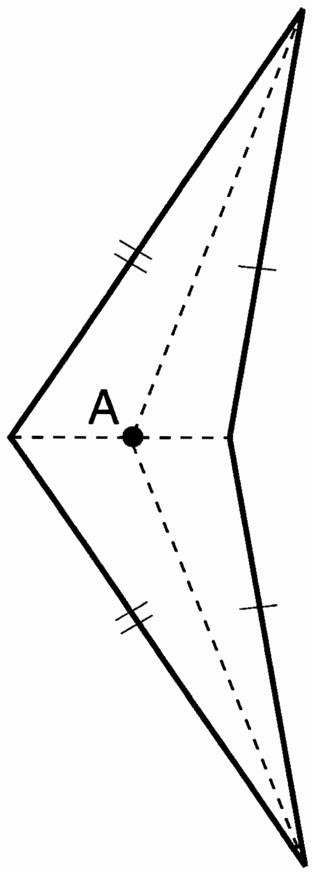




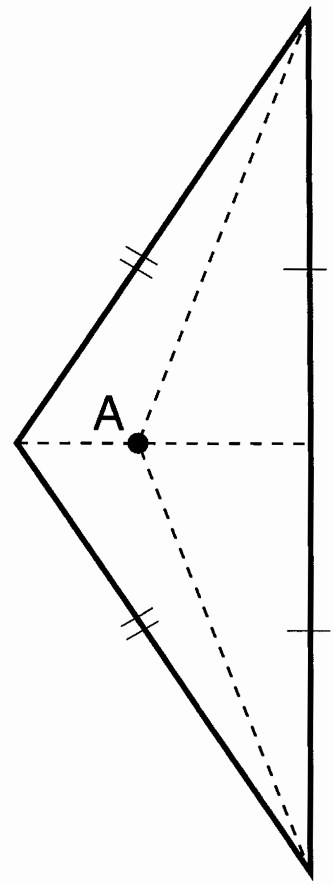




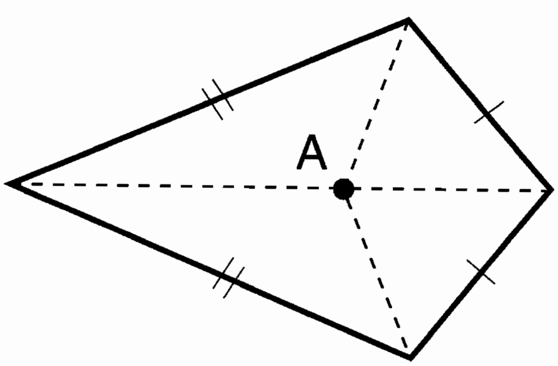

45°
x



(a)



(b)



(c)

Mirrors and Shapes

O V E R V I E W

Students investigate possible shapes that can be seen as the mirror is moved about on various geometric figures.

Prerequisite Activities

Unit X, Activity 1, *Paper Folding*. Some experience with angle measure.

Materials

Mirrors, activity sheets and masters. Protractors may helpful.

Actions

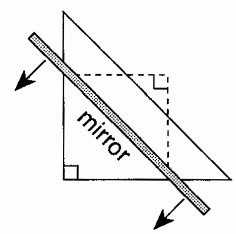
1. (a) Give each student a mirror and a copy of Activity Sheet X-2-A. Ask the students to place the mirror on the isosceles right triangle so they "see" a square. In section I of the activity sheet, have them write a description of where the mirror can be placed to accomplish this.

(b) Ask the students to share their written directions for where to place the mirror. Discuss.

Comments

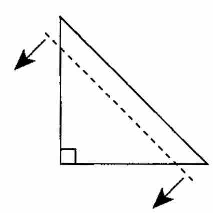
1. A master of the activity sheet is attached.

You may have to clarify what is being asked. The challenge is to place the mirror on the triangle, perpendicular to it, so that a portion of the triangle and its reflection in the mirror forms a square. In the sketch below, the dotted line represents the reflection in the mirror.



There are an infinite number of placements for the mirror. The students will describe these in different ways. Here are three possibilities:

"Place the mirror along any line parallel to the hypotenuse of the triangle."



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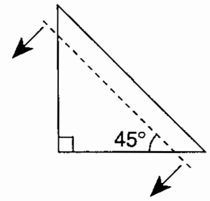
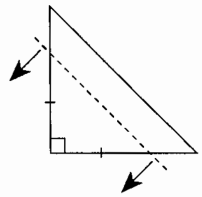
Actions

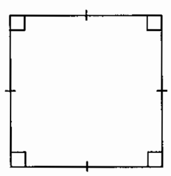
Comments



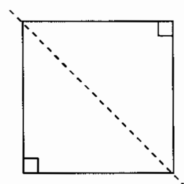
1. *Continued.* "Place the mirror so it intersects one of the legs of the right triangle at a 45° angle."



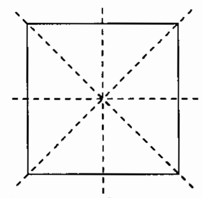
"Place the mirror so that it cuts off equal lengths on the legs of the right triangle."



A square has 4 right angles and 4 sides of the same length.

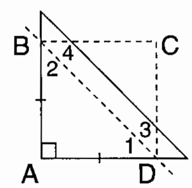


A diagonal divides a square into 2 congruent right triangles.



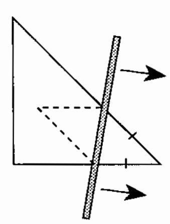
A square has 4 lines of symmetry.

In the discussion, you can ask the students how they know that their placement of the mirror creates a square. This can lead to a review of properties of the square, some of which are pictured to the left, and informal proofs, such as the following:



"I made $AB = AD$. Because of the mirror, $BC = AB$ and $DC = AD$, so I have 4 equal sides. Now $\angle C$ equals 90° since it's the same as $\angle A$. Also, angles 1, 2, 3, 4 are all equal since ΔABD is isosceles. Each is 45°. So $\angle 2 + \angle 4 = 90°$ and $\angle 1 + \angle 3 = 90°$. That makes 4 right angles in *ABCD*. So I've made a square."

2. Repeat Action 1, only this time ask the students to locate the mirror so as to see a rhombus that is not a square. Have them record their descriptions of where to place the mirror in section II of the activity sheet.



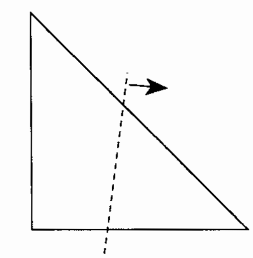
2. A rhombus is a quadrilateral that has 4 congruent sides. While a square is a rhombus, this action asks students to create a rhombus that is not a square.

Students will typically comment that if the mirror reflects a vertex of an acute angle of the right triangle and cuts off equal lengths on the leg and hypotenuse that form that angle, then a rhombus can be seen.

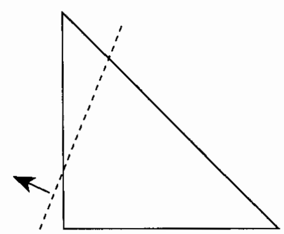
Here are some questions for discussion: What do you notice about this rhombus? What are some of its properties? How is a rhombus like a square? How is it different?

Actions

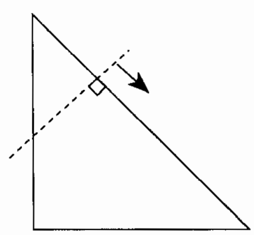
3. Distribute copies of Activity Sheet X-2-B, pages 1–3, to the students. Have them do the activities on page 2. Discuss their results.



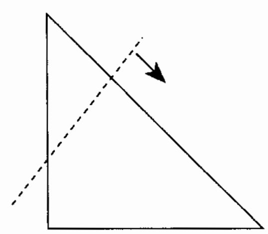
Concave Kite



Convex Kite



Pentagon



Hexagon

Comments

3. Only Column 1 of page 1 of Activity Sheet X-2-B will be used in this Action. Columns 2 and 3 will be used in the next two Actions.

From Actions 1 and 2 above, the students already know that a square and a rhombus can be seen. Thus, they can write "Yes" in the first two spaces of Column 1.

It is assumed that students are familiar with the shapes listed on page 1. You may wish to review the definitions of these shapes. The distinction between convex and concave kites is described in Unit X, Activity 1, Comment 7(b).

In addition to the square and rhombus, both types of kites, an isosceles triangle, a pentagon and a hexagon can be seen in a mirror appropriately placed on an isosceles right triangle. Possible placements for the mirror are shown at the left. The arrow indicates the face of the mirror. Other placements are possible.

Some students may notice that any shape that can be seen has a line of symmetry along the edge of the mirror. Thus, if a shape has no line a symmetry, it cannot be seen with the use of a mirror. Since a parallelogram which is not a rhombus has no line of symmetry, it cannot be seen.

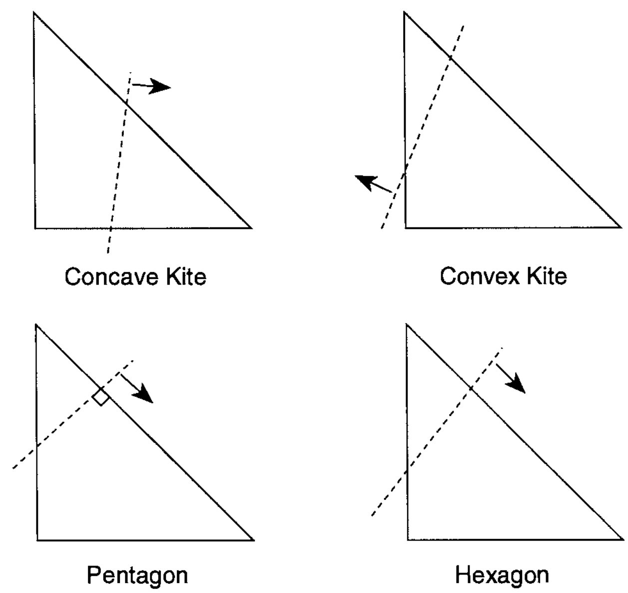
Note that an equilateral triangle, even though it has a line of symmetry, cannot be seen, since an isosceles right triangle has no 60° angle to reflect. Thus, having a line of symmetry is not a sufficient condition for a shape to be seen.

The only other shape that cannot be seen is a decagon. Since a triangle has 3 sides, the most sides that can be seen by placing a mirror on the triangle is twice that, or 6.

During the discussion, you might ask several students to describe how they located the mirror to see a particular shape and to explain how they know this shape has the desired properties. Also, you can ask for volunteers to present the directions they wrote in part 3 without revealing the shape for which they are written and see if the class can correctly identify the shape.

Actions

3. Distribute copies of Activity Sheet X-2-B, pages 1–3, to the students. Have them do the activities on page 2. Discuss their results.



Comments

3. Only Column 1 of page 1 of Activity Sheet X-2-B will be used in this Action. Columns 2 and 3 will be used in the next two Actions.

From Actions 1 and 2 above, the students already know that a square and a rhombus can be seen. Thus, they can write "Yes" in the first two spaces of Column 1.

It is assumed that students are familiar with the shapes listed on page 1. You may wish to review the definitions of these shapes. The distinction between convex and concave kites is described in Unit X, Activity 1, Comment 6(b).

In addition to the square and rhombus, both types of kites, an isosceles triangle, a pentagon and a hexagon can be seen in a mirror appropriately placed on an isosceles right triangle. Possible placements for the mirror are shown at the left. The arrow indicates the face of the mirror. Other placements are possible.

Some students may notice that any shape that can be seen has a line of symmetry along the edge of the mirror. Thus, if a shape has no line a symmetry, it cannot be seen with the use of a mirror. Since a parallelogram which is not a rhombus has no line of symmetry, it cannot be seen.

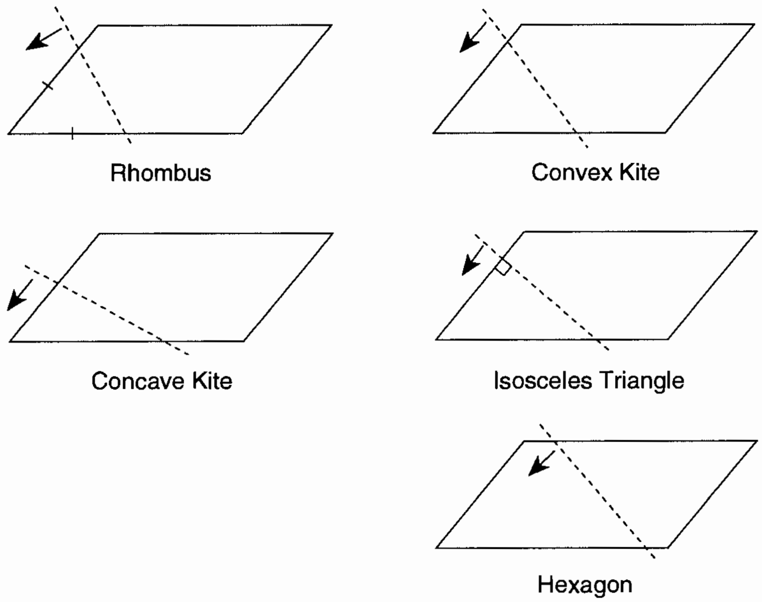
Note that an equilateral triangle, even though it has a line of symmetry, cannot be seen, since an isosceles right triangle has no 60° angle to reflect. Thus, having a line of symmetry is not a sufficient condition for a shape to be seen.

The only other shape that cannot be seen is a decagon. Since a triangle has 3 sides, the most sides that can be seen by placing a mirror on the triangle is twice that, or 6.

During the discussion, you might ask several students to describe how they located the mirror to see a particular shape and to explain how they know this shape has the desired properties. Also, you can ask for volunteers to present the directions they wrote in part 3 without revealing the shape for which they are written and see if the class can correctly identify the shape.

Actions

4. Distribute copies of pages 4 and 5 of Activity Sheet X-2-B to the students. Have them do the activities on page 5. Discuss.



Comments

4. Pages 4 and 5 of Activity Sheet X-2-A are similar to pages 1 and 3, except that a mirror is placed upright on a general parallelogram instead of a right isosceles triangle.

Shown on the left are possible mirror placements for the listed figures that can be seen.

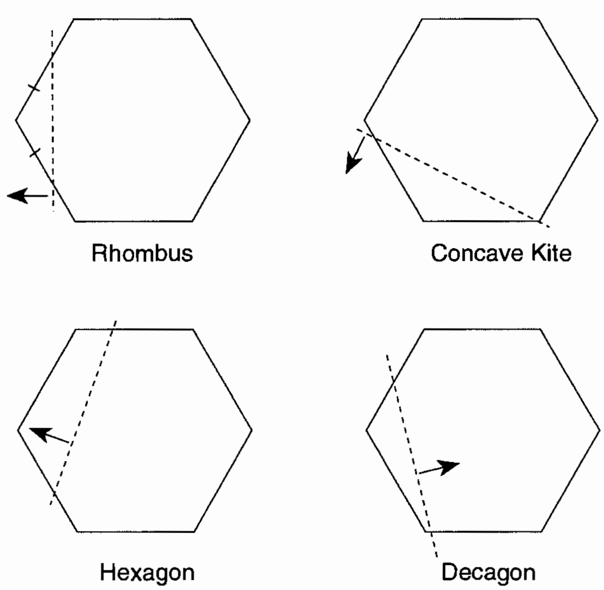
It is not possible to see a square since there is no 90° angle to reflect. Since a parallelogram which is not a rhombus has no line of symmetry, it cannot be seen.

A mirror can be placed on a parallelogram to see an equilateral triangle provided the parallelogram has a base angle of 60°. The acute base angle of the parallelogram pictured on page 5 is closer to 45° than 60°.

In order to see a pentagon, the mirror would have to be placed so that it reflected 3 sides and was perpendicular to exactly one of them. This is not possible.

Since a parallelogram has 4 sides, the maximum number of sides that can be seen, for any placement of the mirror, is 8. Hence, a decagon cannot be seen.

5. Ask the students to put away their mirrors, then distribute copies of pages 6 and 7 of Activity Sheet X-2-B. Have the students do the activities on page 6. Invite volunteers to share their results and explain their reasoning.



5. Activity Sheet X-2-B, page 6, can be assigned as homework and the results discussed later in class.

The students may find it easier to begin by determining those shapes which cannot be seen by placing a mirror upright on a regular hexagon. For example, squares, concave kites, and isosceles triangles, including equilateral ones, cannot be seen because the angles of the hexagon are all obtuse and, hence, there are no right or acute angles to reflect.

A parallelogram that is not a rhombus has no line of symmetry and hence cannot be formed with the use of a mirror.

Forming a pentagon requires placing the mirror so that it reflects 3 sides and is perpendicular to exactly one of them. This is not possible on a regular hexagon.

Shown on the left are possible mirror placements for the listed figures that can be seen.

Continued next page.

Actions

Comments

5. *Continued.* After discussing the students' results, mirrors can be used to verify the conclusions reached.

6. Show the students "Sue's Directions for Seeing A Square", see below. Ask them to critique the directions with two criteria in mind:

(a) Are they adequate?

(b) Are they redundant?

Have the students report their conclusions and give supporting arguments for them. Discuss the reports with the class.

Sue's Directions for Seeing A Square

Step 1. Draw a right angle.

Step 2. Bisect the right angle.

Step 3. Place the mirror perpendicular to the bisector in Step 2 (with the reflecting side facing the vertex of the right angle).

Step 4. Move the mirror, keeping its face towards the angle, until it cuts off equal lengths on the sides of the angle.

6. A transparency of Sue's directions can be made from Master 1 which is attached.

A set of directions is *adequate* if, when followed, its intent is accomplished. In this case, the directions are adequate if, when followed, the result is seeing a square. A set of directions is *redundant* if it contains unnecessary steps.

Sue's directions are both adequate and redundant. Following the directions will lead to seeing a square. Either step 4 may be omitted or steps 2 and 3 may be omitted.

The students arguments in support of their conclusions are likely to be quite intuitive. Their classmates' questions about how they arrived at their conclusions may help them clarify and crystallize their thinking. In providing supporting arguments for their conclusions, the students are, in essence, constructing informal geometric proofs.

7. Divide the class into groups. Ask each group to prepare a set of directions for seeing a rhombus that is not a square. Have the groups post their messages and ask the class to critique them for adequacy and redundancy.

7. Here are examples of sets of directions that students have prepared.

I. Draw an angle that is not 90°. Mark a point *A* on one side of this angle. Measure the distance from *A* to the vertex. Find a point *B* on the other side of the angle that is the same distance from the vertex as *A*. Draw a line from *A* to *B* and place the mirror along this line. Be sure the mirror faces the vertex of the angle. You will then see a rhombus.

II. Make an acute angle *A*. Place the mirror so that it forms equal angles with the sides of angle *A*. Be sure the mirror faces *A* and that it cuts off equal lengths on the sides of angle *A*. You will then see a rhombus.

Note that set II is redundant.

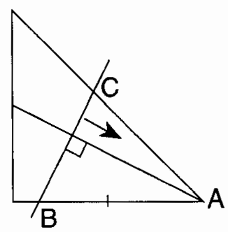
8. Repeat Action 6 only this time critique the directions shown below.

Bob's Directions for Seeing A Rhombus

Step 1. Draw a right angle whose legs are 2 units long.

Step 2. Pick a leg. Draw the line that joins the midpoint of this leg to the vertex opposite it. Label this vertex A.

Step 3. Place a mirror perpendicular to the line drawn in Step 2 so the mirror faces A and intersects both sides of the triangle which form vertex A.



Compare distances *AB* and *AC*

9. Repeat Action 7, this time asking the groups to prepare a set of directions for seeing a concave kite.

8. A master (Master 2) of Bob's directions is attached.

This message does not contain adequate instructions for seeing a rhombus. However, this may not be obvious to the class and there may be a division of opinion. Some students may insist that a rhombus is formed because it "really looks that way." Some may feel that the line drawn bisects the angle at A, thereby permitting one to see a rhombus. Others may dispute that claim and some may be unsure. Discussing their views may help students clarify their thinking.

One way to show the figure seen is not a rhombus is to make a careful drawing (e.g., using a square corner of a sheet of paper to draw a right angle), marking where the mirror cuts the two sides when set perpendicular to the desired line (the mirror is perpendicular to a line when the line and its reflection in the mirror are collinear) and comparing the distances between these marks and vertex *A*. (See the sketch.)

If the students have sufficient background in geometry, they may be able to prove deductively that the figure seen is not a rhombus by showing that $AB = AC$ leads to a contradiction.

9. Here are some sets of directions that students have prepared.

Group A: (1) Draw an acute angle *A*. (2) Place the mirror so that it faces *A* and intersects both sides of angle *A*. (3) Be sure that the mirror forms an obtuse angle with one of the sides of angle *A*.

Group B: (1) Draw an acute angle *A*. (2) Let *B* be a point on one side of angle *A*. Place the mirror so it faces *A* and goes through *B* and any point on the other side of *A*. (3) If you don't see a concave kite, rotate the mirror about *B* and towards *A* until you do.

Group C: (1) Draw a right triangle. Label the vertex at the right angle *A* and let *B* be another vertex of the triangle. (2) Let *C* be a point on the hypotenuse of the right triangle so angle *CAB* measures 30°. (3) Place the mirror facing *B* on the line through *A* and *C*.

Name ______________________

Activity Sheet X-2-B
page 1

	Column 1 **Right Triangle**	Column 2 **Parallelogram**	Column 3 **Hexagon**
Square			
Rhombus (not a square)			
Concave Kite			
Convex Kite (not a rhombus)			
Parallelogram (not a rhombus)			
Isosceles Triangle			
Equilateral Triangle			
Pentagon			
Hexagon			
Decagon			

Name ____________________________

Mirrors on Isosceles Triangles

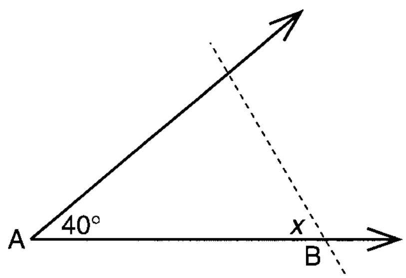
1. If, for a shape listed on page 1, a mirror can be placed upright on an isosceles right triangle so that shape can be seen, write "Yes" in the appropriate space in Column 1, page 1. Indicate on one of the triangles on page 3 where the mirror should be placed to see the shape and write the name of the shape beneath the triangle.

2. In the remaining spaces in Column 1, page 1, indicate briefly why you believe the mirror cannot be placed to see the shape.

3. Pick a shape, other than a square or rhombus, and write directions for placing a mirror on an isosceles right triangle so that shape can be seen. Your directions can be written below or on the back of this sheet.

Actions

10. (Optional) Show the students the following diagram. Tell the students to imagine that the sides of the angle are of infinite length and that a mirror of infinite length is placed on the dotted line facing A. Point out to them that angle x can be made to vary by rotating the mirror about point B. Ask the students to determine what different kinds of polygonal figures can be seen as angle x varies and for what values of x each kind of figure is seen.




```
TO START                              TO LINE
PU HOME PD                            FD 100 BK 200 HOME

TO REFLECT :X :Y                      TO MIRROR :X :Y
LINE                                  IF :X < OR = 0 THEN
LT :X FD 70                           REFLECT :X :Y
RT 180 - :Y                           DRAW
FD (70*SIN :X)/SIN 180 - (:X + :Y)    MIRROR :X - 10 :Y
START   END
RT :X FD 70
LT 180 - :Y
FD (70*SIN :X)/SIN 180 - (:X + :Y)
START
END
```

Type *REFLECT 80 40*
MIRROR 100 40

to see examples of a specific reflection or a series of reflections, respectively.

Comments

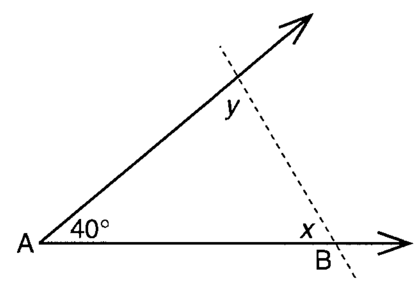
10. A master (Master 3) is attached from which a transparency can be made. You may also wish to make copies of this diagram for the students.

If m is the number of degrees in angle x, then for the indicated values of m, the following figures are seen:

- $0 < m < 50$, concave kite;
- $m = 50$, isosceles triangle;
- $50 < m < 90$, convex kite (when $m = 70$, the kite is a rhombus);
- $m = 90$, isosceles triangle;
- $90 < m < 140$, concave kite.

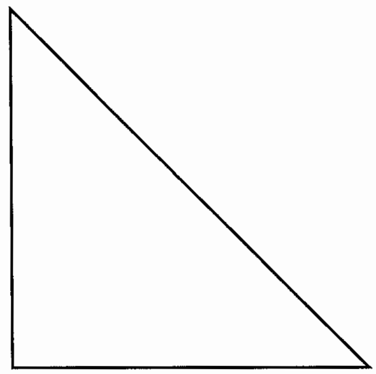
If $m > 140$, no polygonal figure is seen.

The effects of changing x can also be demonstrated with Terrapin Logo procedures such as those on the left below. In these procedures, X and Y refer to the angles shown below.



Activity Sheet X-2-A

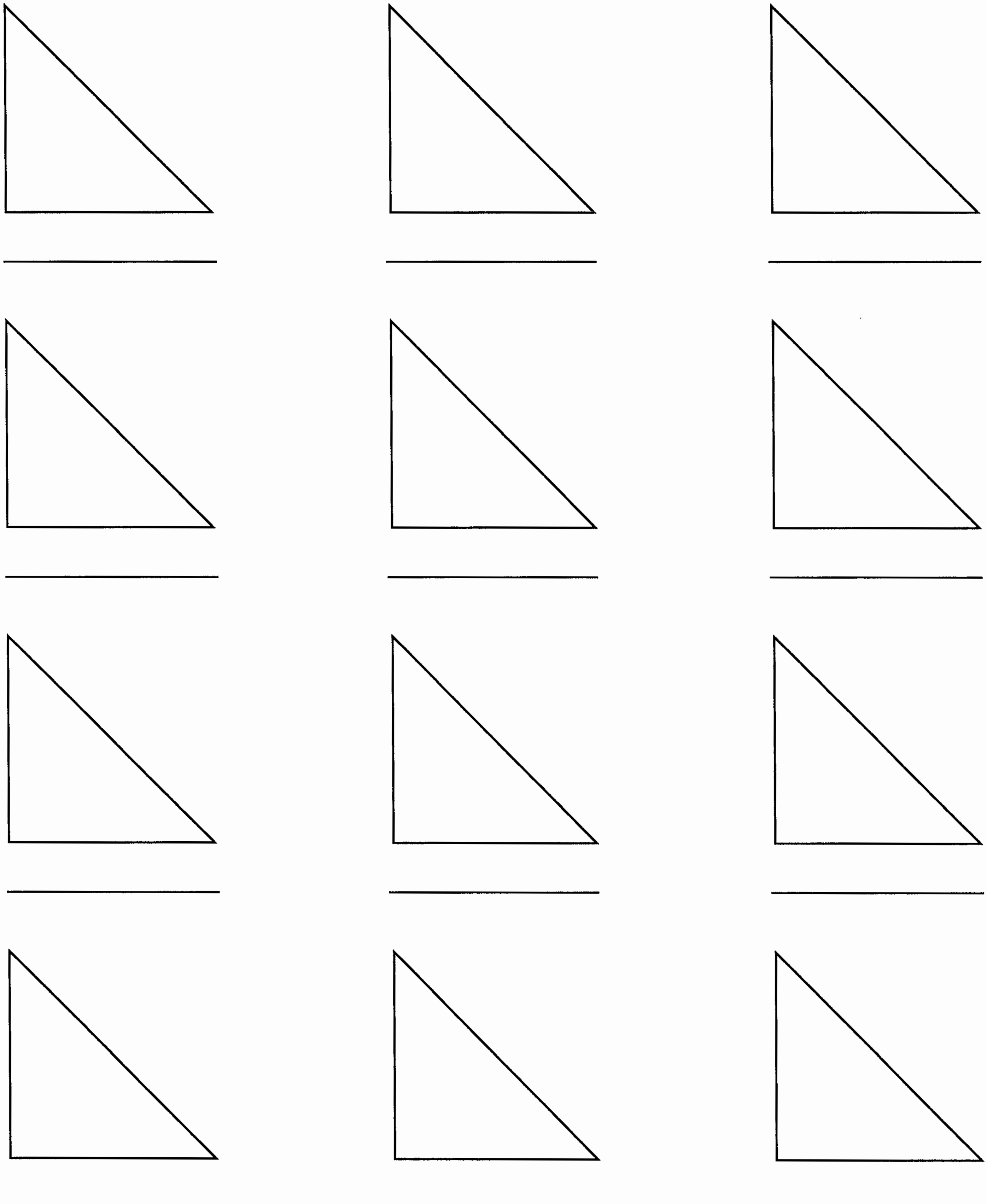
Name ______________________



I

II

Name ______________________________



Name ____________________________

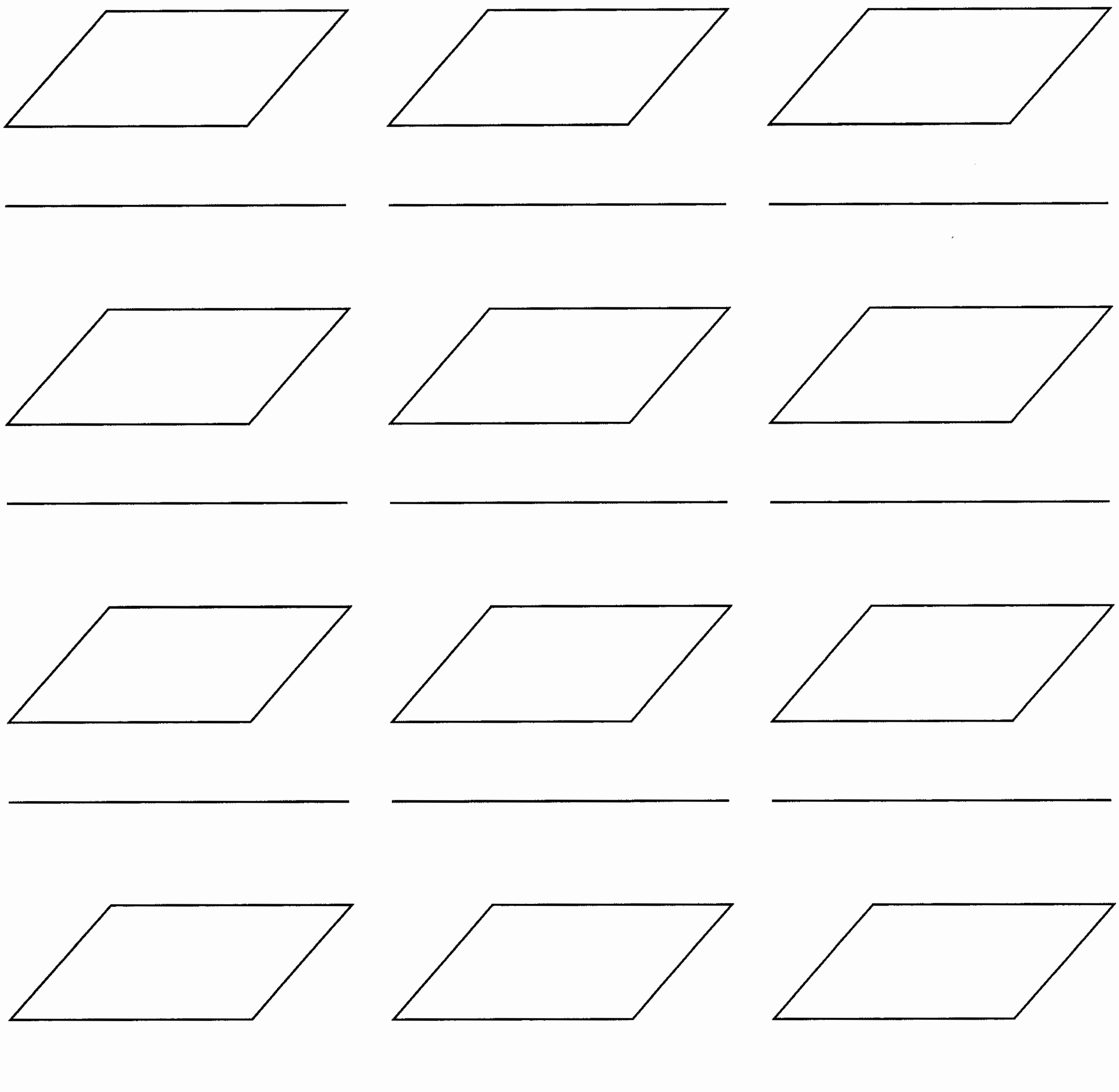
Mirrors on Parallelograms

1. If, for a shape listed on page 1, a mirror can be placed upright on a parallelogram so that shape can be seen, write "Yes" in the appropriate space in Column 2, page 1. Indicate on one of the parallelograms on page 5 where the mirror should be placed to see the shape and write the name of the shape beneath the parallelogram.

2. In the remaining spaces in Column 2, page 1, indicate briefly why you believe the mirror cannot be placed to see the shape.

3. Pick a shape and write directions for placing a mirror on a parallelogram so that shape can be seen. Your directions can be written below or on the back of this sheet.

Name ______________________________



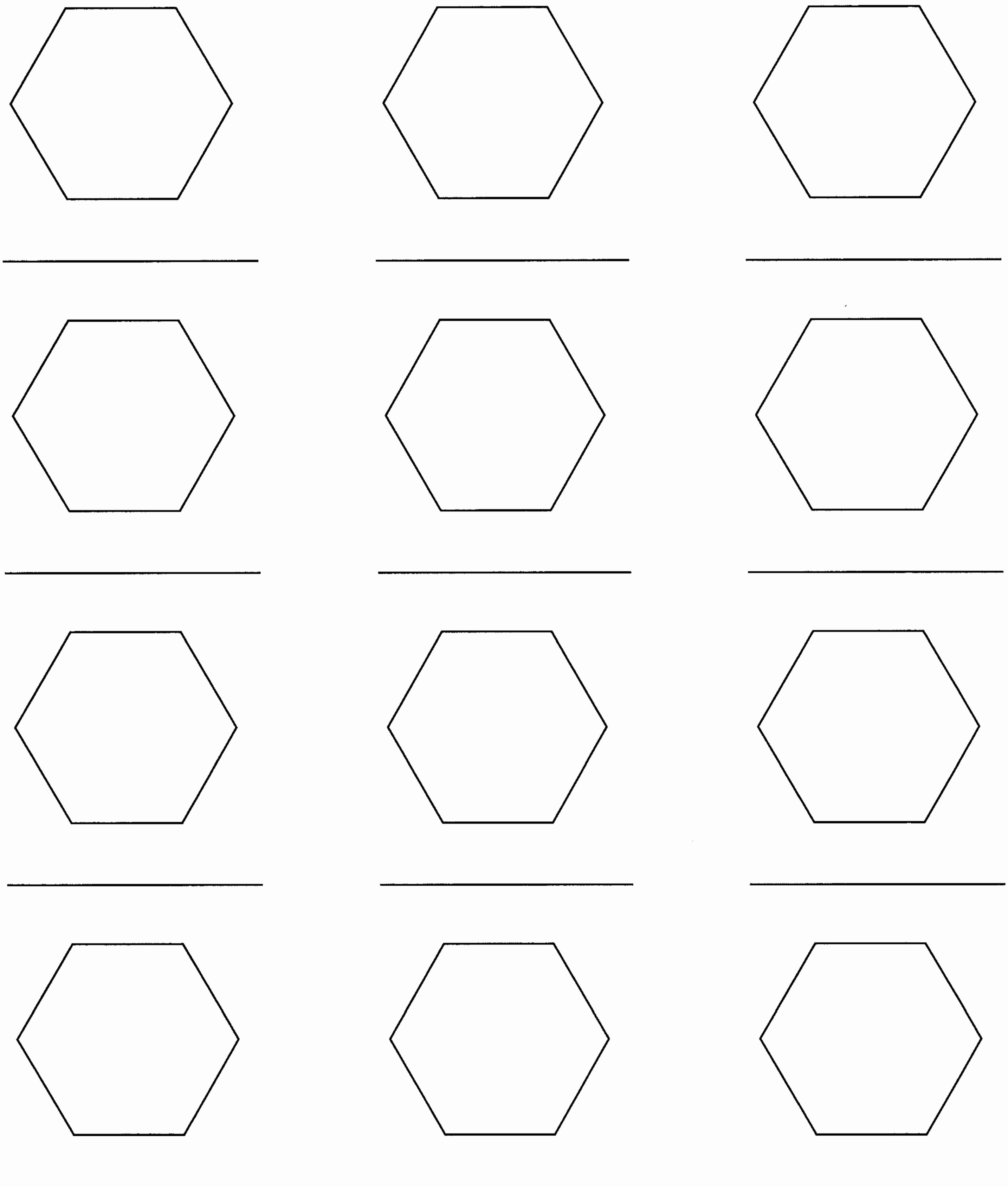
Name ____________________________

Imagining Mirrors on Regular Hexagons

1. If, for a shape listed on page 1, a mirror can be placed upright on a hexagon so that shape can be seen, write "Yes" in the appropriate space in Column 3, page 1. Indicate on one of the hexagons on page 7 where the mirror should be placed to see the shape and write the name of the shape beneath the hexagon.

2. In the remaining spaces in Column 3, page 1, indicate briefly why you believe the mirror cannot be placed to see the shape.

Name ______________________________



Sue's Directions for Seeing A Square

Step 1. Draw a right angle.

Step 2. Bisect the right angle.

Step 3. Place the mirror perpendicular to the bisector in Step 2 (with the reflecting side facing the vertex of the right angle).

Step 4. Move the mirror, keeping its face towards the angle, until it cuts off equal lengths on the sides of the angle.

You will then see a square.

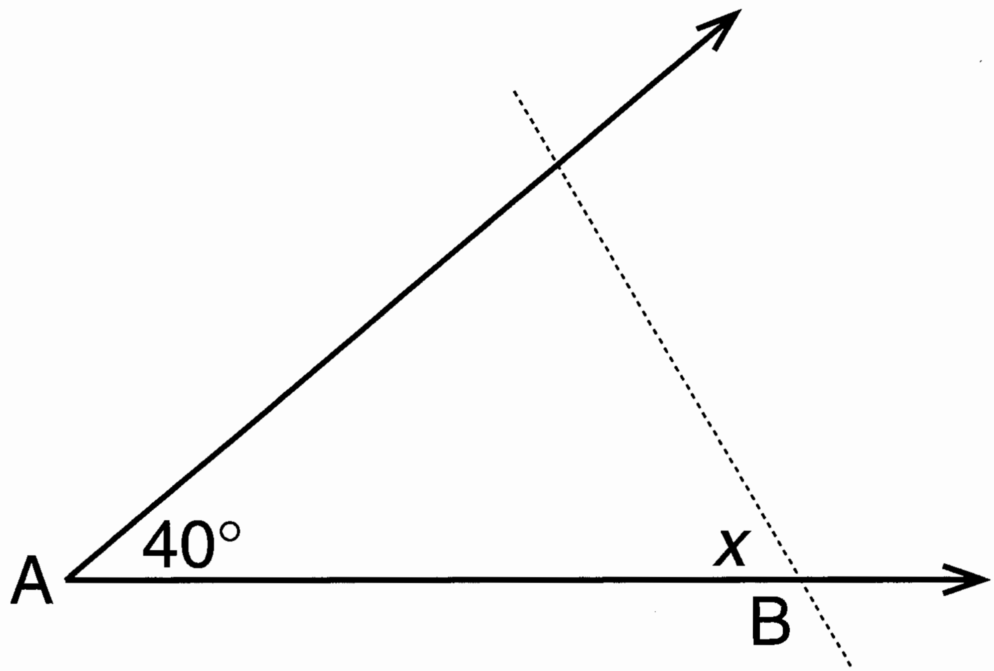
Bob's Directions for Seeing A Rhombus

Step 1. Draw a right angle whose legs are 2 units long.

Step 2. Pick a leg. Draw the line that joins the midpoint of this leg to the vertex opposite it. Label this vertex A.

Step 3. Place a mirror perpendicular to the line drawn in Step 2 so the mirror faces A and intersects both sides of the triangle which form vertex A.

You will then see a rhombus.


40°
x
A
B

Unit X • Activity 3

Shapes and Symmetries

O V E R V I E W

The students begin by drawing "frames" around shapes and exploring different ways of fitting them in. This is followed by identification of lines of reflection and centers of rotation. The students are encouraged to make generalizations, to classify, to make conjectures, and to pose and solve problems.

Prerequisite Activity

Unit X, Activity 1, *Paperfolding*, and Unit X, Activity 2, *Mirrors and Shapes*, are recommended.

Materials

Scissors and activity sheets. Mirrors may be helpful (see Comment 3).

Actions

1. Hold a rectangular sheet of plain paper against a chalkboard, or a larger sheet of paper pinned to the board, and draw a "frame" around it.

(a) Ask the students how many different ways the rectangle can be positioned in its frame.

(b) Ask some students to demonstrate the different ways. Ask how they know the ways are different.

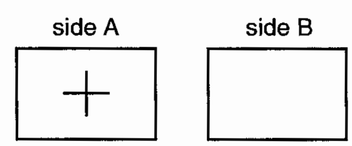
(c) Discuss with the students how the rectangle might be marked so different positions are distinguishable.

Comments

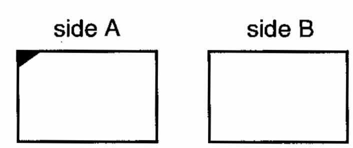
1. (a) The answers here may vary. And if, for example, several students say "two", it does not mean they all see the *same* two.

(b) Since the paper is plain, it may not be absolutely clear that one position is different from another. Allow arguments to develop over this.

(c) Some of the students' suggestions for marking may not distinguish all 4 positions. For example, marking a cross on one side of the rectangle, as shown, will show whether or not the rectangle has been flipped, but not whether it has been rotated.



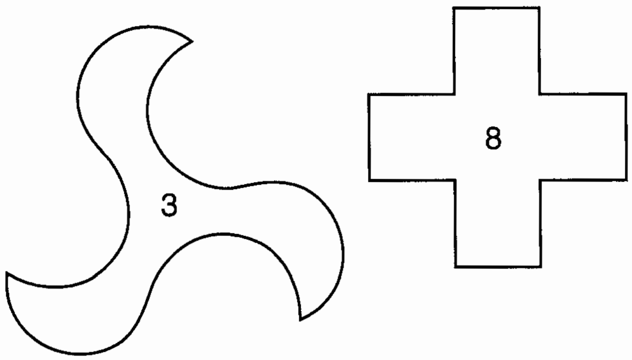
Marking a corner may distinguish rotations on one side, but not on the other side unless the paper is transparent!



Suggestions should be tried and discussed until a satisfactory method is agreed upon.

Actions

2. (a) Pass out copies of Activity Sheet X-3-A. Ask the students to cut out the shapes and then, on a separate sheet of paper, to draw frames around the shapes. Inside each frame have the students write the number of ways in which the corresponding shape can be positioned in its frame, as in the examples below. Discuss.



(b) Ask the students to sketch other types of shapes, and examine the number of ways they can be positioned in their frames. Ask the students for their observations. Encourage any moves to classify and to generalize.

Thus, if a student says, "It looks as if...", then ask whether it is always true, when it is true, for what cases it is true, when else it will be true, and so forth.

Ask if they know—without working with the shape—how many ways a regular 20-gon will fit. Ask them which shapes will fit 4 ways.

Comments

2. (a) This activity is intended to introduce some ideas for students who have had few previous ideas about symmetry, refresh ideas for those who have identified symmetries before, and fill in the gaps and clarify ideas for those with a partial understanding.

It is best to run off copies of Activity Sheet X-3-A on cardstock to make it easier for students to trace frames. If the cardstock is laminated, the shapes can be reused.

A shape which can be positioned in its frame in more than one way is said to have *symmetry* or to be *symmetrical.* The number of different ways a symmetrical shape can be positioned in its frame is called its *order* of symmetry.

At this stage the maturity of the students will determine the sort of vocabulary to use in talking about symmetry. They may want to talk of "flips" and "turns" rather than "reflections" and "rotations". This is not too important here. What is important is a general feel for the different types of transformations that take each shape from one position to another, and the generalizations that come out of the situation.

(b) They may wish to cut out their shapes and draw a frame as in the last action.

A particular observation can often be prompted into a generalization by asking when else it will be true.

If nothing much is happening by way of observation, then invite the students to classify the shapes according to the number of ways they fit. What can they say about those shapes which fit, for example, in 2 ways? in 4 ways?

Some things students have noticed about the number of ways a shape can be positioned in its frame:

- a regular n-gon has $2n$ ways;
- shapes with just 1 axis of symmetry have 2 ways;
- shapes with no symmetry fit in 1 way;
- any shape with at least 1 axis of symmetry will have an even number of ways;

Continued next page.

Actions

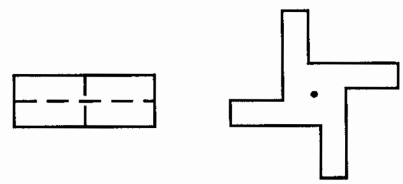
(c) Pass out copies of Activity Sheet X-3-B. Ask the students to cut out the shapes and draw a frame around each. Then have them determine in how many ways each shape can be positioned in its frame?

Comments

2. (b) *Continued.*

- shapes that can be flipped always have an even number of ways—the same number each way up;
- if a shape has in an odd number of ways then it only has rotational symmetry.

The invitation to classify according to number of ways will prompt observations about each category. One way implies the shape has no symmetry. Two ways implies 1 axis of symmetry. Three ways implies 3-fold rotational symmetry. Shapes with 4 ways can be subdivided into two categories: those shapes that have 2 reflections and a half-turn, like the rectangle, and those that have 4-fold rotation. Here are examples of shapes of both categories:



Asking about a shape which the students do not actually have may also encourage generalization; for example, asking in how many different ways a regular 20-gon can be positioned in its frame. They must visualize a regular 20-gon and compare it with other regular polygons. If they can do that, they can then answer the question for any regular polygon.

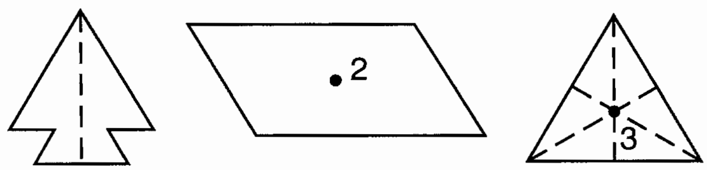
(c) An oval can be placed in its frame 4 ways. There are an unlimited number of ways in which a circle can be placed in its frame.

Students have suggested various things about circles:

- Two ways, either way up.
- You could give it a half-turn, so that would be—4 ways; then you could give it fourth-turns—8 ways; eighth-turns—16 ways; ....
- Turn a degree at a time, 360 turns, 720 ways; but you could turn half a degree at a time, 720 turns, 1440 ways; .... (The 11-year-old student who suggested this eventually wrote "Eternity" in the circle's frame!)

Actions

3. (a) Ask the students to redraw or retrace their shapes from Action 2, marking in axes of symmetry with a dotted or colored line, and indicating orders of rotational symmetry by writing an appropriate number by the center of rotation. For example:



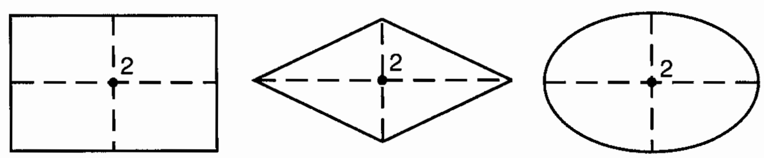
(b) Ask the students to classify the shapes according to their symmetry.

Comments

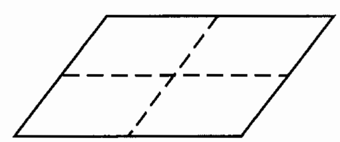
3. (a) This activity can, for younger students, be left until later. But if carried out with students who are meeting these ideas for the first time, then discussion will be necessary about the two types of symmetry, reflective and rotational.

A shape has *reflective* symmetry if it can be moved from one position in its frame to another by flipping it about a line called an *axis of symmetry*. If a mirror is placed on an axis of symmetry, the reflection in the mirror is identical to the part of the figure on the other side of the mirror, hence the name "reflective symmetry". A figure has *rotational* symmetry if it can be moved from one position in its frame to another by rotating it about a point, called the *center of rotation*. The *order* of rotational symmetry of a shape is the number of different positions it can be moved to—in its frame—by rotating it about its center of rotation.

(b) For example, a rectangle, a rhombus, and an oval go together; they have 2 axes of symmetry and 2-fold rotational symmetry. The interesting difference between a rectangle and a rhombus, is that, in one, the axes join midpoints of sides and, in the other, they join vertices.



The symmetry of the parallelogram sometimes cause problems. Students often think it has two axes of symmetry, like the rectangle.



This can be disproved by cutting it out and folding it along one of the "axes", or by placing a mirror along an "axis".

A nice set of problems concerns the minimum number of sides for polygons with various types of symmetry. For example, what is the minimum number of sides for a polygon with 3-fold rotational symmetry only? Is there a maximum number of sides?

Continued next page.

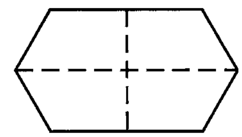
Actions

(c) (Optional) Point out to the students that although a rectangle and a rhombus have the same kind of symmetry—a rotational center of order 2, and 2 separate reflectional axes [see illustration in Comment 2(b)]—that the axes are of two different types, connecting either vertices or edges. Ask the students to check out other polygons for these two types of axes, generalizing their results, if possible.

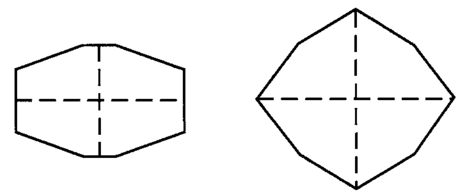
4. Ask the students to classify the capital letters of the alphabet according to symmetry.

Comments

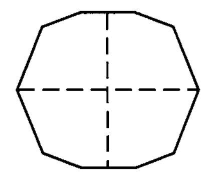
3. *Continued.*
(c) For hexagons, if there are axes, there must be one of each type:



Octagons may have one type or the other, as in the rectangle-rhombus case:



For decagons, as is the case of hexagons, if there are axes, there must be one of each type:

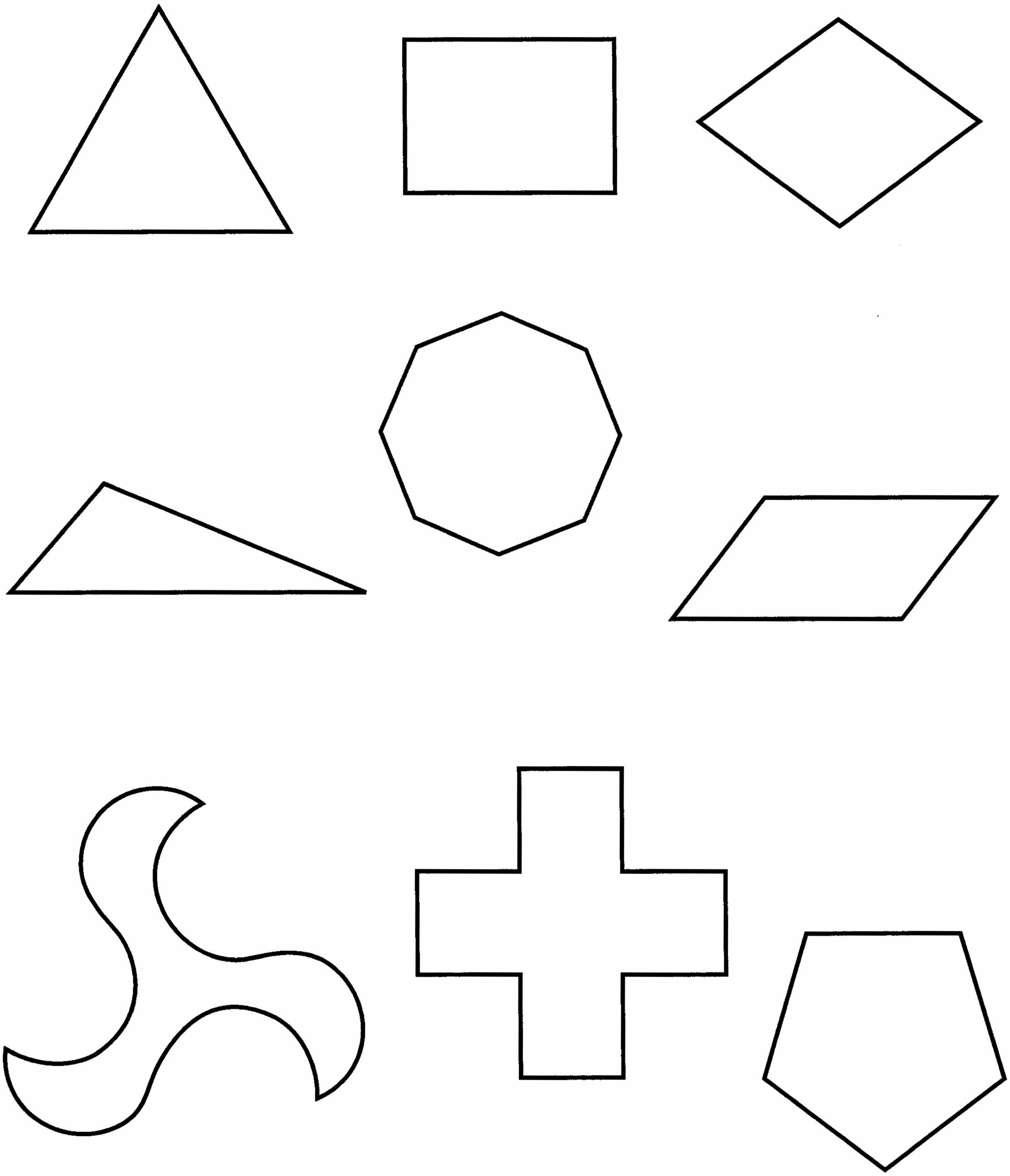


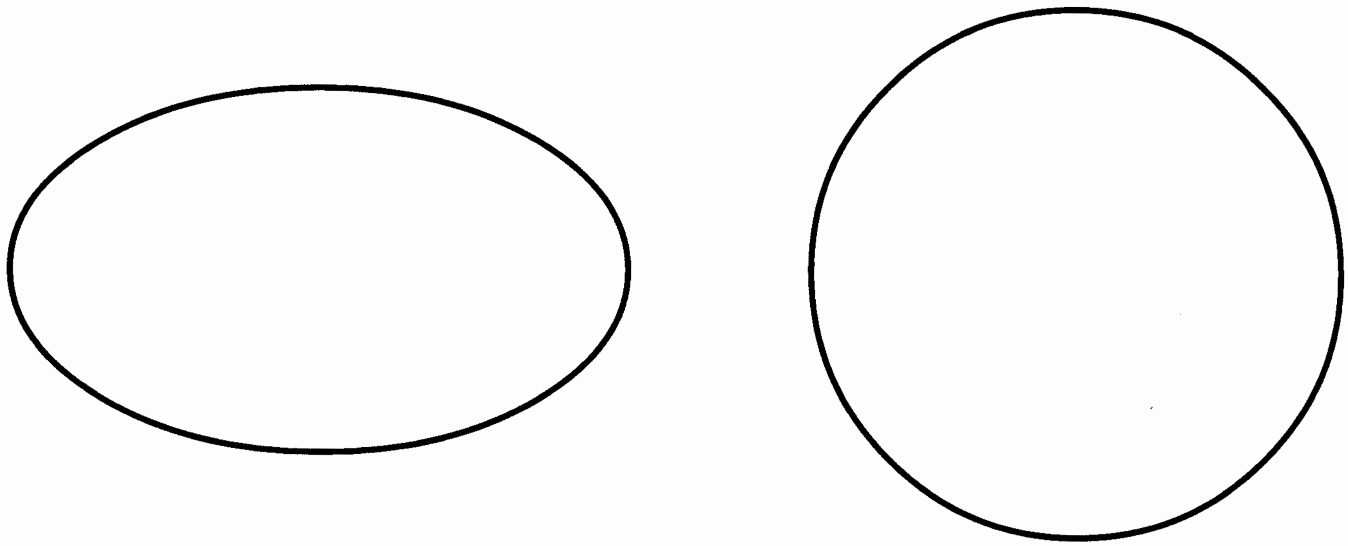
In general, polygons must have an even number of sides to have these types of axes of symmetry. If the number of sides of a polygon is an even number which is a multiple of 4, and if it has axes of symmetry, it must have one type or the other, but not both. If the number of sides is an even number which is not a multiple of 4, and if there are axes of symmetry, there must be one of each kind.

These ideas are taken up and developed more systematically in Unit X, Activity 6, *Symmetries of Polygons.*

4. This activity may be useful as a homework assignment.

It depends on how the letters are written, but generally upper case letters have a horizontal and/or vertical axis or symmetry, or half-turn symmetry, or no symmetry. The lower case letters are not so interesting—most have no symmetry.





Strip Patterns

OVERVIEW

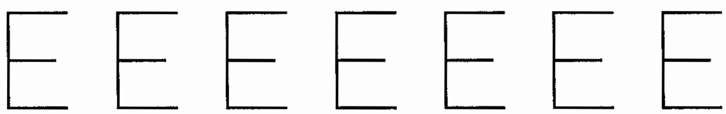
The students examine strip patterns and classify them according to their symmetries.

Prerequisite Activity
Unit X, Activity 3, *Shapes and Symmetry*.

Materials
Overhead transparencies and activity sheet as noted.

Actions

1. (a) Show the students an overhead transparency of the following strip pattern:



Tell the students to suppose the pattern goes on forever in both directions.

(b) Place a second transparency of the pattern on the overhead so it coincides with the first. Ask in what ways this transparency can be moved so that the patterns on the two transparencies still coincide.

(c) Tell the students when a strip pattern is said to *symmetrical*. Ask them to describe the types of symmetry the strip pattern of part (a) possesses.

Comments

1. (a) A master containing two copies of the pattern is attached (Master 1). A transparency of this master can be made and cut in half to obtain to obtain two transparencies of the pattern. The second transparency is needed in Action 1(b).

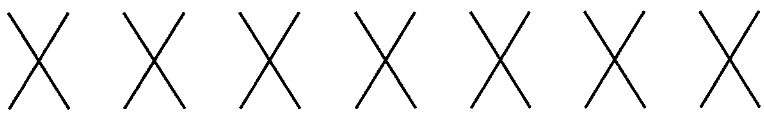
(b) The transparency can be flipped along a horizontal line down the middle of the strip or it can be translated in the direction of the strip, or both.

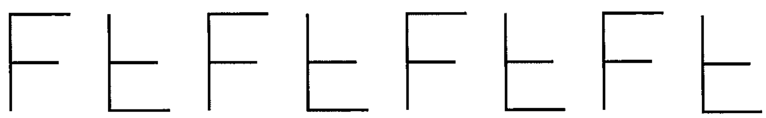
(c) A strip pattern is *symmetrical* if it can be moved from a position in which it coincides with a tracing of itself to another position in which it coincides with the tracing. A transformation of the strip pattern from one position which coincides with a tracing to another coincident position is called a *symmetry* of the strip pattern. (These definitions are similar to those concerning tessellations given in Comment 5 of Unit X, Activity 6.)

The symmetries of the strip pattern shown are a reflection about a horizontal line through the middle of the strip and translations in the direction of the strip, or combinations of these, e.g., a translation followed by a reflection.

Actions

2. (a) Show the students the following strip patterns. Ask them to describe the types of symmetry they possess.

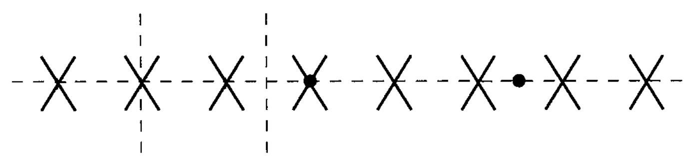




Comments

2. (a) A master of these patterns is attached (Master 2). You may want to prepare an overhead of this master, cut it in half and present the patterns one at a time. A second transparency can be made to serve as a tracing the students can use to describe the symmetries.

A symmetry of the first pattern is either (1) a translation in the direction of the strip, (2) a reflection about a horizontal line through the center of the **X**'s, (3) a reflection about a vertical line through the center of any **X**, (4) a reflection about a vertical line midway between adjacent **X**'s, (5) a half turn about the center of any **X**, (6) a half turn about the midpoint of the line connecting the centers of adjacent **X**'s, or (7) any combination of the above six types. Representative axes of symmetry and centers of rotation are shown below.



(b) Introduce notation for the types of symmetry identified in part (a).

A symmetry of the second pattern is either (1) a translation in the direction of the strip, (2) a translation in the direction of the strip followed by a reflection about a horizontal line through the center of the pattern, or (3) combinations of (1) and (2). A symmetry of the second type—a translation followed by a reflection—is called a *glide reflection.*

(b) Any unambiguous notation will do. You can let the students select a notation or, if you prefer, you can introduce the following notation in which types of symmetry are designated by capital script letters as follows:

T—a translation,
H—a reflection about a horizontal line,
V—a reflection about a vertical line,
R—a half turn,
G—a glide reflection.

The first strip pattern has all of the above symmetry types. The second has types ***T*** and ***G***.

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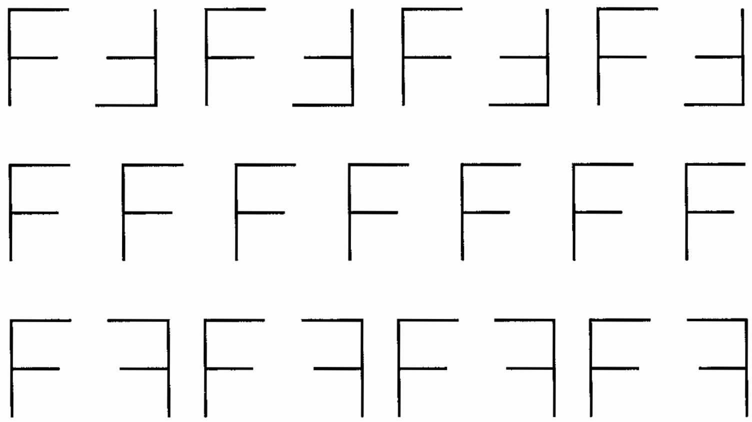
Actions

(c) Show the students the following strip patterns. Ask them to list their symmetries.



Comments

2. *Continued.*

(c) A master of the patterns is attached (Master 3). It can be used to make an overhead transparency and/or copies for the students.

The three strip patterns have, respectively, the following types of symmetry: ***T*** and ***R***; ***T*** only, ***T*** and ***V***.

(d) Ask the students to design a strip pattern which has ***T***, ***G*** and ***R*** type symmetries.

(d) One example is the following:

The pattern also has type ***V*** symmetry. This is the case for every strip pattern which has ***G*** and ***R*** type symmetries, since the combination of a glide reflection followed by a half turn is the same as a reflection about a vertical line.

3. (a) Distribute copies of Activity Sheet X-4. Ask the students to determine the set of symmetries each pattern has.

3. (a) A master of the activity sheet is attached.

The patterns have, respectively, the following sets of symmetries:

T, ***R***, ***V*** and ***G***,
T only,
T and ***V***.

Continued next page.

Actions

(b) Ask the students to design strip patterns which have other sets of symmetries.

Comments

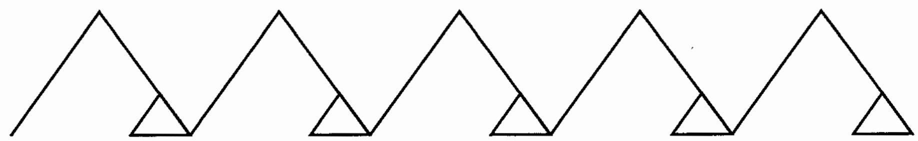
3. *Continued.*

(b) There are seven possible sets of symmetries for a strip pattern. These are:

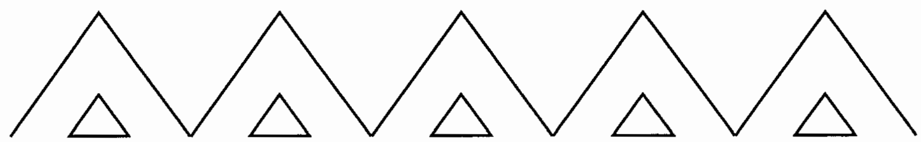
1. ***T*** only
2. ***T*** and ***V***
3. ***T*** and ***R***
4. ***T*** and ***G***
5. ***T***, ***H*** and ***G***
6. ***T***, ***R***, ***V*** and ***G***
7. ***T***, ***R***, ***H***, ***V*** and ***G***

Shown below are examples of strip patterns which have, respectively, these seven sets of symmetries.

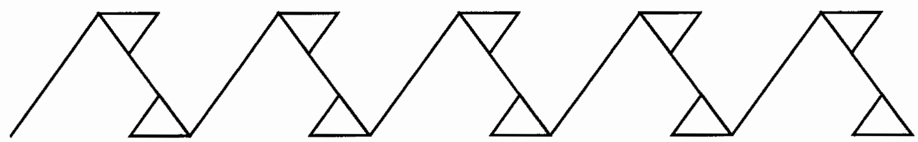
1.



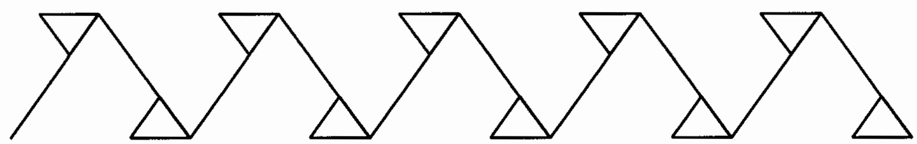
2.



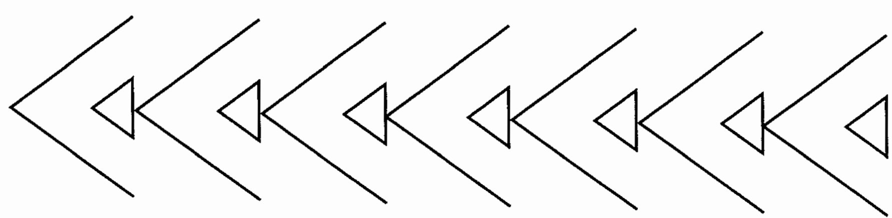
3.



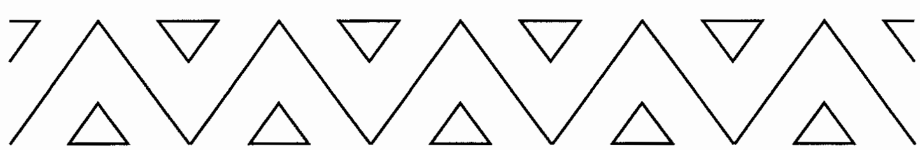
4.



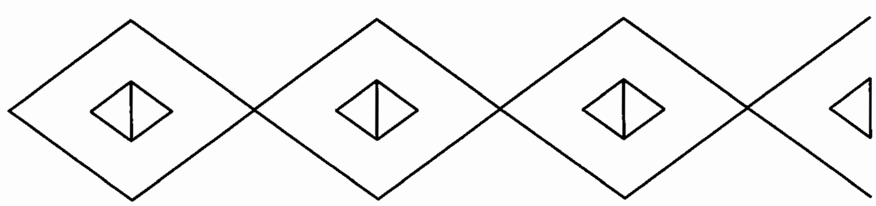
5.



6.



7.



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E E E E E E E

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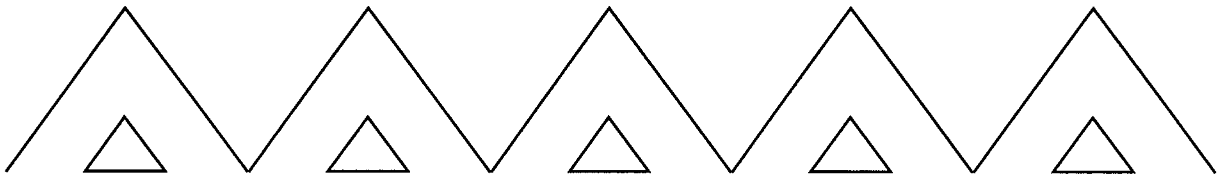
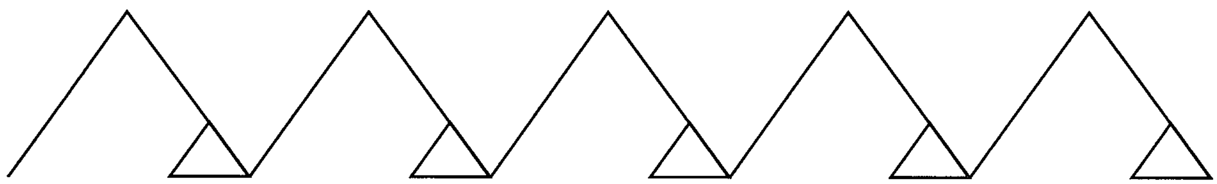
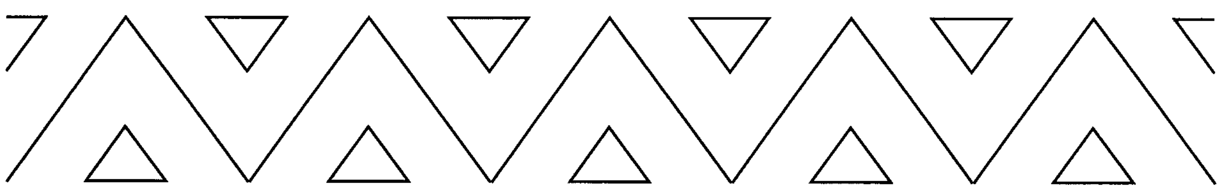
Activity Sheet X-4

Name ______________________________







Combining Shapes

O V E R V I E W

To review and extend ideas about symmetry and develop problem-solving strategies, students explore ways of producing symmetrical figures by joining given shapes together.

Prerequisite Activity

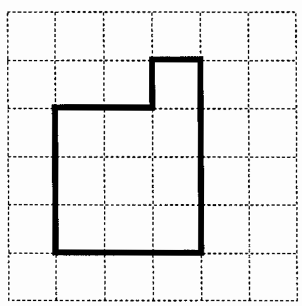
Unit X, Activity 3, *Shapes and Symmetry.*

Materials

Activity sheet, squared and triangular grid paper, scissors.

Actions

1. Draw the following figure on a squared chalkboard or on the overhead.



(a) Ask the students if it is symmetrical.

(b) Ask how the figure could be made symmetrical.

Comments

1. (a) A transparency of squared grid paper may be made from Master 1. Students might need to be reminded when a figure is symmetrical.

(b) Students may suggest first that the small square could be removed. This is certainly acceptable.

If no one suggests it, ask if anything could be "added" to make the figure symmetrical. Any correct suggestions are acceptable, for instance, adding a 2 by 3 rectangle to the right-hand side.

Continued next page.

Actions

(c) Ask the students to determine mentally in how many different ways one square of the grid can be added to the figure to make it symmetrical. Have the students report their conclusions and without revealing the possible positions of the square, describe the system they used to arrive at their answer. Discuss.

(d) When there is some agreement about the number of possible locations for the square, distribute squared grid paper and ask the students to draw the shapes. Discuss the types of symmetry the resulting shapes have.

Comments

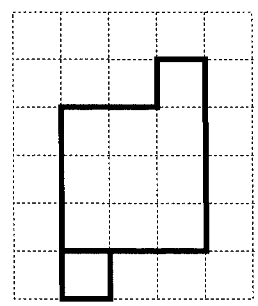
1. *Continued*

(c) One of the aims here is to encourage mental geometry to enhance the students' powers of imagery.

Another aim is to encourage a systematic approach to the problem. One system is to move the small square round the figure, stopping at each position to consider whether a symmetrical figure results.

Discuss any differences in the numbers the students give, again without them revealing the actual locations of the square. One can discuss the idea that even if two students each see four locations, it does not necessarily mean that they see the same four locations.

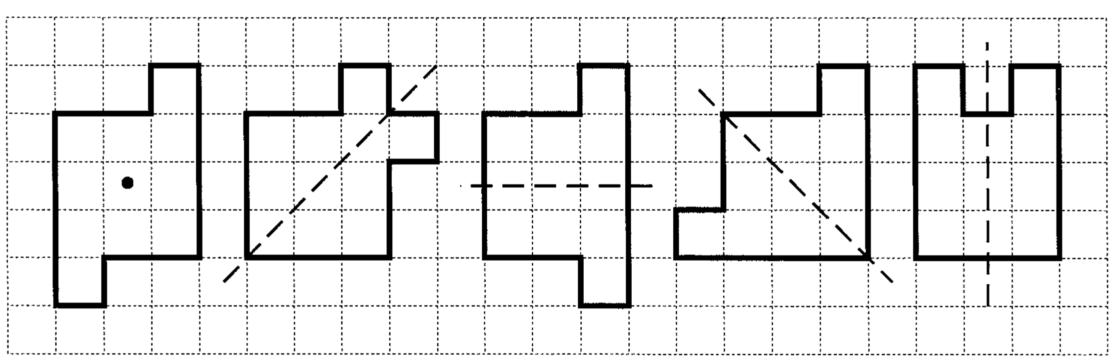
Disagreement may arise because some students do not see the location that leads to rotational symmetry. If you suspect this, you can present this case and ask if the resulting figure is symmetrical.



An addition that creates rotational symmetry.

(d) Squared grid paper can be made from Master 1.

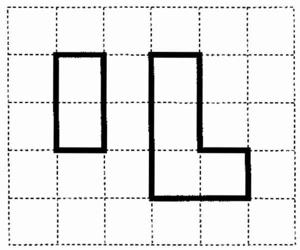
There are five possibilities:



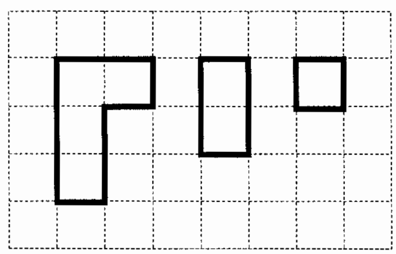
The first of these has rotational symmetry, the others reflective symmetry.

Actions

2. Ask the students to work out, in their heads, how many different symmetrical shapes can be made by joining these two. Discuss the strategies the students use.



3. Show the students the following three shapes.



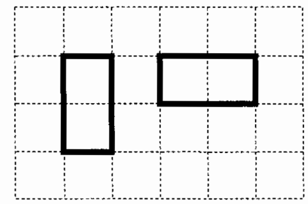
(a) Place cutouts of the shapes on the overhead. Ask a few volunteers to show ways the three shapes can be put together to form a symmetrical shape.

(b) Ask the students to investigate the number of different ways this can be done.

Comments

2. This is a warm-up to Action 3 in which the students are asked to work with three shapes.

Some students may overlook that the oblong can be oriented in two ways.



One strategy is to take each of these orientations of the oblong and mentally place it in different positions around the **L** shape.

3. (a) If there is disagreement whether or not a shape is symmetrical, you can ask if it passes the frame test, i.e., can it be positioned in its frame in more than one way.

(b) Students may wish to work on squared paper or make their own cutouts.

Students will probably begin by working in a random way. As you observe them at work, asking them about their plan of attack may help them become more systematic.

The number of ways is quite large and a complete solution to this problem will take some time. Mathematical problems often take time. Mathematicians may take hours, days, weeks, months or years to solve problems. Some problems take centuries and partial solutions are handed down through the generations. By letting the students experience problems that take more than one class period, they are experiencing one of the realities of problem solving.

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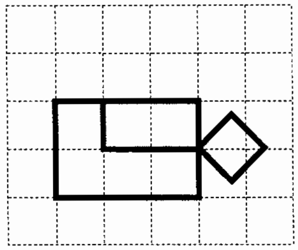
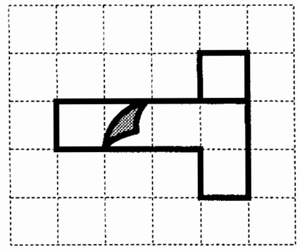
Actions

Comments

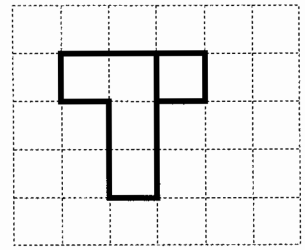
3. (b) *Continued.* Students may ask about the "rules", for example, they may ask if they are allowed to:



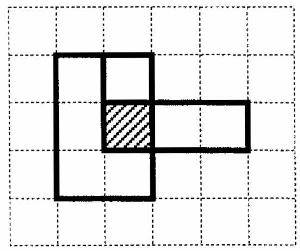
join shapes at the corners?



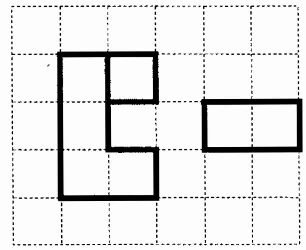
let shapes overlap?



use two shapes?



have gaps?



have shapes that don't touch?

As questions such as these arise, let the students make their own decisions. Encourage them to consider the consequences. For example, if overlaps are allowed, the amount of the overlap may be varied infinitely and the problem becomes unmanageable; if not all three shapes must be used, there will be more possibilities than if all three shapes are always used. Deciding under what conditions an investigation becomes manageable, and determining how the choice of conditions affects the conclusions, are integral parts of carrying out a mathematical investigation.

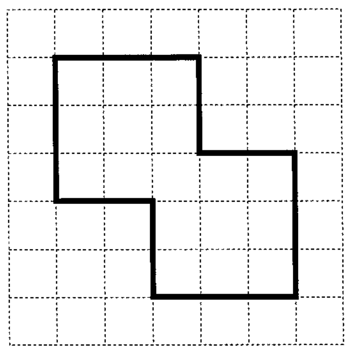
An investigation need not produce complete results. This is of secondary importance to developing a systematic approach, specifying any conditions imposed on the investigation and justifying the conclusions reached.

Actions

4. (a) Distribute copies of Activity Sheet X-5 to the students. Ask them to answer the questions working mentally. Discuss.

(b) Ask the students to create their own problems of this type.

5. Show the students the following shape based on squares.



(a) Ask the students to identify the shape's order of symmetry and the different types of symmetry it possesses.

Comments

4. (a) One strategy is to mentally move the additional square or triangle from one possible location to the next, checking each time whether or not the resulting shape is symmetrical.

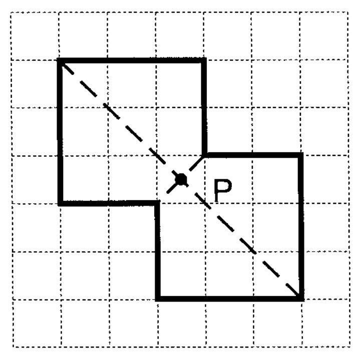
(b) Masters for grid paper—squared or triangular—are attached (Masters 1 and 2).

One challenge is to create shapes that have 1, 2, 3, 4, etc., ways of adding a square or triangle to make them symmetrical.

Students may find that using more complicated shapes tends to limit the number of solutions.

5. (a) You may need to remind the students that the order of symmetry of a shape is the number of different ways it can be positioned in its frame. [See Comment 2(a) in Unit X, Activity 3, *Shapes and Symmetry*.]

If the shape is positioned in its frame, a different positioning of the shape is obtained by flipping it about either of the dashed lines shown below. Thus, the shape has 2-fold reflective symmetry.



Also, a different positioning is obtained by rotating the shape 180° around the point *P*. Rotating the shape through 360° (or 0°) about *P* restores it to its original position. If one adopts the usual convention of referring to this rotation of the shape back to its original position as an instance of rotational symmetry, then there are two cases of rotational symmetry—one involving a rotation of 180° and the other a rotation of 360° (or 0°). Thus, the shape has 2-fold rotational symmetry.

Continued next page.

Actions

(b) Ask the students to investigate the order and types of symmetry of shapes based on squares and/or equilateral triangles.

Comments

5. (a) *Continued.* Altogether, there are 4 ways of positioning the shape in its frame. Thus its order of symmetry is 4. It has 2-fold reflective symmetry and 2-fold rotational symmetry.

(b) Masters for squared and triangular grid paper are attached (Masters 1 and 2).

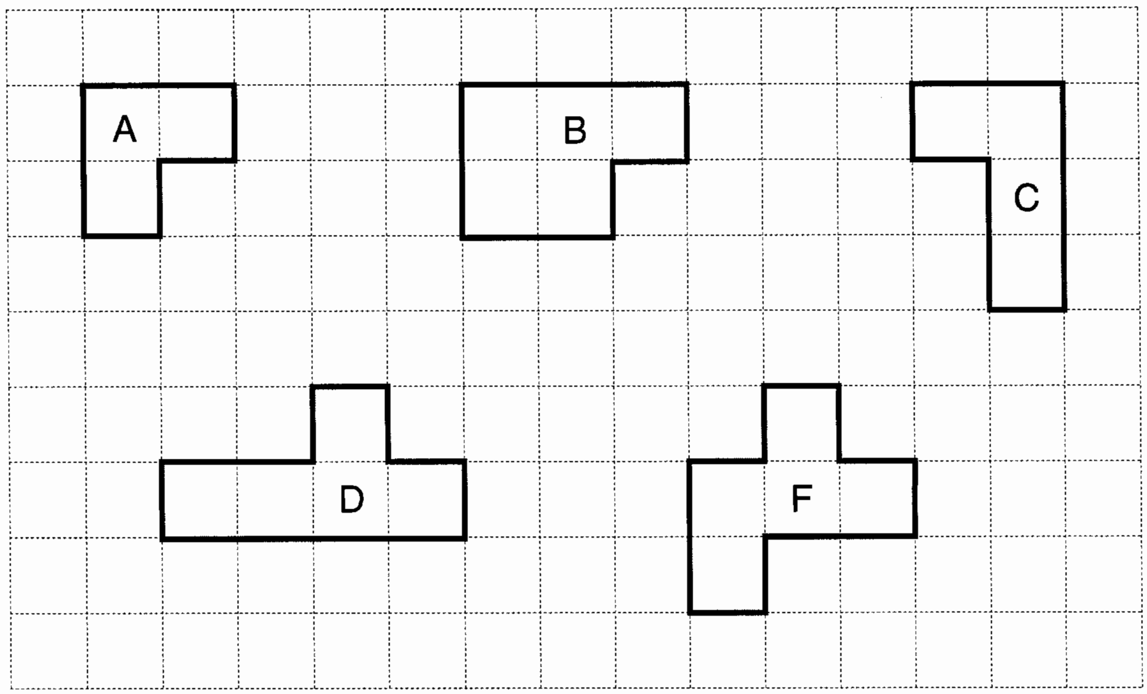
This is intended as an open-ended investigation. In addition to investigating the order and types of symmetry, the students may be interested in investigating related questions, such as:

- For a given order and types of symmetry, what are the smallest number of squares —or equilateral triangles—a shape can have?
- On a squared grid—or a triangular grid—what are possible locations of axes of symmetry? Of centers of rotations?
- What types of symmetry are possible for a shape composed of 2 squares? 3 squares? 4 squares? 5 squares?
- What types of symmetry are possible for a shape composed of 2 equilateral triangles? 3 equilateral triangles? 4 equilateral triangles? 5 equilateral triangles?

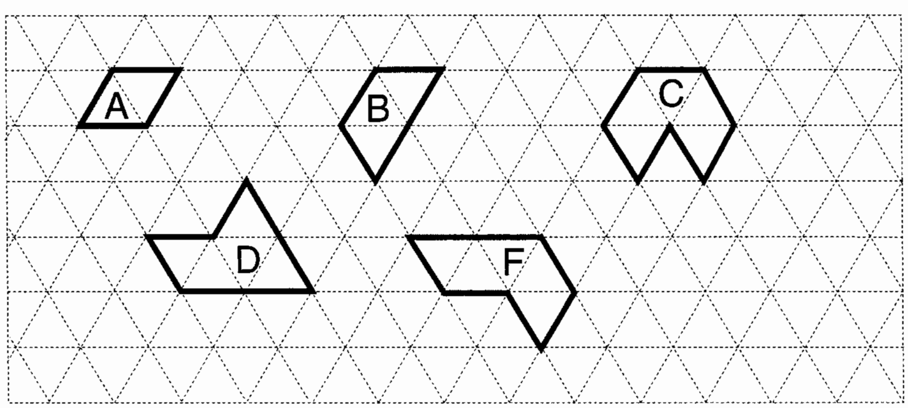
The students may think of other questions as they undertake their investigations.

Name ___________________________

1. For each shape, in how many ways can one square be added to make it symmetrical?



2. For each shape, in how many ways can one triangle be added to make it symmetrical?



Symmetries of Polygons

O V E R V I E W

Students classify hexagons and other polygons according to their symmetries.

Prerequisite Activity

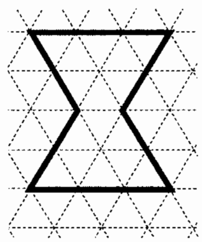
Unit X, Activity 3, *Shapes and Symmetry.*

Materials

Triangular grid paper and squared grid paper.

Actions

1. (a) Show the students the following hexagon. Ask them to describe its symmetries. Then ask them to draw other hexagons which have the same types of symmetries.

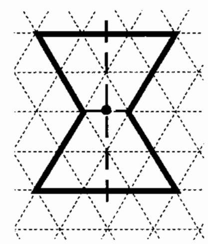


(b) Ask the students to construct examples of hexagons which have: (1) reflective symmetry but no rotational symmetry, (2) rotational symmetry but no reflective symmetry. Discuss their observations.

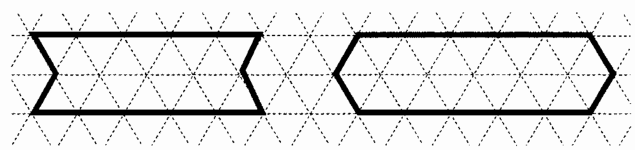
Comments

1. (a) The students may prefer to work on grid paper. Masters for triangular and squared paper are attached to Unit X, Activity 5.

The hexagon has 2-fold reflective and 2-fold rotational symmetry:

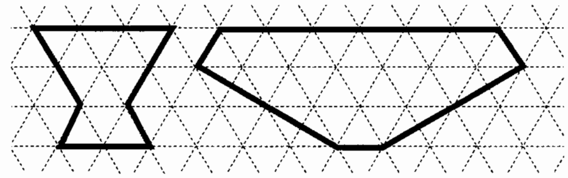


Shown below are two other hexagons, one concave and one convex, with the same types of symmetry.



(b) These hexagons have reflective symmetry and no rotational symmetry:

The students may surmise that it is not possible to have two axes of symmetry and no rotational symmetry (see Comment 2).



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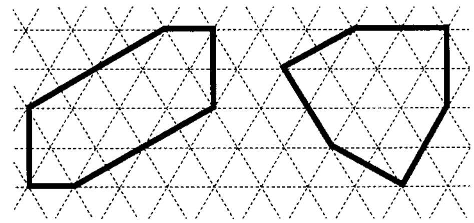
Actions

2. Ask the students to prepare an illustrated chart showing the different types of symmetry a hexagon can have. Discuss the students' conclusions.

Comments

1. (b) *Continued.* The following hexagons have 2-fold and 3-fold rotational symmetry, respectively, but no reflective symmetry.

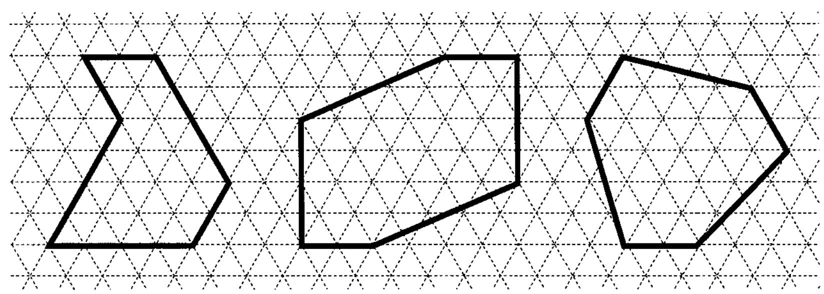


2. The possible orders of symmetry of a hexagon are 1, 2, 3, 4, 6 and 12. You may want to remind the students that the order of symmetry of a shape is the number of different ways it can be positioned in its frame.

It is a moot point whether or not a shape is said to have "no rotational symmetry" or "1-fold rotational symmetry". Whereas the former is more common, the latter is reasonable since a rotation through 360° (or 0°) doesn't change the positioning of a shape in its frame. You can let the students decide on the terminology they wish to use.

The following chart lists, with examples, the possible types of symmetries that a hexagon can have. Note there are seven classifications.

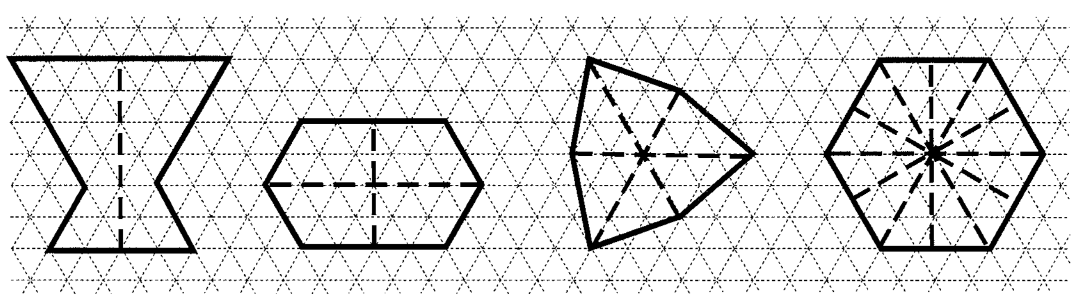
Hexagons Classified by Symmetry Type:



A. no reflective 1-fold rotational

B. no reflective 2-fold rotational

C. no reflective 3-fold rotational



D. 1-fold reflective 1-fold rotational

E. 2-fold reflective 2-fold rotational

F. 3-fold reflective 3-fold rotational

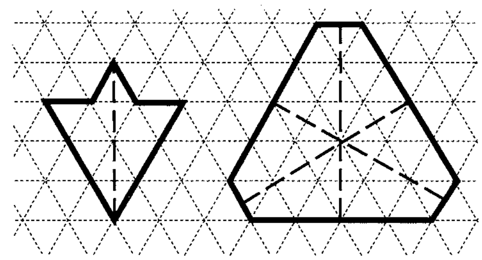
G. 6-fold reflective 6-fold rotational

Continued next page.

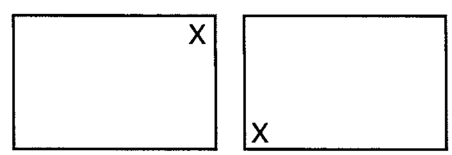
Actions

Comments

2. *Continued.* The example of type D symmetry shown above is a hexagon whose axis of symmetry connects midpoints of opposite sides. Other hexagons which have type D symmetry have an axis of symmetry which connects opposite vertices. An example of such a hexagon is shown on the left below. On the right is an example of a hexagon which has type F symmetry and has axes of symmetry connecting midpoints of opposite sides. The example of type F symmetry given above has axes of symmetry which connect opposite vertices.



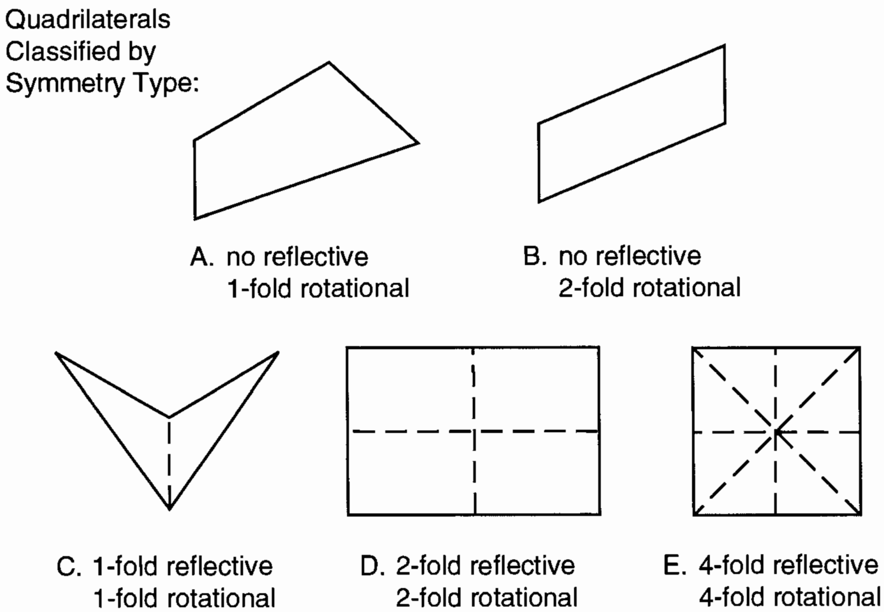
The students may observe that a shape which has 2 or more axes of symmetry also has rotational symmetry. The reason is that changing the position of a shape by first flipping it about one axis and then a second has the same effect as rotating the shape through an angle twice that of the angle between the axes. This can be illustrated by marking an X in the corner of a note card. Flipping the card about its horizontal axis and then its vertical axis (or the other way about) will move the card from the first position shown to the second. The same change can be achieved by rotating the card 180°.



Actions

Comments

3. Repeat Action 2 for quadrilaterals.



3. There are five symmetry types for quadrilaterals as illustrated to the left.

You can discuss with the students the type of symmetry various quadrilaterals have. Squares, and only squares, have type E symmetry. Rectangles and rhombi (other than squares) have type D symmetry (note that the axes of symmetries of rectangles connect midpoints of side and those of rhombi connect vertices). Kites, both convex (other than rhombi) and concave, and isosceles trapezoids have type C symmetry. Parallelograms (other than rectangles and rhombi), and only parallelograms, have type B symmetry.

4. Ask the students to classify triangles according to their symmetries. Compare this classification with other classifications of triangles.

4. There are three symmetry types for triangles:
- no reflective and 1-fold rotational;
- 1-fold reflective and 1-fold rotational;
- 3-fold reflective and 3-fold rotational.

The classification of triangles by symmetry type corresponds to the classification of triangles as *scalene*, i.e., no equal sides; *isosceles*, i.e., 2 equal sides; or *equilateral*, i.e., 3 equal sides.

5. (Optional) Repeat Action 2 for pentagons and/or octagons.

5. There are only three symmetry types for pentagons:
- no reflective and 1-fold rotational;
- 1-fold reflective and 1-fold rotational;
- 5-fold reflective and 5-fold rotational.

The situation is similar for any polygon with a prime number of sides.

Octagons, like hexagons, have seven symmetry types.

Polyominoes and Polyamonds

O V E R V I E W

The students consider shapes made by joining together squares or equilateral triangles, and some tessellations based on these shapes. They classify the shapes by symmetry and extend symmetry concepts to tessellations.

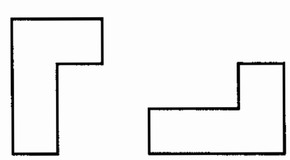
Prerequisite Activity
Unit X, Activity 6, *Symmetries of Polygons*.

Materials
Squared and triangular grid paper, scissors, transparencies as indicated (see Actions 5 and 7).

Actions

1. (a) Distribute grid paper and scissors. Ask the students to cut out a shape made from 4 squares. Ask them to write their names on their shapes. Collect the shapes and display them.

(b) Ask if any of the shapes are the same. For each pair that are alleged to be the same, ask why. Discuss.



Comments

1. (a) The squares of the grid paper should be large enough so students can see the displayed shapes (2-cm or 1" grid paper works well). A master for 2-cm paper is attached.

Some students may cut out shapes made from squares joined at corners and not at edges. They can be included for purposes of this Action. In the next Action, shapes will be limited to those made of squares joined edgewise.

The students' names serve two purposes: one, they identify the creator of the shape; two, they indicate which way is up when the shapes are displayed.

The shapes can be displayed by pinning them to a bulletin board or taping them to a chalkboard or some other surface.

(b) "The same" is purposely ambiguous. Students may disagree on when two shapes are "the same"; some students may say the two shapes to the left are the same because they can be made to coincide, others may claim they are different because they differ in orientation. At this stage, it is not important that all students adopt the same criterion for "sameness". It is more important that a student's criterion is unambiguous and clear to others.

If a student experiences difficulty in describing why two shapes are the same, the question "How can I make this shape look like that shape?" may be useful.

Continued next page.

Actions

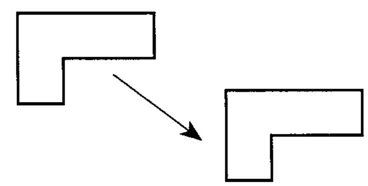
(c) Show the students a pair of shapes for which one shape can be moved to coincide with the other. Ask the students to identify the motions needed to do this. Tell them that these motions are called *transformations*. Where two successive transformations are suggested, ask if they can be replaced by just one. Discuss.

Comments

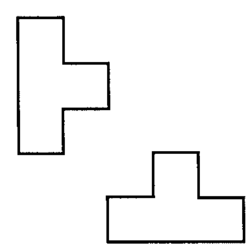
1. *Continued.*
(c) The shapes can be displayed on a bulletin board or shown on the overhead.

The students' responses may be ambiguous, such as "Turn it around" or "Turn it over." One way to help students sharpen their language is to deliberately misunderstand. If, by "turn it around," the student means a rotation about a point, flip the shape about an axis, or vice versa. If the student intends a flip and doesn't mention an axis, flip about a different axis than the one intended. Gradually the students' answers will become more precise.

In addition to rotating or flipping, a shape can be transformed by sliding it to a different position. A slide is also called a *translation.*



The transformations that will move one shape to coincide with another are not unique. For the pair shown below, the shape on the left can be transformed to the one on the right by either (1) rotating it 90°



to the right and then flipping it about a horizontal axis, (2) rotating it 90° to the left and then flipping it about a vertical axis or (3) making just one transformation, namely, a flip about an oblique axis.

Note that performing transformations is not commutative; doing a flip and then a rotation gives a different result than doing the rotation first and then the flip.

Shapes that are transformable into each other are called *congruent.*

Actions

2. (a) Discard all duplicates (i.e., shapes which are congruent to other shapes) and all shapes in which the squares are not joined edgewise. Ask the students if there are any other shapes made from 4 squares joined edgewise. Display any new shapes found.

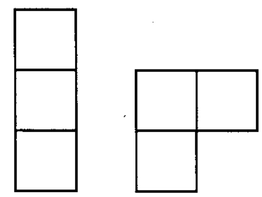
(b) Ask the students to explain how they know they have found all the tetrominoes.

3. Ask the students to construct all the pentominoes.

Comments

2. (a) A shape made of 4 squares joined edgewise is called a *tetromino* from an analogy with *domino*, a shape made of 2 squares joined edgewise. A 3-square shape is called a *triomino* or *tromino*, a 5-square shape, a *pentomino,* and so on.

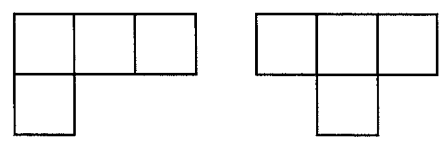
(b) One approach is to consider the ways in which 1 more square can be attached to the only two possible arrangements of 3 squares:



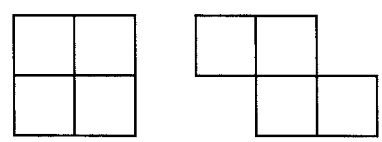
Attach 1 more square to these arrangements.

Another is to consider the cases in which 4 squares are in a row,

then the cases where the maximum number of squares in a row or column is 3,



and, finally, the cases where the maximum in a row or column is 2.



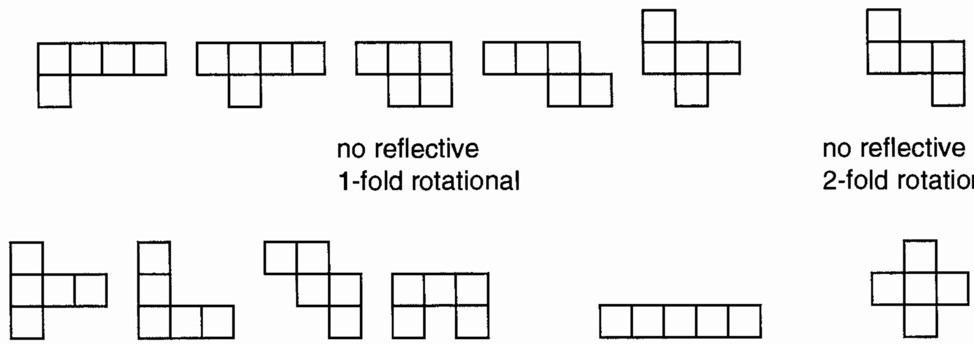
3. There are twelve of these; the students can establish this themselves.

Actions

4. (a) Ask the students to classify the pentominoes by symmetry type.

Comments

4. (a) The twelve pentominoes fall into five different symmetry types. Note that half of the pentominoes have no reflective symmetry and half do.



(b) Repeat Action 4(a) for tetrominoes.

(b) Note how the tetrominoes fit one into each of the five types of symmetry listed above.

(c) For each of the five symmetry types found in (a), ask the students to see if they construct a hexomino of that type.

(c) Shown below are hexominoes for four of the types. No hexomino has the fifth type.

5. (a) Place a transparency of Master 1 on the overhead. Tell the students they are to imagine that the pattern shown continues infinitely in all directions. Ask the students for their observations.

5. (a) This is an example of a *tiling* or *tessellation*, that is, a covering of the plane without gaps or overlaps by a set of one or more congruent figures.

If nothing is forthcoming from the students, you can ask specific questions. For example: What shape is the tessellation made from? Are there other ways of arranging this shape to get a tessellation? How many shapes come together at each vertex?

(b) Place a second transparency of Master 1 on the overhead so it coincides with the first one. Ask for volunteers to move the second transparency to another position in which it coincides with the first.

(b) There are a number of ways to do this. The top transparency can be slid to a new location, it can be flipped over, it can be rotated or it can be moved by a combination of these motions. After three or four examples you can move on to the next part of this Action.

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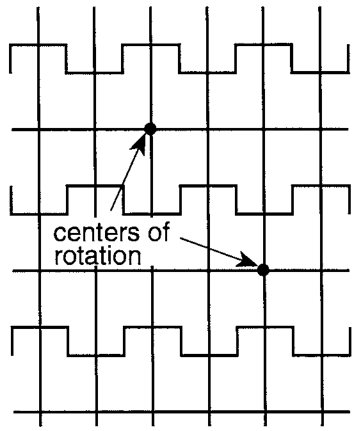
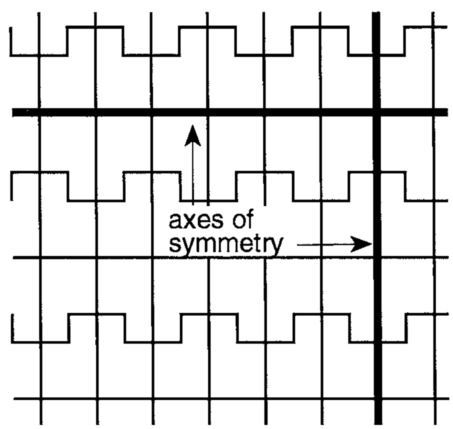
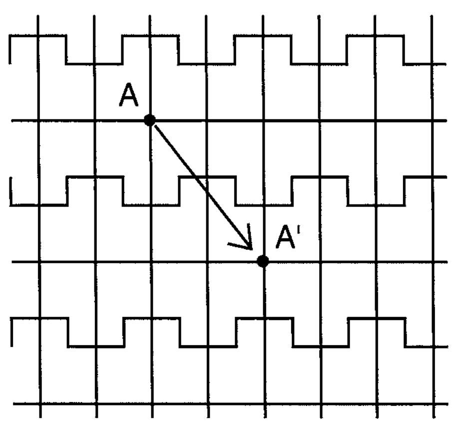
Actions

(c) Tell the students when a tessellation is said to be *symmetrical.* Ask them to describe some of the symmetries the tessellation of part (a) has.







Comments

5. *Continued.*
(c) A tessellation is *symmetrical* if a tracing of the tessellation can be moved from one position in which it is coincident with the tessellation to another position coincident with it.

The tessellation of part (a) has *rotational* symmetry—a tracing can be rotated 180° about either of the points indicated below (or a similar point) and the resulting position of the tracing will again be coincident with the tessellation.

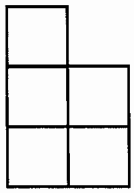
It also has *reflective* symmetry—a tracing can be flipped about any vertical or horizontal line of the tessellation and the result will be coincident with the tracing.

In addition, the tessellation has *translational* symmetry—for example, a tracing can be moved by sliding the point of the tracing coinciding with point *A* along the arrow shown, without any rotation, until it reaches point *A*'. Such a motion is called a slide or a translation.

Other symmetries exist which are combinations of the symmetries described above, for example, a translation followed by a flip.

Actions

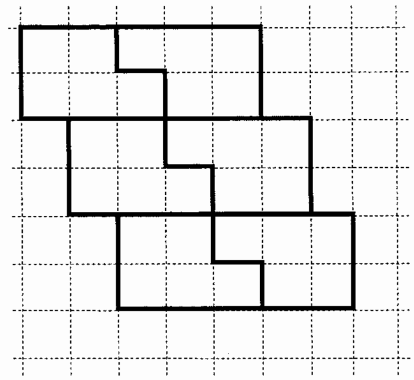
6. (a) Hand out squared grid paper to the students. Show them the pentomino below, which is the basic tile for the tessellation of Action 5. Ask them to draw other tessellations based on this tile, including one that has no reflective symmetry.

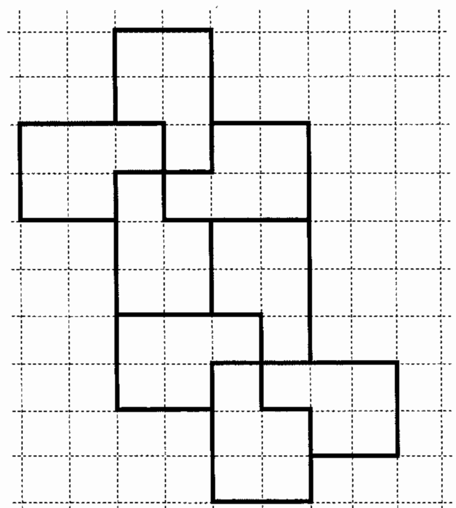


(b) (Optional) Ask the students to draw tessellations based on other pentominoes.

Comments

6. (a) Here are two possibilities that have no reflective symmetry:





Some students may make patterns that have gaps in them or are non-repeating. In this case, remind the students that a tessellation covers all points of the plane, so it has no gaps, and it has translational symmetry, which means it has a repetitive pattern.

(b) This can serve as a special project. The activity can be focused by asking the students to draw tessellations whose symmetries satisfy certain constraints. For example, to draw tessellations which, other than translational symmetry, have:

- only 360° rotational symmetry;
- only 2-fold rotational symmetries;
- only 4-fold rotational symmetries;
- axes of symmetry in one direction only;
- 4-fold rotational symmetries and axes of symmetries in 2 directions.

Actions

7. (Optional)
(a) Hand out triangular grid paper to the students. Ask them to cut out all shapes made of 2, 3, 4 or 5 triangles joined edgewise. Ask them to classify these shapes by their symmetries.

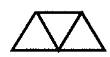
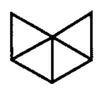
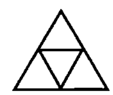
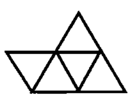
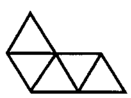
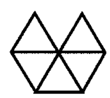
Comments

7. (a) A master for triangular grid paper is attached.

Shapes made from equilateral triangles are named analogously to the *diamond*, which is made from 2 triangles. So we have *triamond*, *tetramond*, *pentamond*, *hexamond*, etc. In general, a shape made of equilateral triangles joined edgewise is called a *polyamond.*

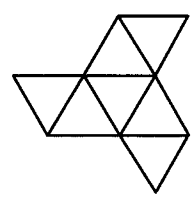
There are 1 diamond, 1 triamond, 3 tetramonds and 4 pentamonds. They are classified in the table below by their symmetries. The students may wish to find the hexamonds, of which there are 12, and enlarge the table to contain them.

Symmetries

	no reflective 1-fold rotational	no reflective 2-fold rotational	1-fold reflective 1-fold rotational	2-fold reflective 2-fold rotational	3-fold reflective 3-fold rotational
triamond and diamond					
tetramonds					
pentamonds	 		 		

(b) Point out to the students that the triangular tetramond has 3-fold rotational symmetry and 3-fold reflective symmetry. Ask them to find a polyamond that has 3-fold rotational symmetry but no reflective symmetry.

(b) The smallest polymond that has 3-fold rotational symmetry without reflective symmetry contains 7 triangles:

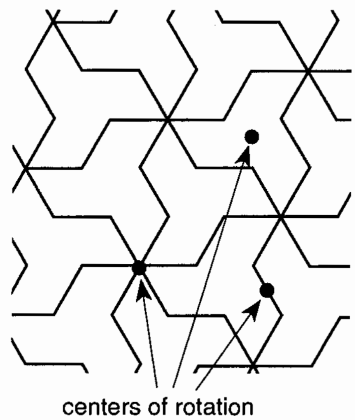


Continued next page.

(c) Show the students the tessellation on Master 2. Ask them to describe its symmetries.

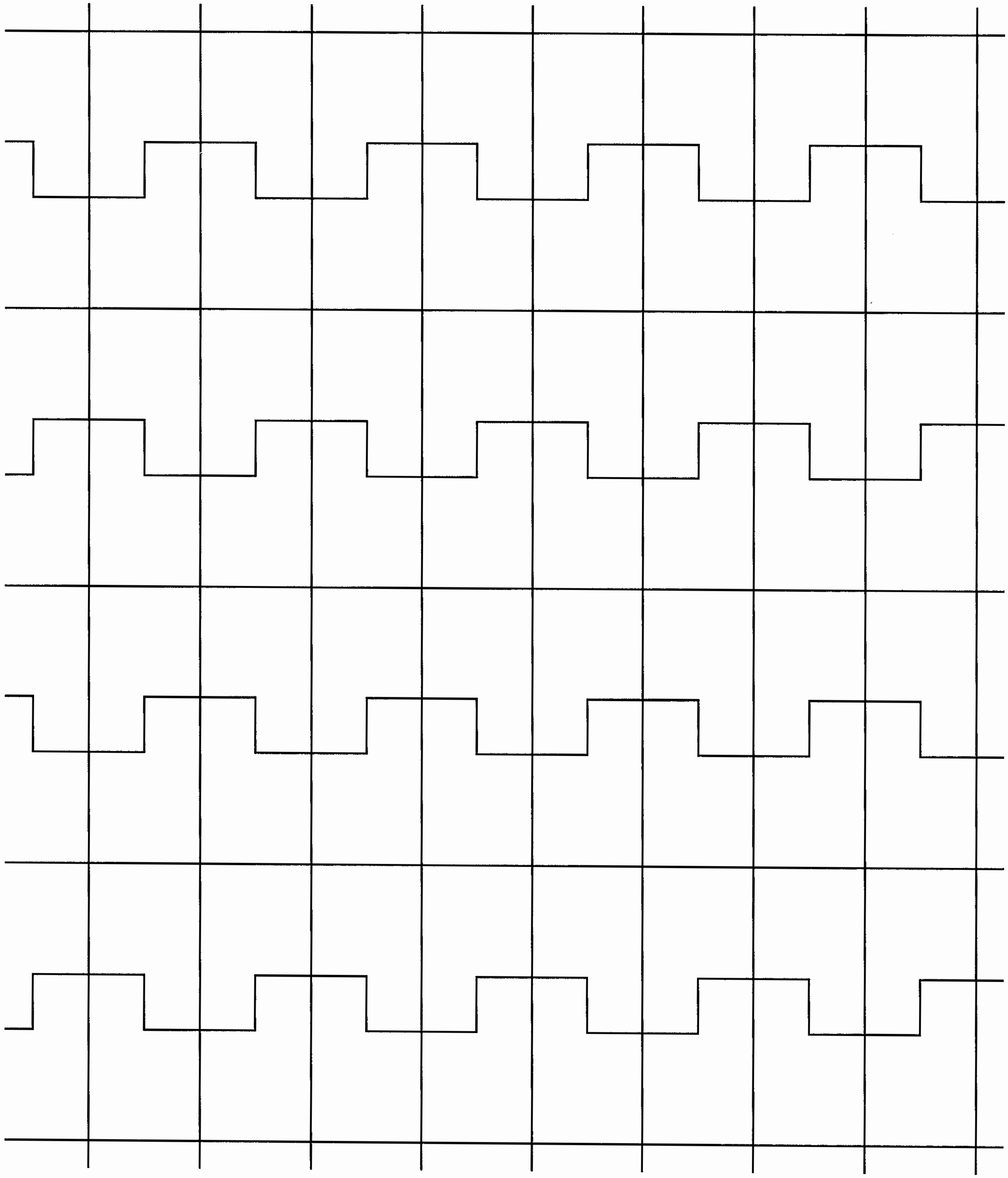
7. Continued.

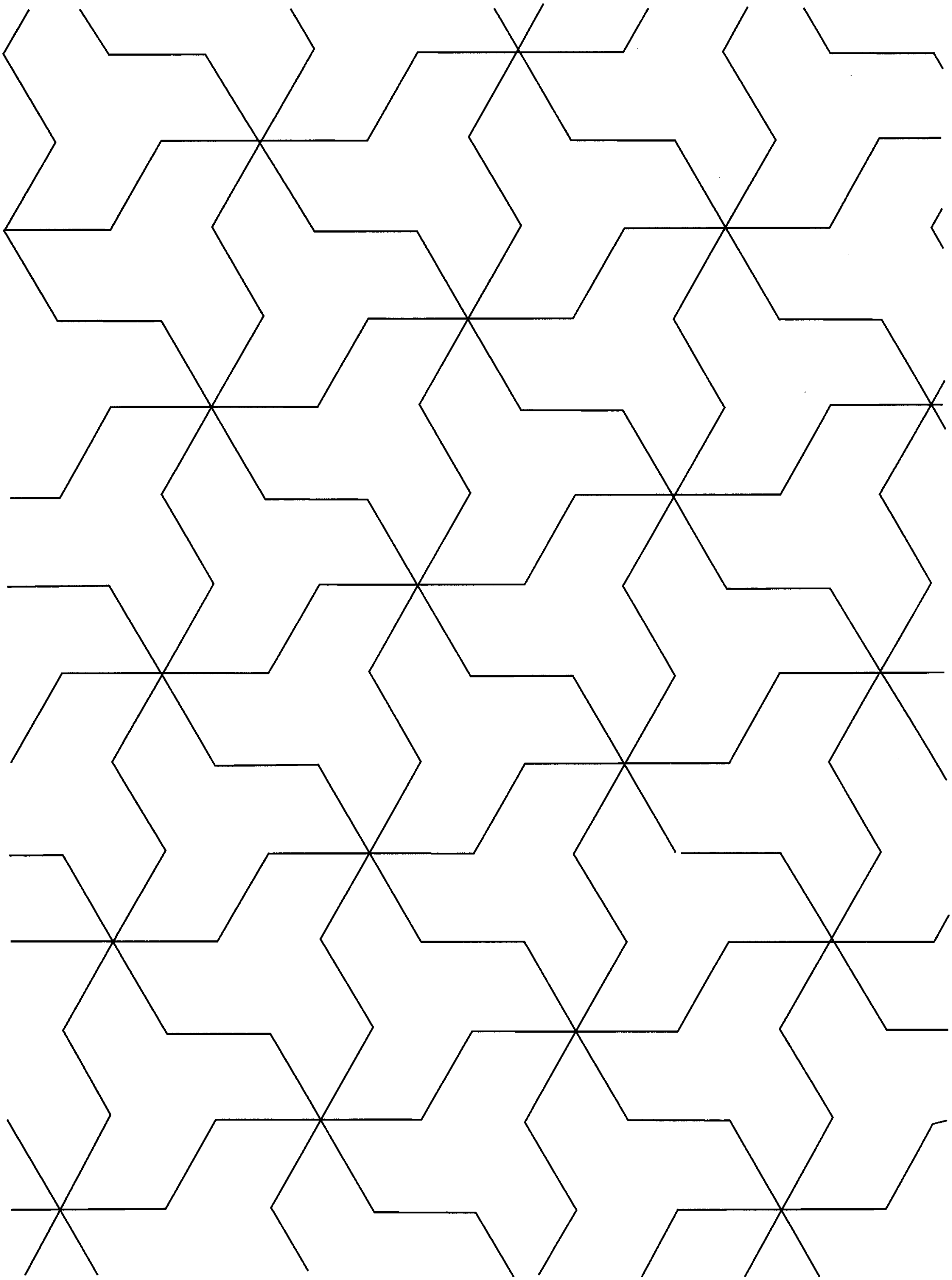
(c) The basic tile of the tessellation is the heptamond shown in Comment 7. In addition to translational symmetry, the tessellation has 2-fold, 3-fold and 6-fold rotational symmetries. This can be demonstrated by taking a second transparency, placing it to coincide with the first transparency and rotating it about the centers of rotation of the tessellation. A rotation can be facilitated by sticking a pin through its center of rotation.



(d) Ask the students to draw other tessellations based on polyamonds and describe their symmetries.

(d) This, again, can be a special project. The students may wish to display their work and describe how they developed their tessellations.



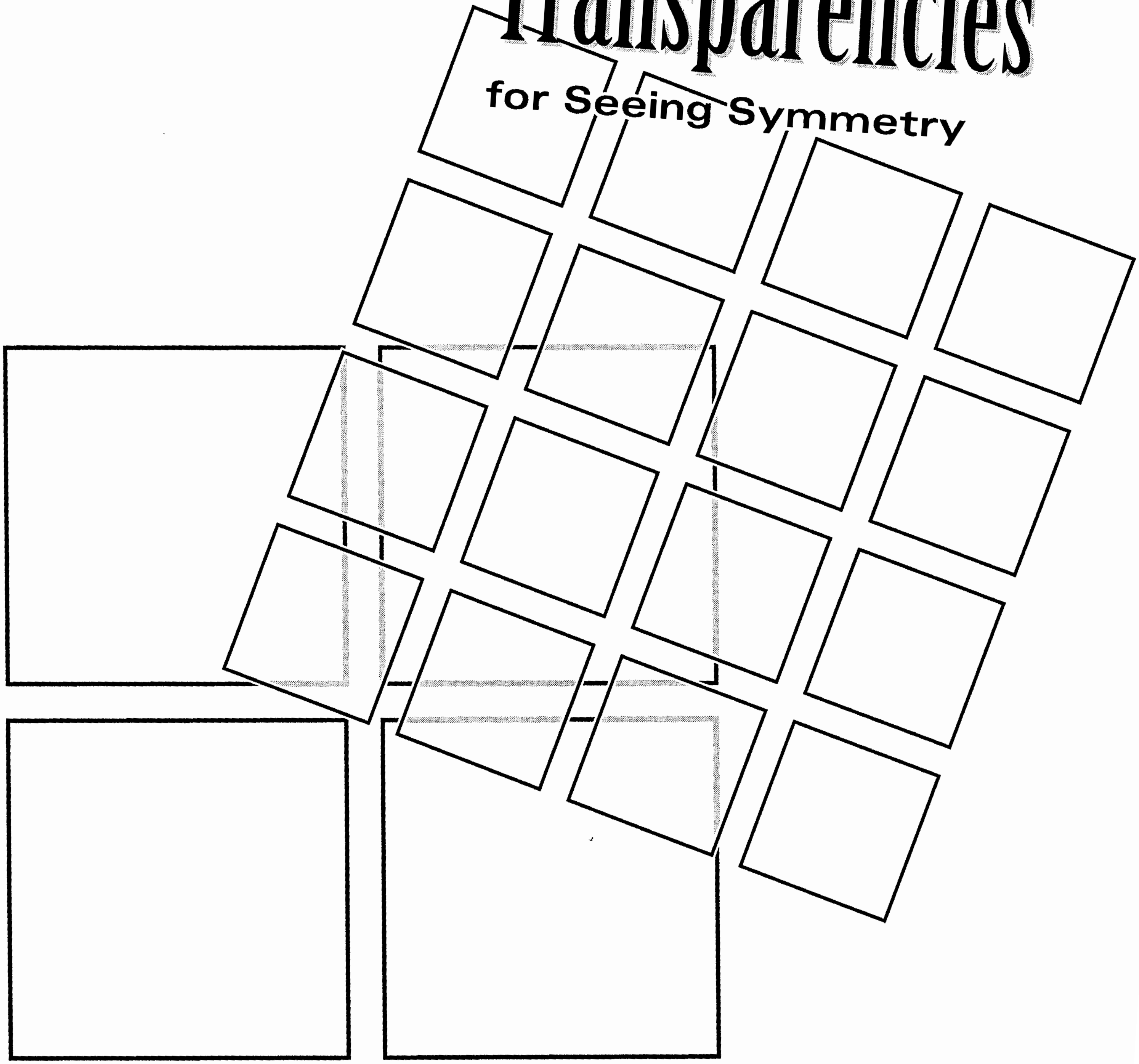


2-cm Grid Paper

Unit X / Math and the Mind's Eye

Transparencies

for Seeing Symmetry



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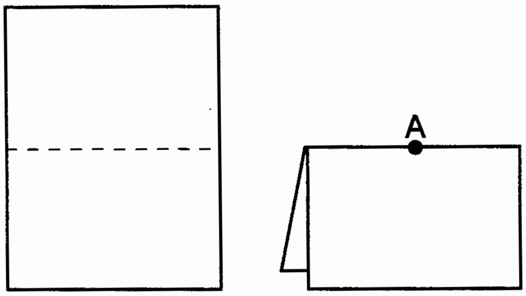
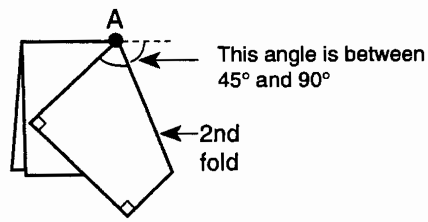
Catalog #MET10

Name ____________________________

Fold a piece of paper in half and mark a point *A* on the fold as shown here:



Now, make a second fold, through *A*, at an angle between 45° and 90° to the first fold:



1. By making a single, straight cut across the folds, it is possible to unfold the cutoff piece into an arrowhead. Describe how to make such a cut.

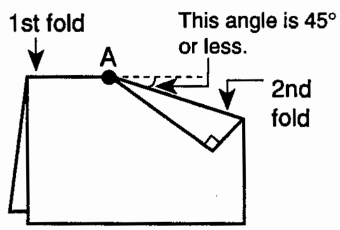
2. How should the cut be made so as to unfold a kite that is not a concave kite?

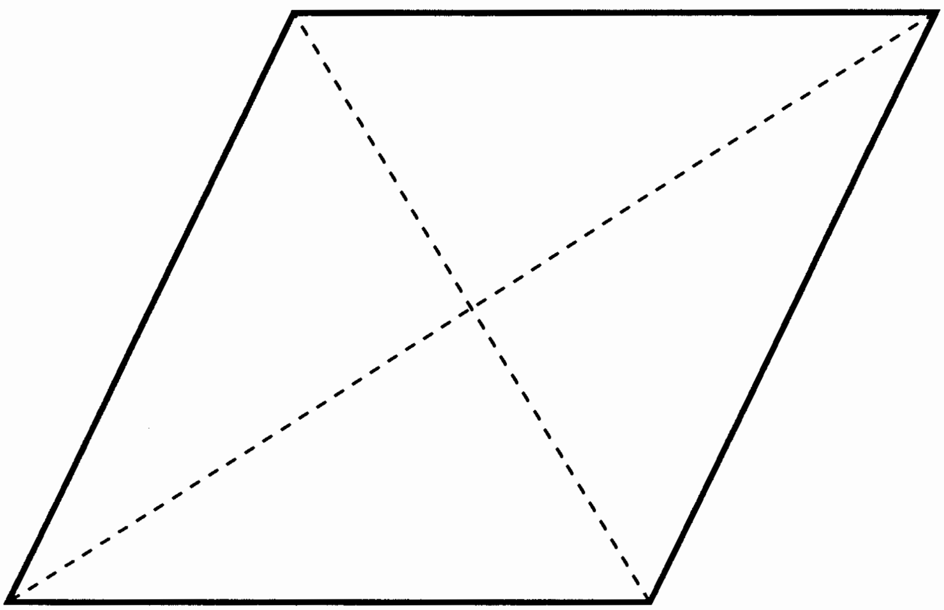
3. How should the cut be made so as to unfold a triangle?

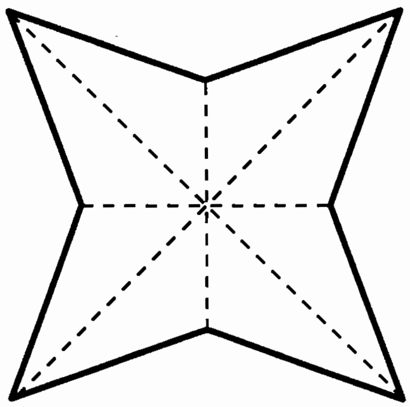
Name ______________________________

4. Describe how to make the second fold and the cut so as to unfold an equilateral triangle.

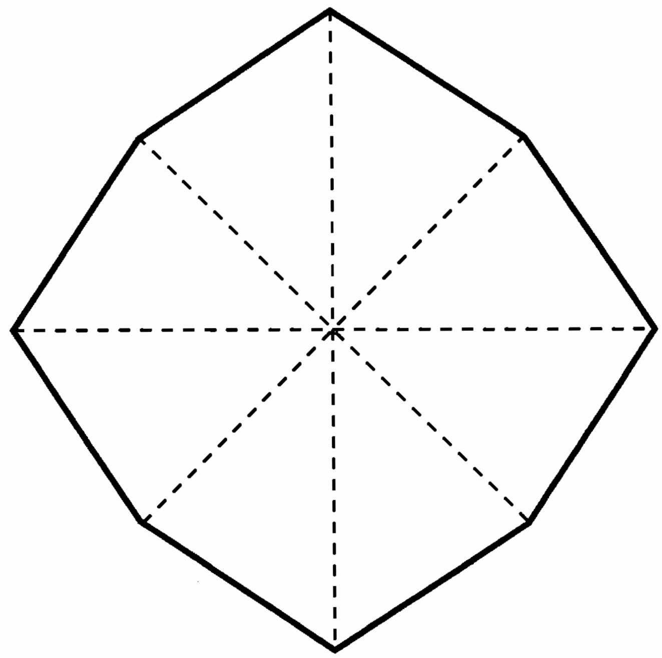
5. Investigate this situation: Make the second fold through *A* at an angle of 45° or less to the first fold (as illustrated below). What shapes(s) can be formed by making a single, straight cut across the folds and unfolding the cutoff piece?



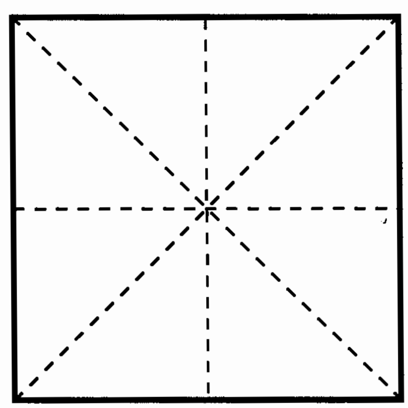




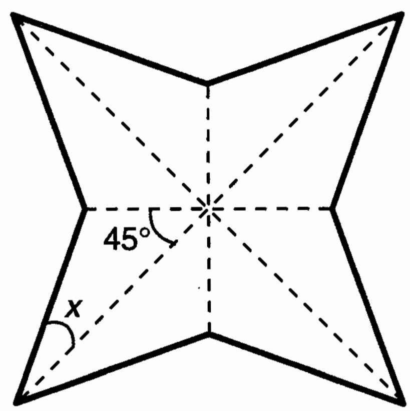
(a)

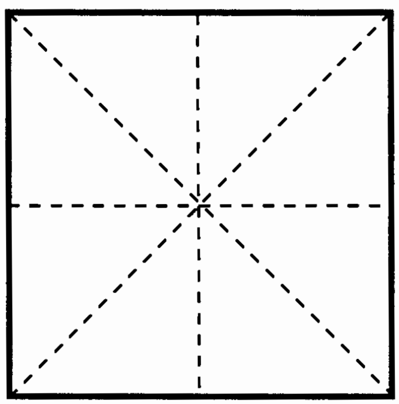


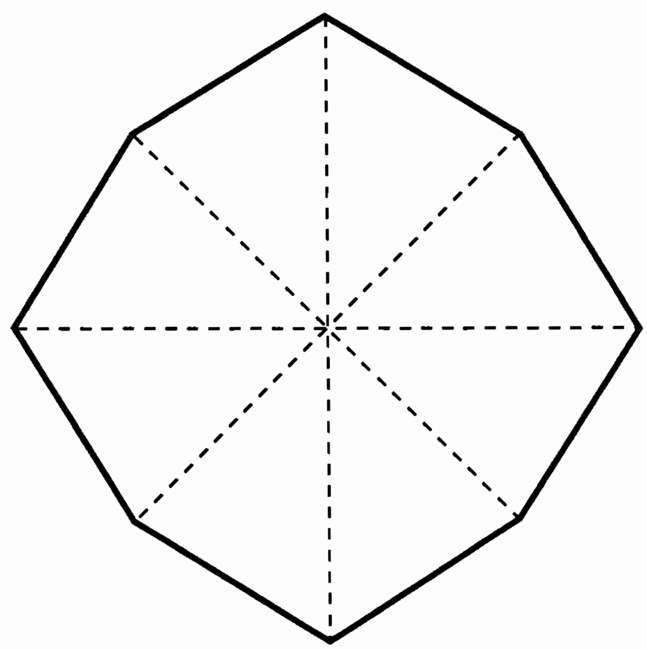
(b)

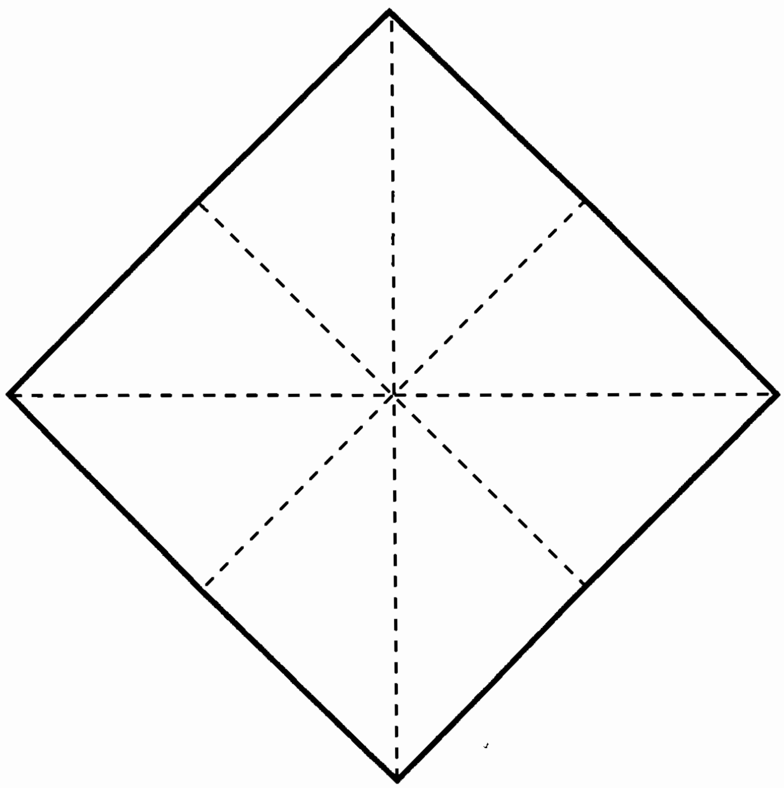


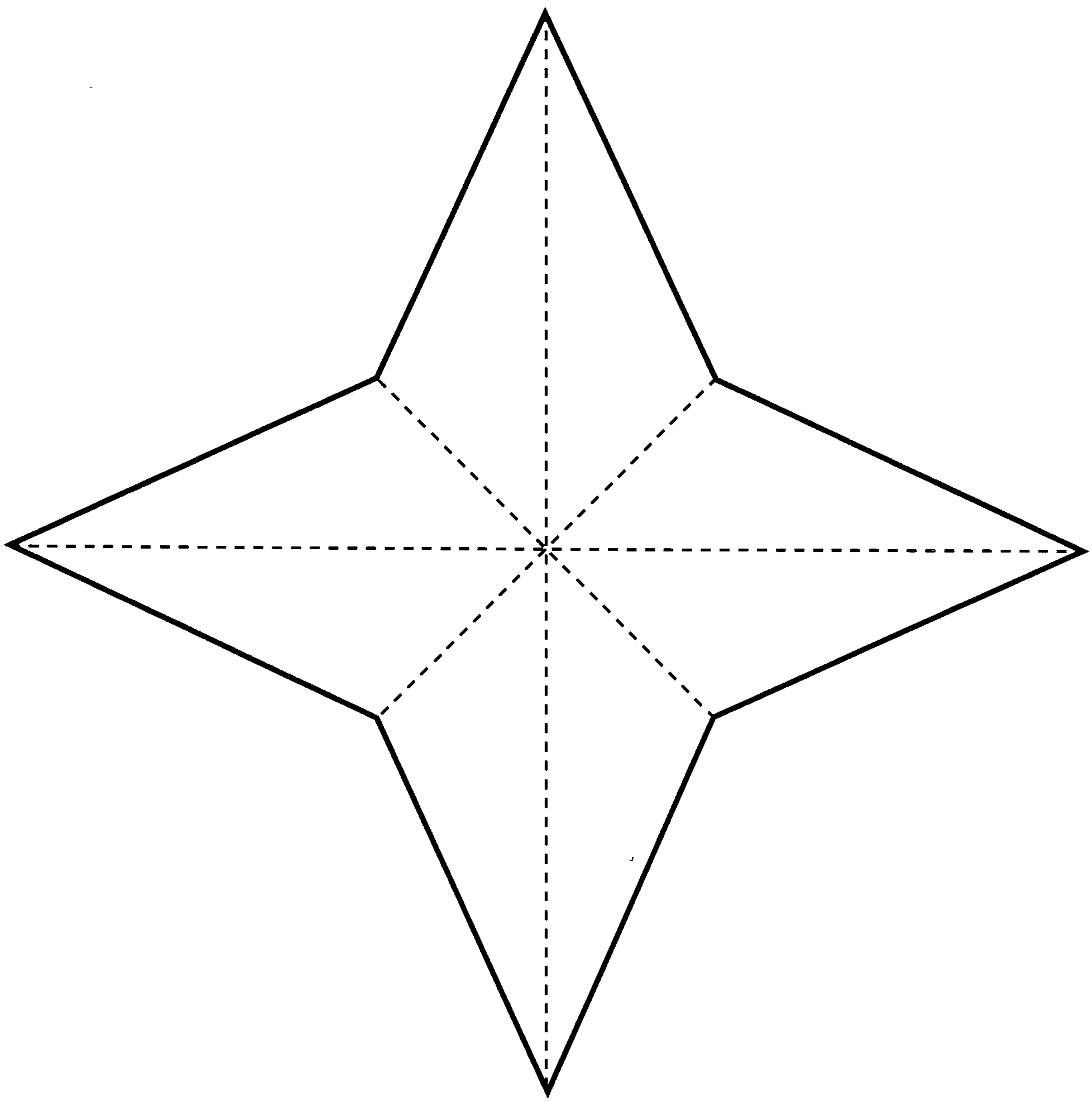
(c)

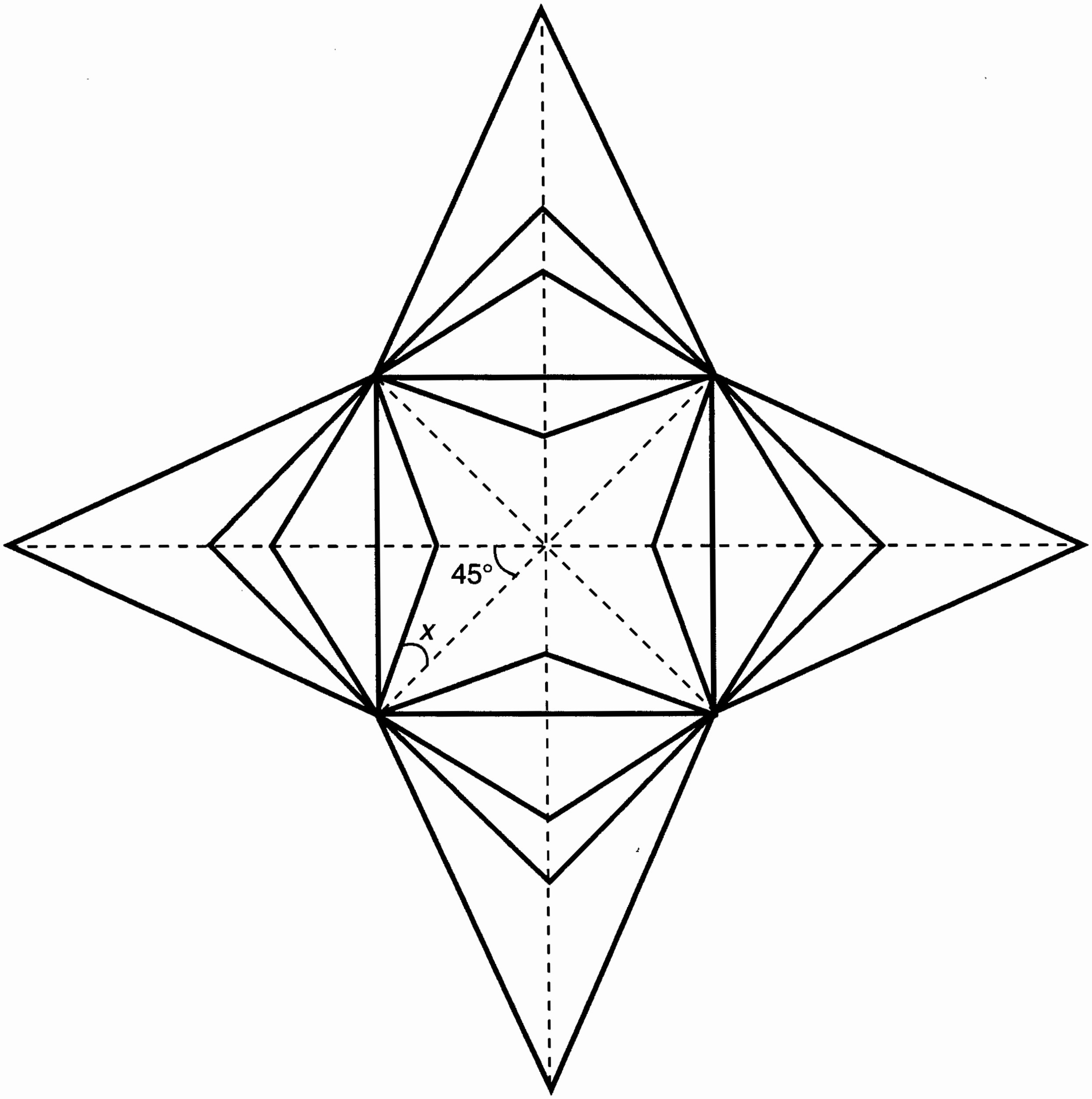

45°
x

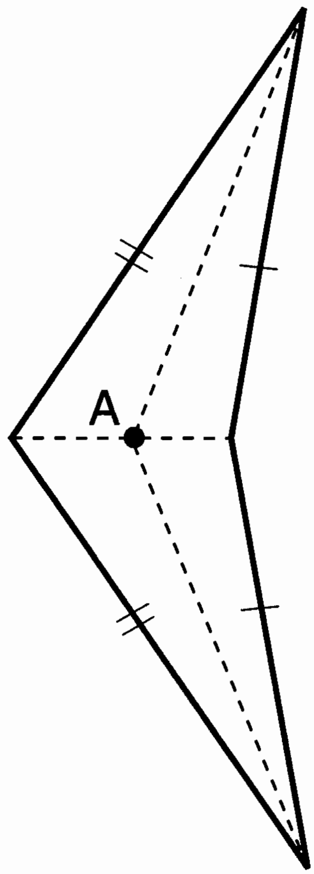




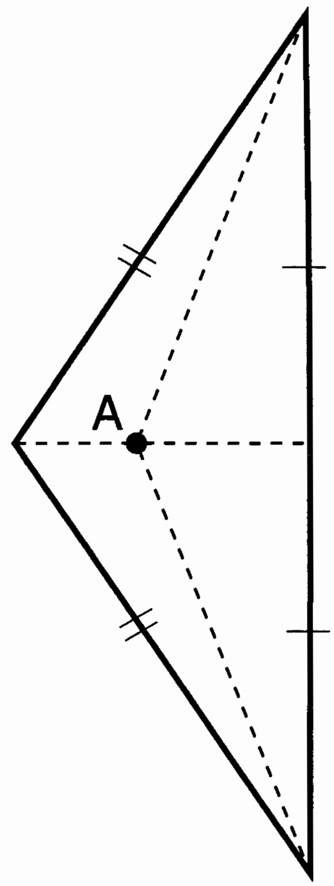




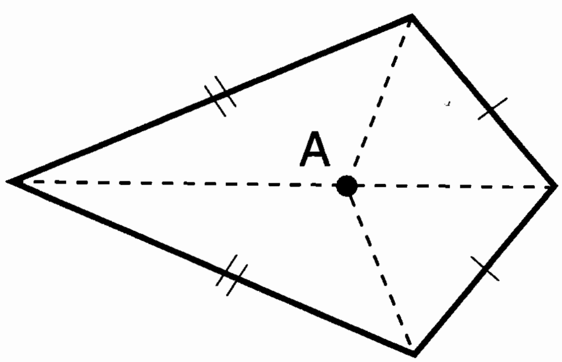

45°
x



(a)



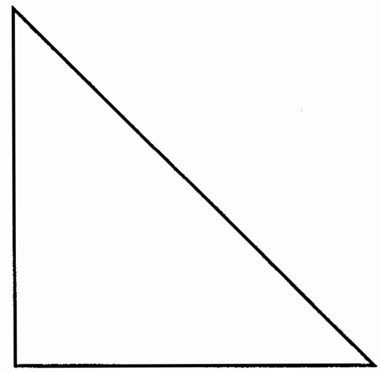
(b)



(c)

Name ______________________

Activity Sheet X-2-A



I

II

Name ______________________

Activity Sheet X-2-B
page 1

	Column 1 **Right Triangle**	Column 2 **Parallelogram**	Column 3 **Hexagon**
Square			
Rhombus (not a square)			
Concave Kite			
Convex Kite (not a rhombus)			
Parallelogram (not a rhombus)			
Isosceles Triangle			
Equilateral Triangle			
Pentagon			
Hexagon			
Decagon			

Name ___________________________

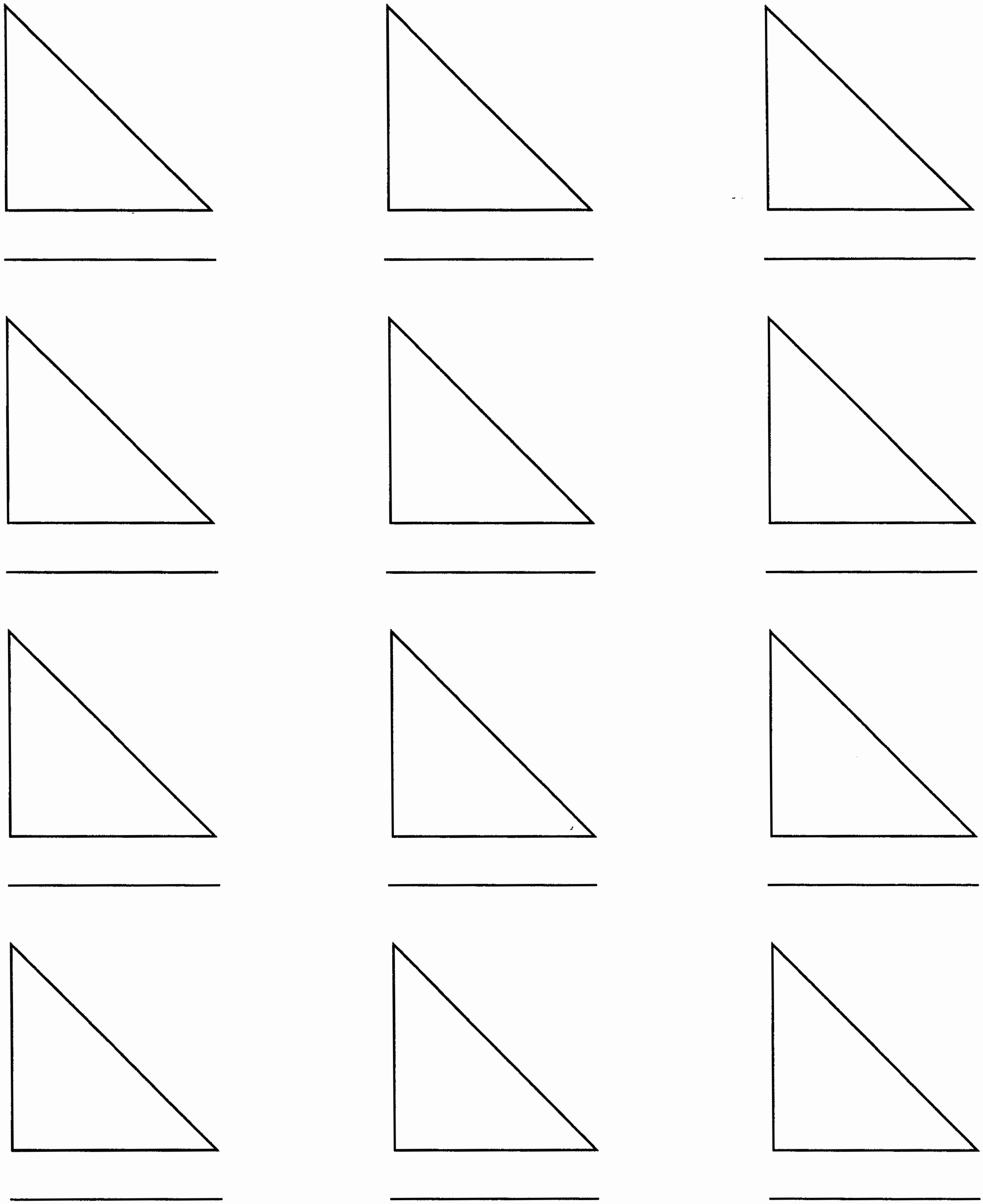
Mirrors on Isosceles Triangles

1. If, for a shape listed on page 1, a mirror can be placed upright on an isosceles right triangle so that shape can be seen, write "Yes" in the appropriate space in Column 1, page 1. Indicate on one of the triangles on page 3 where the mirror should be placed to see the shape and write the name of the shape beneath the triangle.

2. In the remaining spaces in Column 1, page 1, indicate briefly why you believe the mirror cannot be placed to see the shape.

3. Pick a shape, other than a square or rhombus, and write directions for placing a mirror on an isosceles right triangle so that shape can be seen. Your directions can be written below or on the back of this sheet.

Name ______________________________



Name ___________________________

Mirrors on Parallelograms

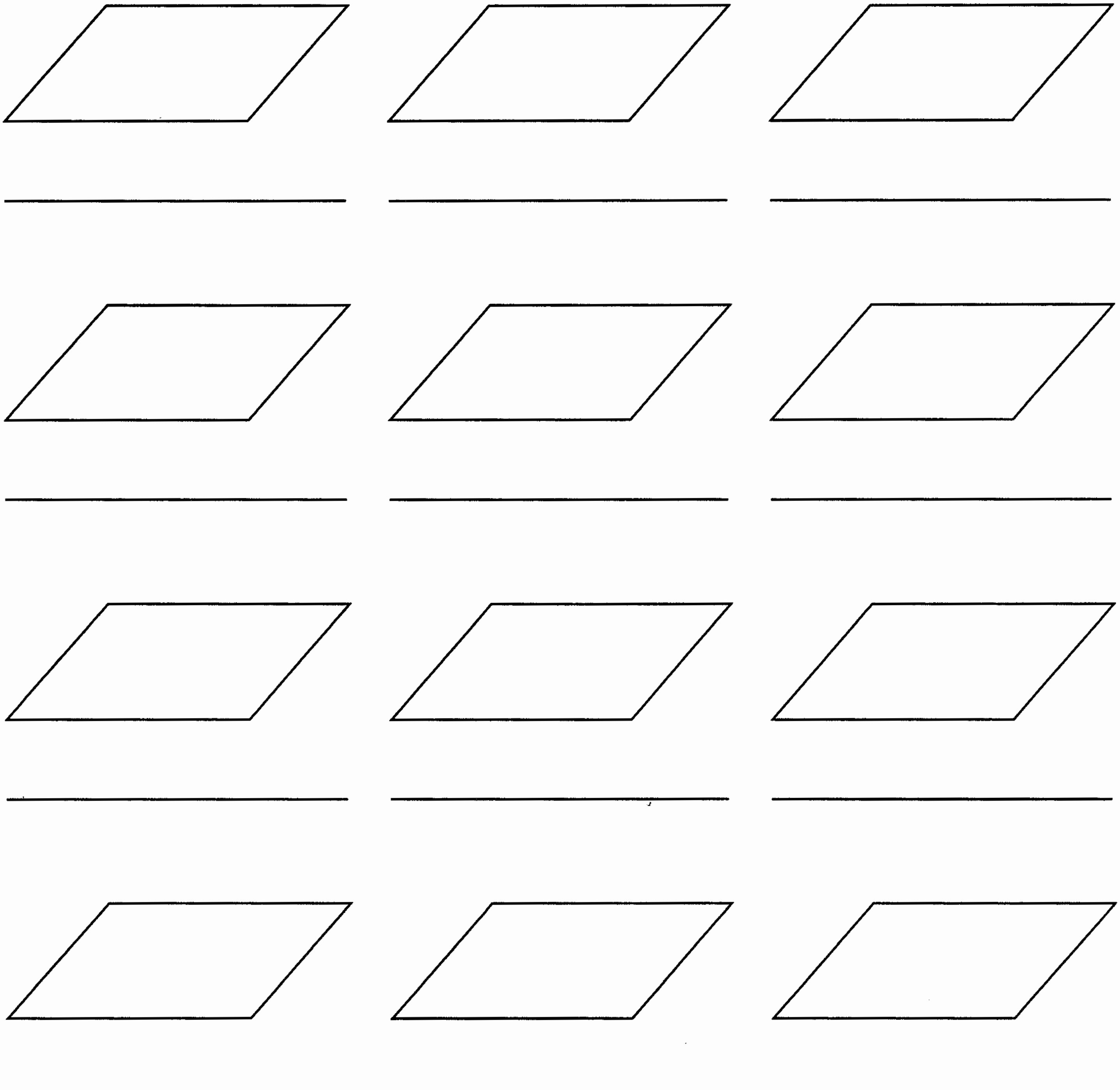
1. If, for a shape listed on page 1, a mirror can be placed upright on a parallelogram so that shape can be seen, write "Yes" in the appropriate space in Column 2, page 1. Indicate on one of the parallelograms on page 5 where the mirror should be placed to see the shape and write the name of the shape beneath the parallelogram.

2. In the remaining spaces in Column 2, page 1, indicate briefly why you believe the mirror cannot be placed to see the shape.

3. Pick a shape and write directions for placing a mirror on a parallelogram so that shape can be seen. Your directions can be written below or on the back of this sheet.

Name ____________________

Activity Sheet X-2-B



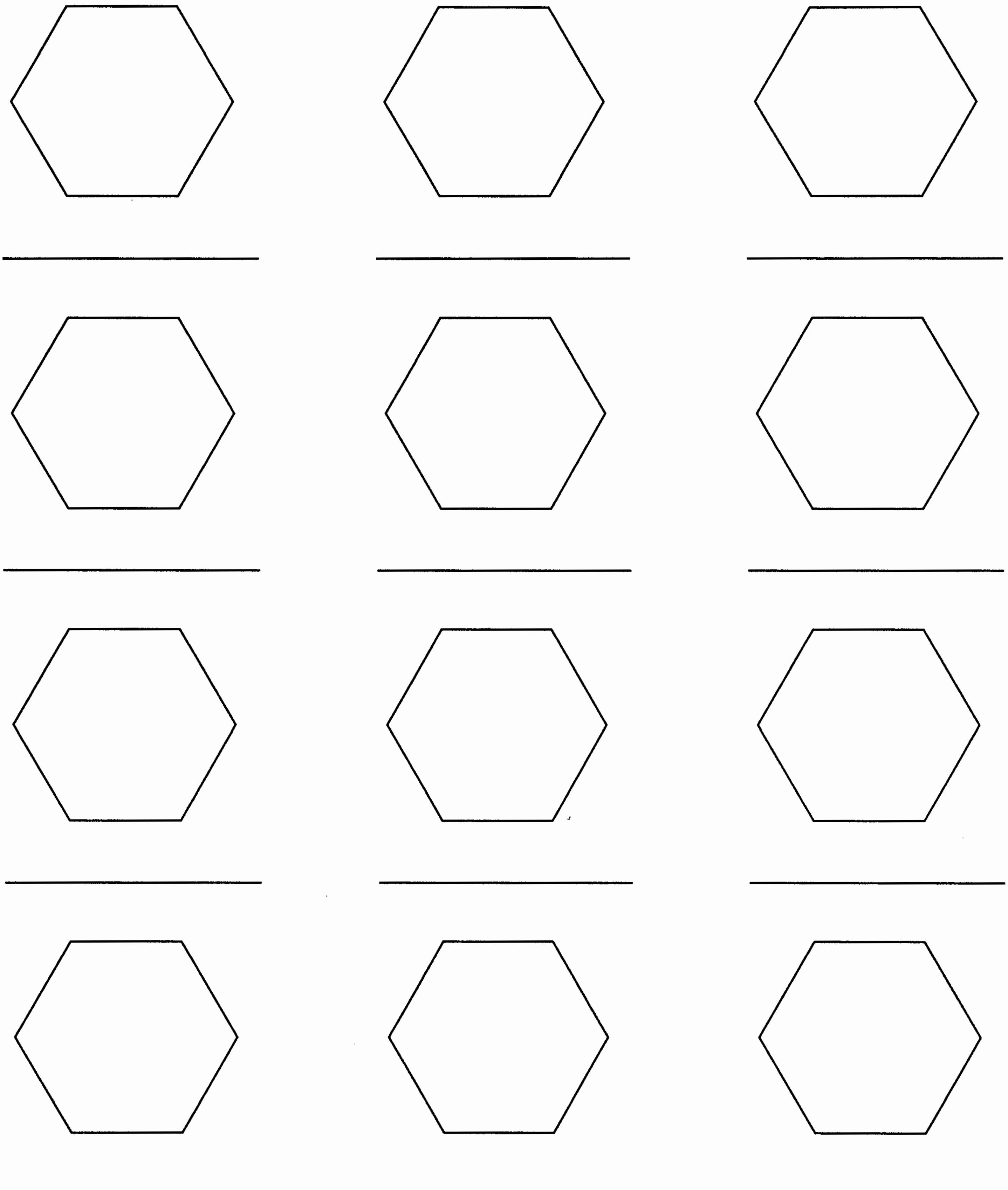
Name ____________________________

Imagining Mirrors on Regular Hexagons

1. If, for a shape listed on page 1, a mirror can be placed upright on a hexagon so that shape can be seen, write "Yes" in the appropriate space in Column 3, page 1. Indicate on one of the hexagons on page 7 where the mirror should be placed to see the shape and write the name of the shape beneath the hexagon.

2. In the remaining spaces in Column 3, page 1, indicate briefly why you believe the mirror cannot be placed to see the shape.

Name ______________________



Sue's Directions for Seeing A Square

Step 1. Draw a right angle.

Step 2. Bisect the right angle.

Step 3. Place the mirror perpendicular to the bisector in Step 2 (with the reflecting side facing the vertex of the right angle).

Step 4. Move the mirror, keeping its face towards the angle, until it cuts off equal lengths on the sides of the angle.

You will then see a square.

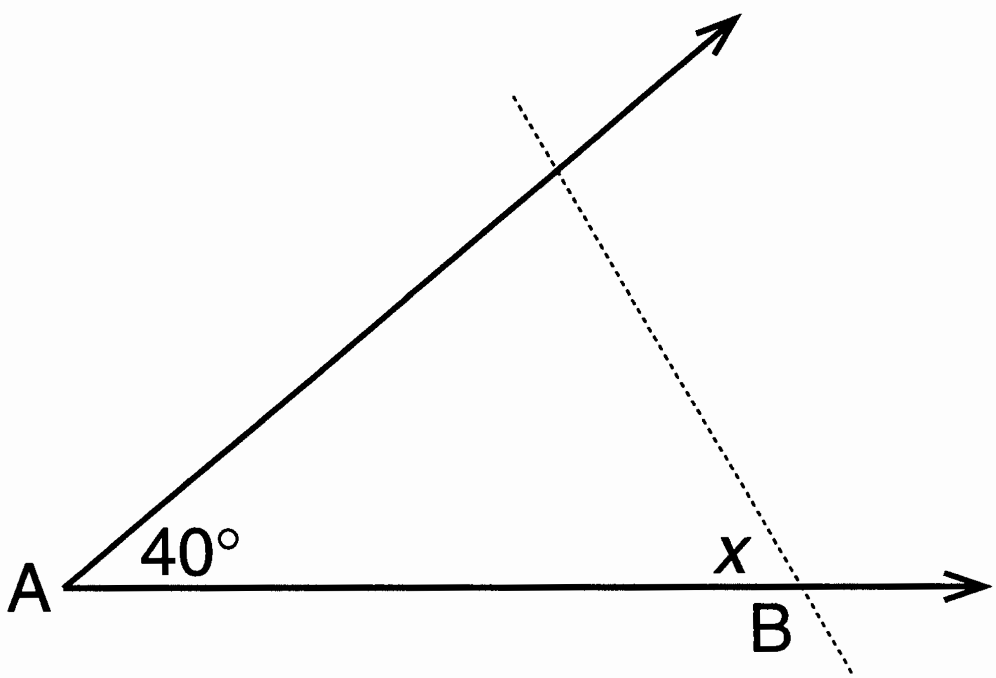
Bob's Directions for Seeing A Rhombus

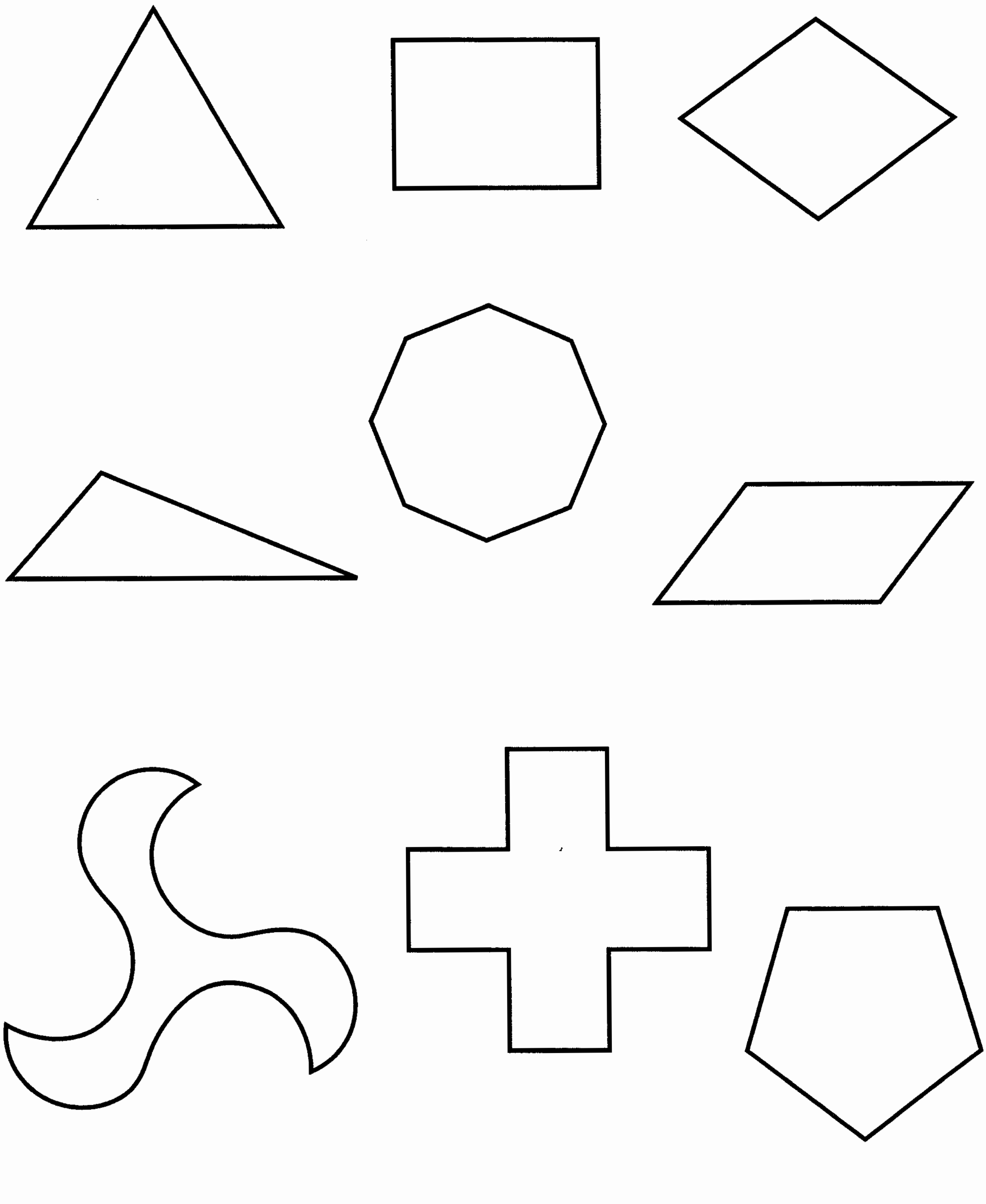
Step 1. Draw a right angle whose legs are 2 units long.

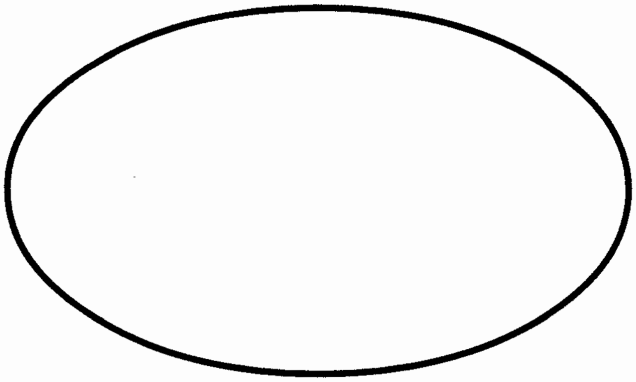
Step 2. Pick a leg. Draw the line that joins the midpoint of this leg to the vertex opposite it. Label this vertex A.

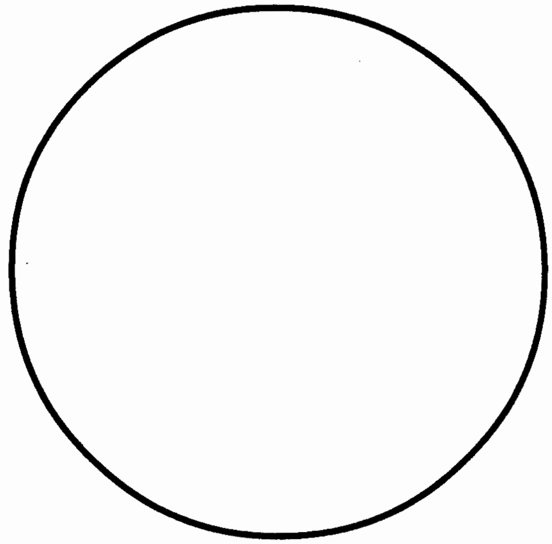
Step 3. Place a mirror perpendicular to the line drawn in Step 2 so the mirror faces A and intersects both sides of the triangle which form vertex A.

You will then see a rhombus.


40°
x
A
B







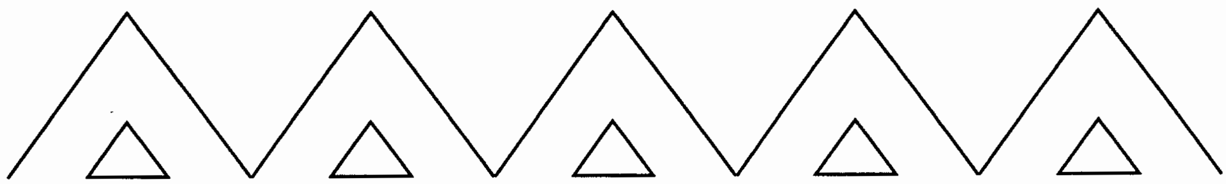
E E E E E E E

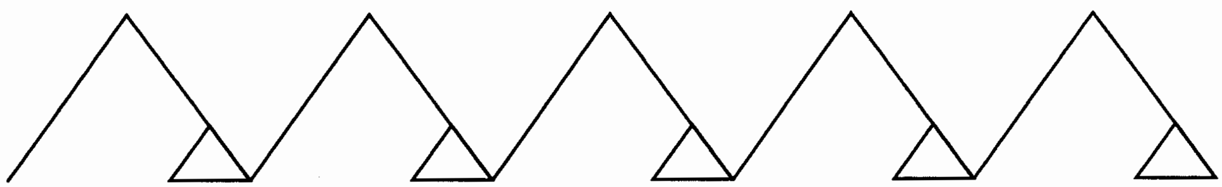
E E E E E E E

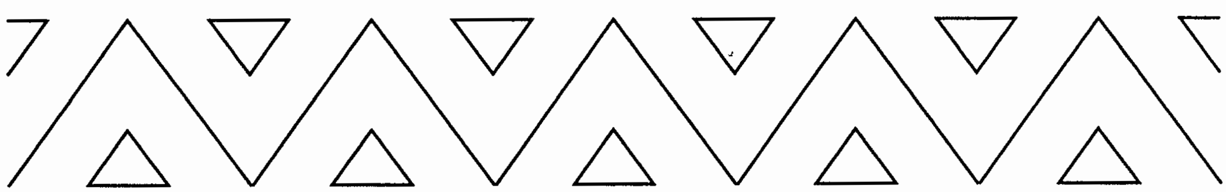
X X X X X X X

F F F F F F F

Name ______________________________

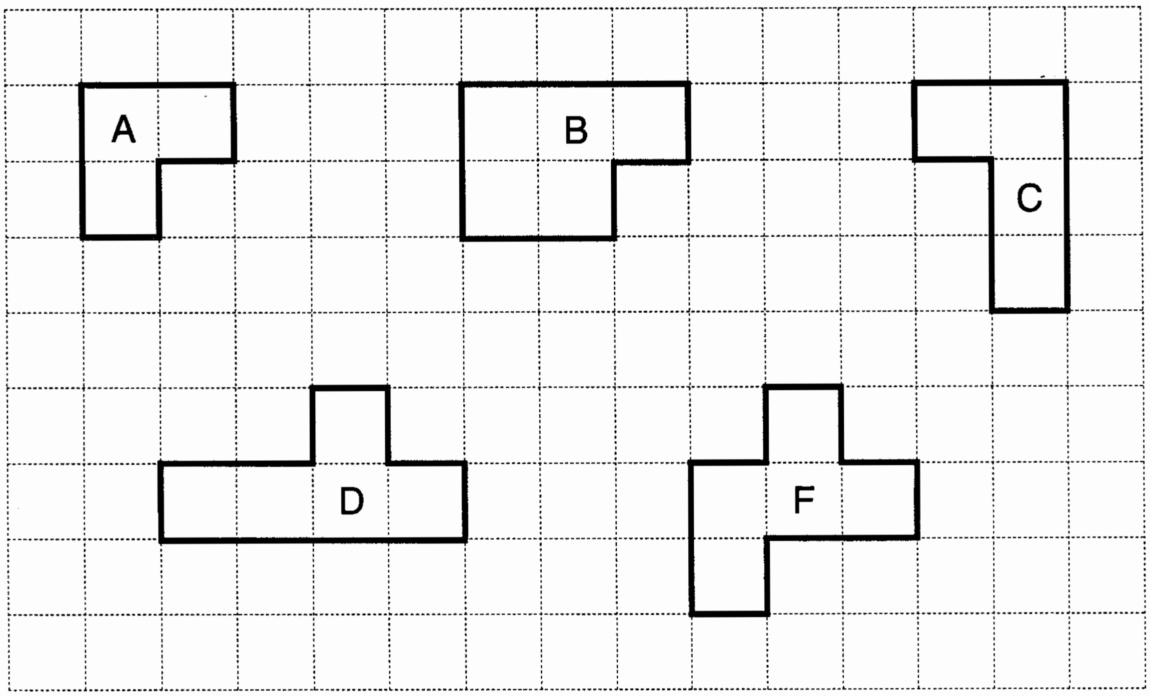




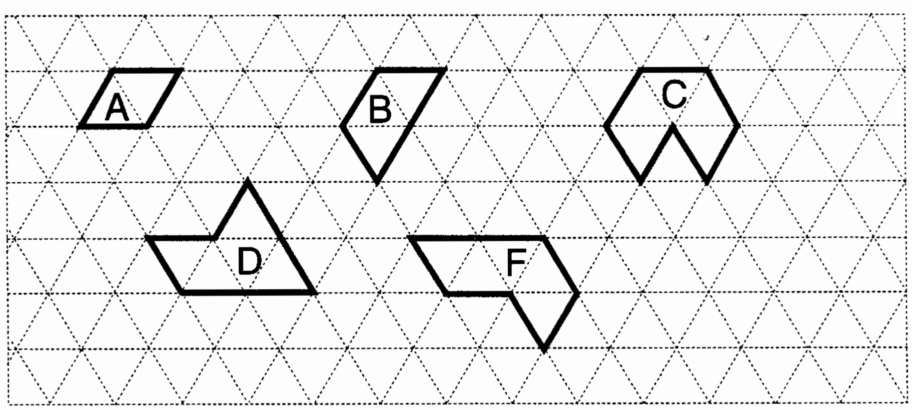


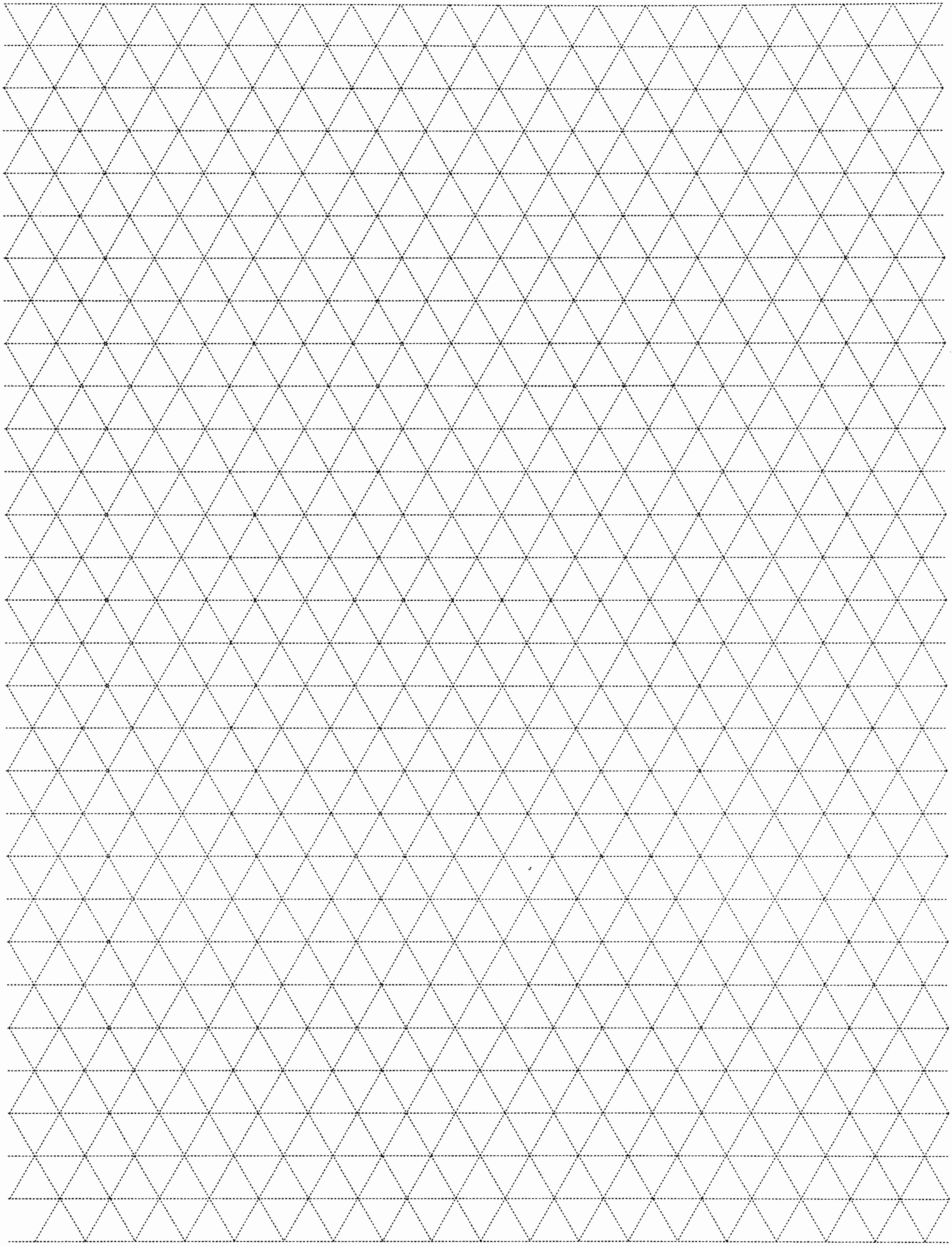
Name ____________________________

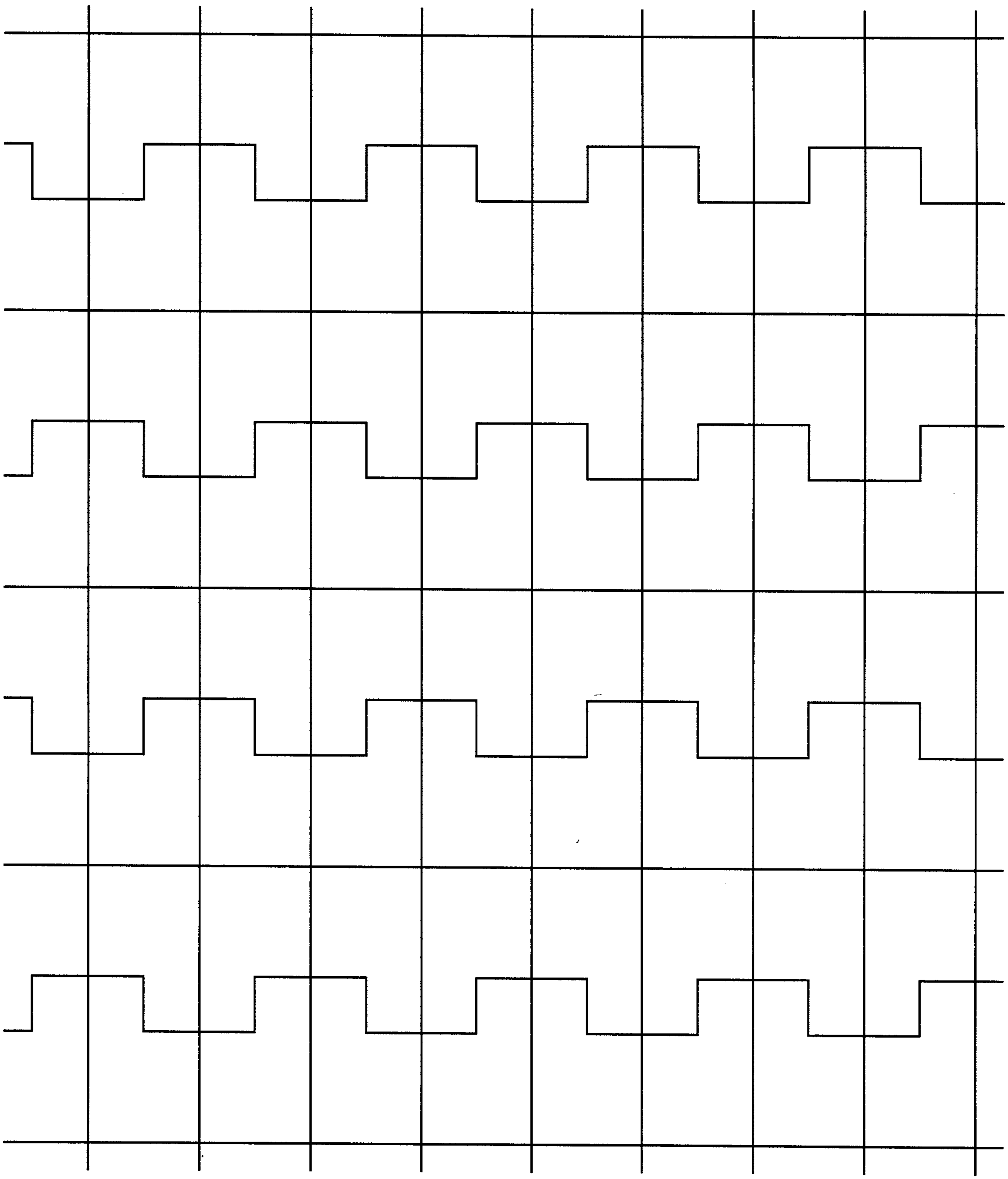
1. For each shape, in how many ways can one square be added to make it symmetrical?

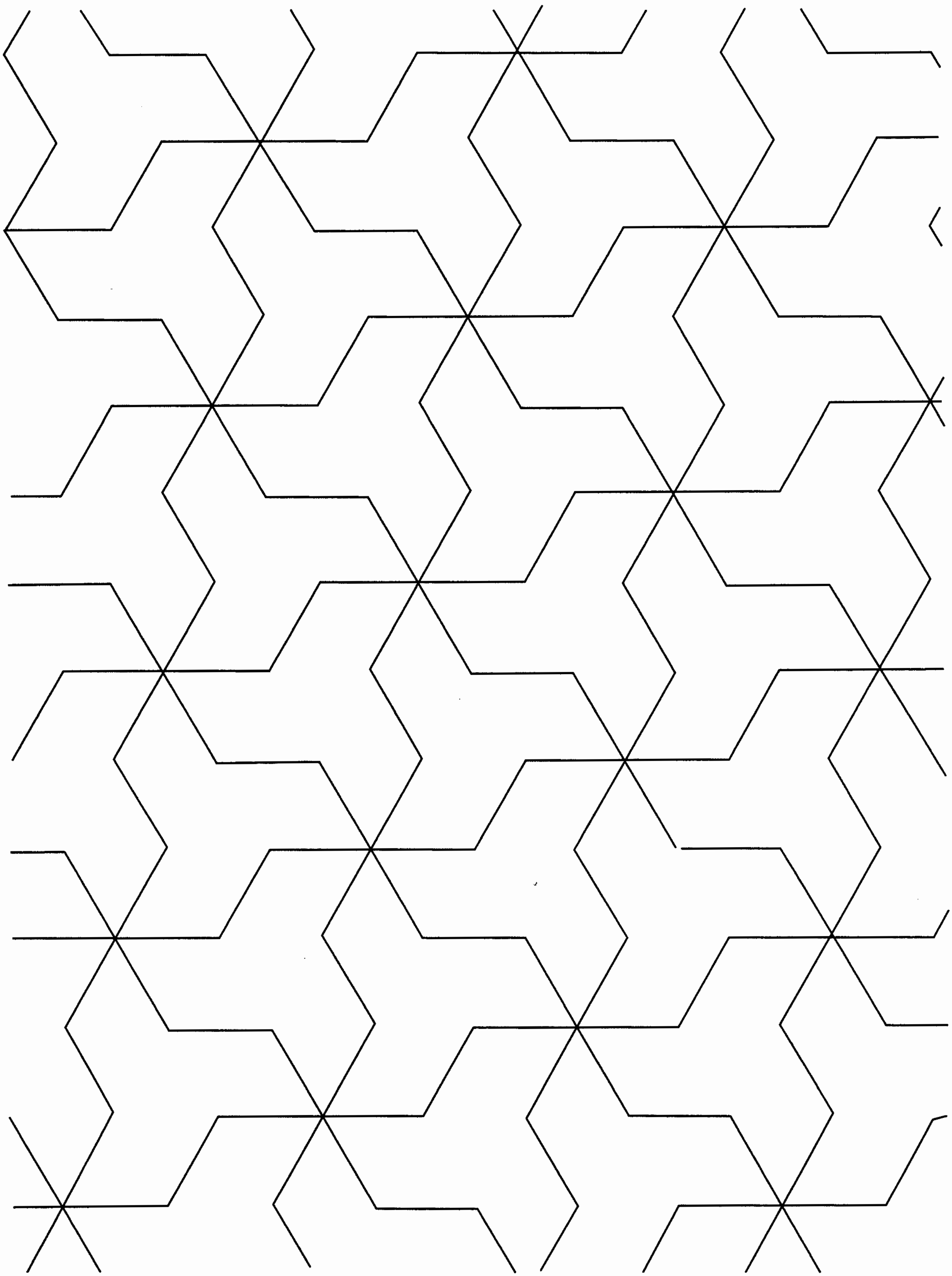


2. For each shape, in how many ways can one triangle be added to make it symmetrical?









2-cm Grid Paper

Triangular Grid Paper